Navier-Stokes-Fourier system: weak solutions, relative entropies, weak strong uniqueness

Summer school, Prague, August 27-31 2012

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Compilation of joint works with *Eduard Feireisl*

Plan of the mini course

1. Weak solutions
2. Relative entropy inequality
3. Weak strong uniqueness

1 Navier-Stokes-Fourier system

1.1 Classical formulation

• $T > 0$, $t \in [0, T]$ is time variable, Ω is a bounded domain in R^3, $x \in \Omega$ is a space variable. We are searching for unknown functions $\varrho(t, x)$ - density, $\vartheta(t, x)$- absolute temperature, $\mathbf{u}(t, x)$ - velocity vector satisfying

$$\partial_t \varrho + \mathrm{div}_x(\varrho \mathbf{u}) = 0, \tag{1.1}$$

$$\partial_t(\varrho \mathbf{u}) + \mathrm{div}_x(\varrho \mathbf{u} \otimes \mathbf{u}) + \nabla_x p(\varrho, \vartheta) = \mathrm{div}_x \mathbb{S}(\vartheta, \nabla_x \mathbf{u}), \tag{1.2}$$

$$\partial_t(\varrho s(\varrho, \vartheta)) + \mathrm{div}_x(\varrho s(\varrho, \vartheta)\mathbf{u}) + \mathrm{div}_x \left(\frac{\mathbf{q}(\vartheta, \nabla_x \vartheta)}{\vartheta} \right) = \sigma, \tag{1.3}$$

$$\sigma = \frac{\mathbb{S}(\vartheta, \nabla_x \mathbf{u})}{\vartheta} : \nabla_x \mathbf{u} - \frac{\mathbf{q}(\vartheta, \nabla_x \vartheta)}{\vartheta^2} \cdot \nabla_x \vartheta, \tag{1.4}$$

$$\mathbb{S}(\vartheta, \nabla_x \mathbf{u}) = \mu(\vartheta)\Big(\nabla_x \mathbf{u} + (\nabla_x \mathbf{u})^T - \frac{2}{3}\mathrm{div}_x \mathbf{u}\mathbb{I}\Big) + \eta(\vartheta)\mathrm{div}_x \mathbf{u}\mathbb{I}, \tag{1.5}$$

$$\mathbf{q}(\vartheta, \nabla_x \vartheta) = -\kappa(\vartheta)\nabla_x \vartheta \tag{1.6}$$

$$\mathbf{u}|_{\partial\Omega} = \mathbf{q} \cdot \mathbf{n}|_{\partial\Omega} = 0. \tag{1.7}$$

$$\mathrm{d}e - \vartheta \mathrm{d}s = \frac{p}{\varrho^2}\mathrm{d}\varrho.$$

1.2 Constitutive relations

1.2.1 Pressure, Internal energy, entropy

• Gibbs relations

We assume that the thermodynamic functions p, e, and s are interrelated through Gibbs' equation

$$\vartheta \mathrm{d}s(\varrho,\vartheta) = \mathrm{d}e(\varrho,\vartheta) - \frac{p(\varrho,\vartheta)}{\varrho^2}\mathrm{d}\varrho. \tag{1.8}$$

• Pressure

$$p(\varrho,\vartheta) = \vartheta^{\gamma/(\gamma-1)} P\left(\frac{\varrho}{\vartheta^{1/(\gamma-1)}}\right) + \frac{a}{3}\vartheta^4, \ a > 0, \ \gamma > 3/2, \tag{1.9}$$

where

$$P \in C^1[0,\infty), \ P(0) = 0, \ P'(Z) > 0 \text{ for all } Z \geq 0. \tag{1.10}$$

• Internal Energy

$$e(\varrho,\vartheta) = \frac{1}{\gamma-1}\frac{\vartheta^{\gamma/(\gamma-1)}}{\varrho} P\left(\frac{\varrho}{\vartheta^{1/(\gamma-1)}}\right) + a\frac{\vartheta^4}{\varrho}. \tag{1.11}$$

$$0 < \frac{\gamma P(Z) - P'(Z)Z}{Z} < c \text{ for all } Z > 0. \tag{1.12}$$

Relation (1.12) implies that the function $Z \mapsto P(Z)/Z^\gamma$ is decreasing, and we suppose that

$$\lim_{Z\to\infty} \frac{P(Z)}{Z^\gamma} = P_\infty > 0. \tag{1.13}$$

• Specific entropy

$$s(\varrho,\vartheta) = S\left(\frac{\varrho}{\vartheta^{1/(\gamma-1)}}\right) + \frac{4a}{3}\frac{\vartheta^3}{\varrho}, \tag{1.14}$$

where, in accordance with Third law of thermodynamics,

$$S'(Z) = -\frac{1}{\gamma-1}\frac{\gamma P(Z) - P'(Z)Z}{Z^2} < 0, \ \lim_{Z\to\infty} S(Z) = 0. \tag{1.15}$$

From the point of view of statististical mechanics, the above hypotheses are physically reasonable at least in two cases: if $\gamma = 5/3$ they modelize the monoatomic gas, if $\gamma = 4/3$ they modelize the so called relativistic gas.

1.2.2 Transport coefficients

$$\mu, \eta \in C^1[0,\infty) \text{ are globally Lipschitz and } \mu_0(1+\vartheta) \leq \mu(\vartheta),\ 0 \leq \eta(\vartheta), \quad \mu_0 > 0, \tag{1.16}$$

$$\kappa \in C^1[0,\infty), \quad \kappa_0(1+\vartheta^3) \leq \kappa(\vartheta) \leq \kappa_1(1+\vartheta^3),\ 0 < \kappa_0 \leq \kappa_1. \tag{1.17}$$

1.3 Weak solutions

• Weak solutions

Let $\Omega \subset R^3$ be a bounded Lipschitz domain. We say that a trio $\{\varrho, \vartheta, \mathbf{u}\}$ is a *weak solution* to the Navier-Stokes-Fourier system (1.1 - 1.7) emanating from the initial data

$$\varrho(0,\cdot) = \varrho_0,\ \varrho\mathbf{u}(0,\cdot) = \varrho_0\mathbf{u}_0,\ \varrho s(\varrho,\vartheta)(0,\cdot) = \varrho_0 s(\varrho_0,\vartheta_0), \quad \varrho_0 \geq 0,\ \vartheta_0 > 0 \tag{1.18}$$

if:

We say that a trio $(\varrho, \vartheta, \mathbf{u})$ is a weak solution *to the Navier-Stokes-Fourier system (1.1– 1.7) if:*

(i) the density and the absolute temperature satisfy $\varrho(t,x) \geq 0$, $\vartheta(t,x) > 0$ for a.a. $(t,x) \in (0,T)\times\Omega$, $\varrho \in L^\gamma(\Omega)$, $\varrho\mathbf{u} \in L^\infty(0,T; L^{2\gamma/(\gamma+1)}(\Omega;R^3))$, $\varrho\mathbf{u}^2 \in L^\infty(0,T;L^1(\Omega))$, $\vartheta \in L^4(\Omega))\cap L^2(0,T;W^{1,2}(\Omega))$, $\log\vartheta \in L^2(0,T;W^{1,2}(\Omega;R^3))$ and $\mathbf{u} \in L^2(0,T;W_0^{1,2}(\Omega;R^3))$;

(ii) $\varrho \in C_{\rm weak}([0,T];L^\gamma(\Omega))$ and equation (1.1) is replaced by a family of integral identities

$$\int_\Omega \varrho\varphi \ {\rm d}x\Big|_0^\tau = \int_0^\tau \int_\Omega \Big(\varrho\partial_t\varphi + \varrho\mathbf{u}\cdot\nabla_x\varphi\Big) \ {\rm d}x \ {\rm d}t \tag{1.19}$$

for all $\tau \in [0,T]$ and for any $\varphi \in C^1([0,T]\times\overline{\Omega})$;

(iii) $\varrho\mathbf{u} \in C_{\rm weak}([0,T];L^{2\gamma/(\gamma+1)}(\Omega;R^3))$ and momentum equation (1.2) is satisfied in the sense of distributions, specifically,

$$\int_\Omega \varrho\mathbf{u}\cdot\varphi \ {\rm d}x\Big|_0^\tau = \int_0^\tau \int_\Omega \Big(\varrho\mathbf{u}\cdot\partial_t\varphi + \varrho\mathbf{u}\otimes\mathbf{u} : \nabla_x\varphi + p(\varrho,\vartheta){\rm div}_x\varphi - \mathbb{S}(\vartheta,\nabla_x\mathbf{u}) : \nabla_x\varphi\Big) \ {\rm d}x \ {\rm d}t \tag{1.20}$$

for all $\tau \in [0,T]$ and for any $\varphi \in C_c^1([0,T]\times\Omega;R^3)$;

(iv) the entropy balance (1.3), (1.4) is replaced by a family of integral inequalities

$$-\int_\Omega \varrho s(\varrho,\vartheta)\varphi \ {\rm d}x\Big|_0^\tau + \int_0^\tau \int_\Omega \frac{\varphi}{\vartheta}\left(\mathbb{S}(\vartheta,\nabla_x\mathbf{u}) : \nabla_x\mathbf{u} - \frac{\mathbf{q}(\vartheta,\nabla_x\vartheta)\cdot\nabla_x\vartheta}{\vartheta}\right) \ {\rm d}x \ {\rm d}t \tag{1.21}$$

$$\leq -\int_0^\tau \int_\Omega \left(\varrho s(\varrho,\vartheta)\partial_t\varphi + \varrho s(\varrho,\vartheta)\mathbf{u}\cdot\nabla_x\varphi + \frac{\mathbf{q}(\vartheta,\nabla_x\vartheta)\cdot\nabla_x\varphi}{\vartheta}\right) \ {\rm d}x \ {\rm d}t$$

for a.a. $\tau \in (0,T)$ and for any $\varphi \in C^1([0,T]\times\overline{\Omega})$, $\varphi \geq 0$;

(v) the conservation of total energy in the volume Ω is verified

$$\int_\Omega \left(\frac{1}{2}\varrho|\mathbf{u}|^2 + \varrho e(\varrho,\vartheta)\right)(\tau,\cdot)\ \mathrm{d}x = \int_\Omega \left(\frac{1}{2}\varrho_0|\mathbf{u}_0|^2 + \varrho_0 e(\varrho_0,\vartheta_0)\right)\ \mathrm{d}x \tag{1.22}$$

for a.a. $\tau \in (0,T)$.

Here and hereafter, the symbol $\int_\Omega g\mathrm{d}x\Big|_0^\tau$ means $\int_\Omega g(x,\tau)\mathrm{d}x - \int_\Omega g_0(x)\mathrm{d}x$.

• **Renormalized weak solutions**

We say that the triplet $(\varrho,\vartheta,\mathbf{u})$ is a renormalized weak solution to the Navier-Stokes-Fourier system (1.1 - 1.7) if it is a very weak solution, if $b(\varrho) \in C_{\mathrm{weak}}([0,T],L^1(\Omega))$ and if the couple $(\varrho,\mathbf{u})$ satisfies the continuity equation in the renormalized sense,

$$\int_\Omega b(\varrho)\varphi\ \mathrm{d}x\Big|_0^\tau = \int_0^\tau\int_\Omega b(\varrho)\big(\partial_t\varphi + \mathbf{u}\cdot\nabla_x\varphi\big)\mathrm{d}x\mathrm{d}t + \int_0^\tau\int_\Omega \big(\varrho b'(\varrho) - b(\varrho)\big)\mathrm{div}\mathbf{u}\varphi\mathrm{d}x\mathrm{d}t \tag{1.23}$$

for any $\tau \in [0,T]$, and any

$$b \in C[0,\infty), \quad b' \in C_c[0,\infty) \quad \text{and} \quad \varphi \in C_c^1([0,T)\times\overline{\Omega}).$$

Notice that the set of admissible renormalizing functions b can be extended by density and Lebesgue dominated convergence theorem to

$$b \in C[0,\infty)\cap C^1(0,\infty),\ zb' \in L^\infty(0,1),\ b/z^{5\gamma/6},\ zb'/z^{\gamma/2} \in L^\infty(1,\infty).$$

1.4 Existence theorem

Theorem 1.1 *Let $\Omega \subset R^3$ be a bounded Lipschitz domain. Suppose that the thermodynamic functions p, e, s satisfy hypotheses (1.9 - 1.15), and that the transport coefficients μ, η, and κ obey (1.16), (1.17). Finally assume that the initial data (1.18) verify*

$$\int_\Omega \Big(\frac{1}{2}\varrho_0\mathbf{u}_0^2 + \varrho_0 e(\varrho_0,\vartheta_0) + \varrho_0|s(\varrho_0,\vartheta_0)|\Big)\ \mathrm{d}x < \infty. \tag{1.24}$$

Then the complete Navier-Stokes-Fourier system (1.1–1.7) admits at least one renormalized weak solution.

1.5 Suitability of the definition

- Weak solutions that are regular satisfy the NSF system in the classical sense
- One can prove existence of renormalized weak solutions for the NSF system
- Robustness: the definition is stable with respect to a great number of singular limits (low Mach, low Mach/high Reynolds) for ill prepared initial data
- Enjoys stability with respect to large time behavior
- Weak strong uniqueness: Weak solutions coincide with strong solutions with the same initial data as long as the strong solutions exist
- Beale-Kato-Majda type regularity criterion: Weak solutions are regular as soon as $\nabla_x \mathbf{u}$ remains bounded.

2 Relative entropy inequality

2.1 Motivation - Navier-Stokes-Poisson system

2.1.1 Germain's result

Let $p(\varrho) = \varrho^\gamma$, $\gamma > 3/2$ and $\Omega = T^3$ or $\mathbb{R}^3$.

Let $(\varrho, \mathbf{u})$ be a weak solution to the Navier-Stokes-Poisson system (6.1–6.4) with additional regularity

$$\nabla_x \varrho \in L^{2\gamma}(0, T; L^{\left(\frac{1}{2\gamma}+\frac{1}{3}\right)^{-1}}(\Omega)) \tag{2.1}$$

emanating from the initial data $(\varrho_0, \mathbf{u}_0)$.

Let $(r, \mathbf{U})$ be a classical (sufficiently) smooth solution emanating from the initial data $(\varrho_0, \mathbf{u}_0)$.

Let

$$E(\varrho, r) = H(\varrho) - H'(r)(\varrho - r) - H(r).$$

Then the following inequality holds:

$$\frac{d}{dt}\Big(\int_\Omega \frac{1}{2}\varrho(\mathbf{u}-\mathbf{U})^2 + E(\varrho, r)\Big) + \int_\Omega \mathbb{S}(\nabla_x(\mathbf{u}-\mathbf{U})) : \nabla_x(\mathbf{u}-\mathbf{U}) \leq \int_\Omega \varrho(\mathbf{u}-\mathbf{U})\cdot\nabla_x\mathbf{U}\cdot(\mathbf{u}-\mathbf{U})$$

$$+ \int_\Omega \frac{\varrho - r}{r}\mathrm{div}\mathbb{S}(\nabla_x\mathbf{U})\cdot(\mathbf{u}-\mathbf{U}) + \int_\Omega \mathrm{div}\mathbf{U}E(\varrho, r).$$

In particular

$$\varrho = r, \quad \mathbf{u} = \mathbf{U}.$$

2.1.2 Motivation for an intrinsic definition

Recall

$$E(\varrho, r) = H(\varrho) - H'(r)(\varrho - r) - H(r), \text{ where } H(\varrho) = \varrho \int_1^\varrho \frac{p'(s)}{s^2} \mathrm{d}s$$

Let $r = r(t,x)$, $\mathbf{U} = \mathbf{U}(t,x)$ be smooth functions defined on $[0,T] \times \overline{\Omega}$,

$$r > 0 \text{ on } [0,T] \times \overline{\Omega}, \ \mathbf{U}|_{\partial\Omega} = 0. \tag{2.2}$$

Suppose that ϱ, $\mathbf{u}$ is a smooth solution of the Navier-Stokes-Poisson system (6.1 - 6.4). A tedious but straightforward computation yields the following integral inequality

$$\int_\Omega \left(\frac{1}{2} \varrho |\mathbf{u} - \mathbf{U}|^2 + E(\varrho, r) \right) (\tau, \cdot) \ \mathrm{d}x + \int_0^\tau \int_\Omega [\mathbb{S}(\nabla_x \mathbf{u}) - \mathbb{S}(\nabla_x \mathbf{U})] : \nabla_x (\mathbf{u} - \mathbf{U}) \ \mathrm{d}x \ \mathrm{d}t \tag{2.3}$$
$$\leq \int_\Omega \left(\frac{1}{2} \varrho_0 |\mathbf{u}_0 - \mathbf{U}(0,\cdot)|^2 + E(\varrho_0, r(0,\cdot)) \right) \ \mathrm{d}x + \int_0^\tau \mathcal{R}\left(\varrho, \mathbf{u}, r, \mathbf{U}\right) \ \mathrm{d}t \text{ for a.a. } \tau \in (0,T),$$

where

$$\mathcal{R}\left(\varrho, \mathbf{u}, r, \mathbf{U}\right) = \int_\Omega \Big(\varrho \Big(\partial_t \mathbf{U} + \mathbf{u} \nabla_x \mathbf{U} \Big) \cdot (\mathbf{U} - \mathbf{u}) + \mathrm{div}_x \mathbb{S}(\nabla_x \mathbf{U})(\mathbf{u} - \mathbf{U}) \Big) \ \mathrm{d}x \tag{2.4}$$
$$+ \int_\Omega \Big((r - \varrho) \partial_t P(r) + \nabla_x P(r) \cdot (r \mathbf{U} - \varrho \mathbf{u}) - \mathrm{div}_x \mathbf{U} \Big(\varrho \Big(P(\varrho) - P(r) \Big) - E(\varrho, r) \Big) \Big) \ \mathrm{d}x,$$

and

$$P = H'.$$

For

$$r = \frac{1}{|\Omega|} \int_\Omega \varrho_0 \ \mathrm{d}x, \ \mathbf{U} \equiv 0$$

relation (2.3) reduces to (6.12). This motivates the following definition:

2.1.3 Definition of suitable weak solutions

We shall say that ϱ, $\mathbf{u}$ *represent a* suitable weak solution *to the Navier-Stokes system (6.1 - 6.4) if:*

- *the couple of functions* $(\varrho, \mathbf{u})$ *is a (bounded energy) weak solution to (6.1 - 6.4);*
- *the integral inequality (2.3) holds for any smooth functions* r, $\mathbf{U}$ *satisfying (2.2).*

2.1.4 Existence of suitable weak solutions

Feireisl, Sun, N.,2010

Theorem 2.1 *Let* $\Omega \subset R^3$ *be a bounded domain of class* $C^{2+\nu}$, $\nu > 0$. *Suppose that the pressure* p *is continuously differentiable on* $[0, \infty)$, *and*

$$p(0) = 0,\ p'(\varrho) > 0 \text{ for all } \varrho > 0,\ \lim_{\varrho \to \infty} \frac{p'(\varrho)}{\varrho^{\gamma - 1}} = a > 0 \tag{2.5}$$

for a certain $\gamma > 3/2$. *Let the initial data* ϱ_0, $\mathbf{u}_0$ *satisfy*

$$\varrho_0 \geq 0,\ \varrho_0 \not\equiv 0,\ \varrho_0 \in L^\gamma(\Omega),\ \varrho_0 |\mathbf{u}_0|^2 \in L^1(\Omega).$$

Then the Navier-Stokes system (6.1 - 6.4) possesses a suitable weak solution.

2.1.5 Any finite energy weak solution is a suitable one

Feireisl, Jin, N. (2011)

Theorem 2.2 *Let $\Omega \subset R^3$ be a bounded domain of class $C^{2+\nu}$, $\nu > 0$. Suppose that the pressure p is continuously differentiable on $[0, \infty)$, and*

$$p(0) = 0,\ p'(\varrho) > 0 \text{ for all } \varrho > 0,\ \lim_{\varrho \to \infty} \frac{p'(\varrho)}{\varrho^{\gamma - 1}} = a > 0 \tag{2.6}$$

for a certain $\gamma > 3/2$.

Then any weak solution emanating from the initial data

$$\varrho_0 \geq 0,\ \varrho_0 \not\equiv 0,\ \varrho_0 \in L^\gamma(\Omega),\ \varrho_0 |\mathbf{u}_0|^2 \in L^1(\Omega).$$

is a suitable weak solution.

3 Relative entropy inequality for the Navier-Stokes-Fourier system

3.1 Navier-Stokes-Fourier system

$$\partial_t \varrho + \mathrm{div}_x(\varrho \mathbf{u}) = 0, \tag{3.1}$$

$$\partial_t(\varrho \mathbf{u}) + \mathrm{div}_x(\varrho \mathbf{u} \otimes \mathbf{u}) + \nabla_x p(\varrho, \vartheta) = \mathrm{div}_x \mathbb{S}(\vartheta, \nabla_x \mathbf{u}), \tag{3.2}$$

$$\partial_t(\varrho s(\varrho, \vartheta)) + \mathrm{div}_x(\varrho s(\varrho, \vartheta)\mathbf{u}) + \mathrm{div}_x \left(\frac{\mathbf{q}(\vartheta, \nabla_x \vartheta)}{\vartheta} \right) = \sigma, \tag{3.3}$$

$$\sigma \geq \frac{\mathbb{S}(\vartheta, \nabla_x \mathbf{u})}{\vartheta} : \nabla_x \mathbf{u} - \frac{\mathbf{q}(\vartheta, \nabla_x \vartheta)}{\vartheta^2} \cdot \nabla_x \vartheta, \tag{3.4}$$

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega \left(\frac{1}{2} \varrho |\mathbf{u}|^2 + \varrho e(\varrho, \vartheta) \right) \, \mathrm{d}x = 0, \tag{3.5}$$

$$\mathbb{S}(\vartheta, \nabla_x \mathbf{u}) = \mu(\vartheta) \Big(\nabla_x \mathbf{u} + \nabla_x^t \mathbf{u} - \frac{2}{3} \mathrm{div}_x \mathbf{u} \mathbb{I} \Big) + \eta(\vartheta) \mathrm{div}_x \mathbf{u} \mathbb{I}, \tag{3.6}$$

$$\mathbf{q}(\vartheta, \nabla_x \vartheta) = -\kappa(\vartheta) \nabla_x \vartheta \tag{3.7}$$

$$\mathbf{u}|_{\partial\Omega} = \mathbf{q} \cdot \mathbf{n}|_{\partial\Omega} = 0. \tag{3.8}$$

$$\mathrm{d}e - \vartheta \mathrm{d}s = \frac{p}{\varrho^2} \mathrm{d}\varrho.$$

3.2 Dissipation inequality, Helmholtz function

Any weak solution satisfies the so called dissipation inequality:

$$\int_\Omega \left(\frac{1}{2} \varrho |\mathbf{u}|^2 + H_{\overline{\vartheta}}(\varrho, \vartheta) \right) (\tau, \cdot) \mathrm{d}x \tag{3.9}$$

$$+ \int_0^\tau \int_\Omega \frac{\overline{\vartheta}}{\vartheta} \left(\mathbb{S}(\vartheta, \nabla_x \mathbf{u}) : \nabla_x \mathbf{u} - \frac{\mathbf{q}(\vartheta, \nabla_x \vartheta) \cdot \nabla_x \vartheta}{\vartheta} \right) \mathrm{d}x\mathrm{d}t \leq$$

$$\int_\Omega \left(\frac{1}{2} \varrho_0 |\mathbf{u}_0|^2 + H_{\overline{\vartheta}}(\varrho_0, \vartheta_0) \right) \mathrm{d}x$$

for a.a. $\tau \in (0, T)$, where

$$H_{\overline{\vartheta}}(\varrho, \vartheta) = \varrho e(\varrho, \vartheta) - \overline{\vartheta} \varrho s(\varrho, \vartheta).$$

Indeed, the dissipation inequality (5.2) is obtained from the sum of identity (1.22) and the entropy balance (1.21) multiplied by $\overline{\vartheta}$.

We suppose that the fluid verifies the *thermodynamic stability conditions,*

$$\frac{\partial p(\varrho, \vartheta)}{\partial \varrho} > 0, \ \frac{\partial e(\varrho, \vartheta)}{\partial \vartheta} > 0 \text{ for all } \varrho, \vartheta > 0. \tag{3.10}$$

We easily verify by using (1.8), that

$$\frac{\partial H_{\overline{\vartheta}}}{\partial \vartheta}(\varrho, \vartheta) = \varrho \frac{\vartheta - \overline{\vartheta}}{\vartheta} \frac{\partial e}{\partial \vartheta}(\varrho, \vartheta) \text{ and } \frac{\partial^2 H_{\overline{\vartheta}}}{\partial \varrho^2}(\varrho, \overline{\vartheta}) = \frac{1}{\varrho} \frac{\partial p}{\partial \varrho}(\varrho, \overline{\vartheta}). \tag{3.11}$$

Thus, the thermodynamic stability in terms of the function $H_{\overline{\vartheta}}$, can be reformulated as follows:

$$\varrho \mapsto H_{\overline{\vartheta}}(\varrho, \overline{\vartheta}) \text{ is strictly convex,} \tag{3.12}$$

while

$$\vartheta \mapsto H_{\overline{\vartheta}}(\varrho, \vartheta) \text{ attains its global minimum at } \vartheta = \overline{\vartheta}. \tag{3.13}$$

We set

$$\mathcal{E}(\varrho, \vartheta \mid r, \Theta) = H_\Theta(\varrho, \vartheta) + \partial_\varrho H_\Theta(r, \Theta)(\varrho - r) + H_\Theta(r, \Theta)$$

and notice that

$$\mathcal{E}(\varrho, \vartheta \mid r, \Theta) \geq 0 \ \text{ and } \mathcal{E}(\varrho, \vartheta \mid r, \Theta) = 0 \Leftrightarrow (\varrho, \vartheta) = (r, \Theta).$$

We observe that the dissipation inequality (3.9) can be modified as follows

$$\int_\Omega \left(\frac{1}{2} \varrho |\mathbf{u}|^2 + \mathcal{E}\left(\varrho, \vartheta \,\middle|\, \overline{\varrho}, \overline{\vartheta}\right) \right) (\tau, \cdot) \mathrm{d}x \tag{3.14}$$

$$+ \int_0^\tau \int_\Omega \frac{\overline{\vartheta}}{\vartheta} \left(\mathbb{S}(\vartheta, \nabla_x \mathbf{u}) : \nabla_x \mathbf{u} - \frac{\mathbf{q}(\vartheta, \nabla_x \vartheta) \cdot \nabla_x \vartheta}{\vartheta} \right) \mathrm{d}x \mathrm{d}t \leq$$

$$\int_\Omega \left(\frac{1}{2} \varrho_0 |\mathbf{u}_0|^2 + \mathcal{E}\left(\varrho_0, \vartheta_0 \,\middle|\, \overline{\varrho}, \overline{\vartheta}\right) \right) \mathrm{d}x.$$

for a.a. $\tau \in (0, T)$.

3.3 Relative entropy inequality

Under the assumptions of the existence theory, any weak solution to the Navier-Stokes-Fourier system is a suitable weak one, meaning that it satisfies the relative entropy inequality:

$$\int_\Omega \left(\frac{1}{2}\varrho|\mathbf{u}-\mathbf{U}|^2 + \mathcal{E}(\varrho,\vartheta|r,\Theta)\right)(\tau,\cdot)\ \mathrm{d}x + \int_0^\tau \int_\Omega \frac{\Theta}{\vartheta}\left(\mathbb{S}(\vartheta,\nabla_x\mathbf{u}):\nabla_x\mathbf{u} - \frac{\mathbf{q}(\vartheta,\nabla_x\vartheta)\cdot\nabla_x\vartheta}{\vartheta}\right)\ \mathrm{d}x\ \mathrm{d}t \tag{3.15}$$

$$\leq \int_\Omega \left(\frac{1}{2}\varrho_0|\mathbf{u}_0 - \mathbf{U}(0,\cdot)|^2 + \mathcal{E}(\varrho_0,\vartheta_0|r(0,\cdot),\Theta(0,\cdot))\right)\ \mathrm{d}x$$

$$+\int_0^\tau \int_\Omega \left(\mathbb{S}(\vartheta,\nabla_x\mathbf{u}):\nabla_x\mathbf{U} - \frac{\mathbf{q}(\vartheta,\nabla_x\vartheta)}{\vartheta}\cdot\nabla_x\Theta\right)\ \mathrm{d}x\mathrm{d}t$$

$$+\int_0^\tau \int_\Omega \left(\varrho\Big(\partial_t\mathbf{U} + \mathbf{u}\cdot\nabla_x\mathbf{U}\Big)\cdot(\mathbf{U}-\mathbf{u}) - p(\varrho,\vartheta)\mathrm{div}_x\mathbf{U}\right)\ \mathrm{d}x\ \mathrm{d}t$$

$$-\int_0^\tau \int_\Omega \varrho\Big(s(\varrho,\vartheta) - s(r,\Theta)\Big)\Big(\partial_t\Theta + \mathbf{u}\cdot\nabla_x\Theta\Big)\ \mathrm{d}x\ \mathrm{d}t$$

$$+\int_0^\tau \int_\Omega \left(\left(1-\frac{\varrho}{r}\right)\partial_t p(r,\Theta) - \frac{\varrho}{r}\mathbf{u}\cdot\nabla_x p(r,\Theta)\right)\ \mathrm{d}x\ \mathrm{d}t,$$

$$r,\Theta,\mathbf{U}\in C^1_c([0,T]\times\overline{\Omega}),\quad \mathbf{U}|_{\partial\Omega}=0.$$

$$\mathcal{E}(\varrho,\vartheta\,|\,r,\Theta) = H_\theta(\varrho,\vartheta) - \partial_\varrho H_\Theta(r,\Theta)(\varrho - r) - H_\Theta(r,\Theta).$$

$$H_\Theta(\varrho,\vartheta) = \varrho\Big(e(\varrho,\vartheta) - \Theta s(\varrho,\vartheta)\Big)$$

3.4 An equivalent form of the relative entropy inequality

$$
\begin{aligned}
&\int_\Omega \left(\frac{1}{2} \varrho |\mathbf{u} - \mathbf{U}|^2 + \mathcal{E}(\varrho, \vartheta | r, \Theta) \right) (\tau, \cdot) \ \mathrm{d}x \\
&+ \int_0^\tau \int_\Omega \Theta \frac{\mathbb{S}(\vartheta, \nabla_x \mathbf{u})}{\vartheta} : \nabla_x \mathbf{u} \ \mathrm{d}x\mathrm{d}t - \int_0^\tau \int_\Omega \mathbb{S}(\vartheta, \nabla_x \mathbf{u}) : \nabla_x \mathbf{U} \mathrm{d}x\mathrm{d}t \\
&- \int_0^\tau \int_\Omega \Theta \frac{\mathbf{q}(\vartheta, \nabla_x \vartheta)}{\vartheta^2} : \nabla_x \vartheta \ \mathrm{d}x\mathrm{d}t + \int_0^\tau \int_\Omega \frac{\mathbf{q}(\vartheta, \nabla_x \vartheta)}{\vartheta} \cdot \nabla_x \Theta \mathrm{d}x\mathrm{d}t \\
&\leq \int_\Omega \left(\frac{1}{2} \varrho_0 |\mathbf{u}_0 - \mathbf{U}(0, \cdot)|^2 + \mathcal{E}(\varrho_0, \vartheta_0 | r(0, \cdot), \Theta(0, \cdot)) \right) \ \mathrm{d}x \\
&+ \int_0^\tau \int_\Omega \varrho \Big(\partial_t \mathbf{U} + \mathbf{u} \cdot \nabla_x \mathbf{U} \Big) \cdot (\mathbf{U} - \mathbf{u}) \ \mathrm{d}x \ \mathrm{d}t \\
&+ \int_0^\tau \int_\Omega \varrho (\mathbf{U} - \mathbf{u}) \cdot \frac{\nabla_x p(r, \Theta)}{r} \ \mathrm{d}x \ \mathrm{d}t, \\
&+ \int_0^\tau \int_\Omega \Big(p(r, \Theta) - p(\varrho, \vartheta) \Big) \mathrm{div}_x \mathbf{U} \ \mathrm{d}x\mathrm{d}t \\
&+ \int_0^\tau \int_\Omega \varrho \Big(s(r, \Theta) - s(\varrho, \vartheta) \Big) \Big(\partial_t \Theta + \mathbf{U} \cdot \nabla_x \Theta \Big) \ \mathrm{d}x \ \mathrm{d}t \\
&+ \int_0^\tau \int_\Omega \left(1 - \frac{\varrho}{r} \right) \Big(\partial_t p(r, \Theta) + \mathbf{U} \cdot \nabla_x p(r, \Theta) \Big) \ \mathrm{d}x \ \mathrm{d}t \\
&+ \int_0^\tau \int_\Omega \varrho \Big(s(r, \Theta) - s(\varrho, \vartheta) \Big) \Big(\mathbf{u} - \mathbf{U} \Big) \cdot \nabla_x \Theta \ \mathrm{d}x \ \mathrm{d}t.'
\end{aligned}
\tag{3.16}
$$

$$\int_\Omega \varrho\varphi \ \mathrm{d}x\Big|_0^\tau = \int_0^\tau \int_\Omega \Big(\varrho\partial_t\varphi + \varrho\mathbf{u}\cdot\nabla_x\varphi\Big) \ \mathrm{d}x \ \mathrm{d}t \tag{3.17}$$

$$\int_\Omega \varrho\mathbf{u}\cdot\varphi \ \mathrm{d}x\Big|_0^\tau = \int_0^\tau \int_\Omega \Big(\varrho\mathbf{u}\cdot\partial_t\varphi + \varrho\mathbf{u}\otimes\mathbf{u} : \nabla_x\varphi + p(\varrho,\vartheta)\mathrm{div}_x\varphi - \mathbb{S}(\vartheta,\nabla_x\mathbf{u}) : \nabla_x\varphi\Big) \ \mathrm{d}x \ \mathrm{d}t \tag{3.18}$$

$$\begin{aligned} -\int_\Omega \varrho s(\varrho,\vartheta)\varphi \ \mathrm{d}x\Big|_0^\tau + \int_0^\tau \int_\Omega \frac{\varphi}{\vartheta}\left(\mathbb{S}(\vartheta,\nabla_x\mathbf{u}) : \nabla_x\mathbf{u} - \frac{\mathbf{q}(\vartheta,\nabla_x\vartheta)\cdot\nabla_x\vartheta}{\vartheta}\right) \ \mathrm{d}x \ \mathrm{d}t \\ \leq -\int_0^\tau \int_\Omega \left(\varrho s(\varrho,\vartheta)\partial_t\varphi + \varrho s(\varrho,\vartheta)\mathbf{u}\cdot\nabla_x\varphi + \frac{\mathbf{q}(\vartheta,\nabla_x\vartheta)\cdot\nabla_x\varphi}{\vartheta}\right) \ \mathrm{d}x \ \mathrm{d}t \end{aligned} \tag{3.19}$$

$$\int_\Omega \left(\frac{1}{2}\varrho|\mathbf{u}|^2 + \varrho e(\varrho,\vartheta)\right)(\tau,\cdot) \ \mathrm{d}x = \int_\Omega \left(\frac{1}{2}\varrho_0|\mathbf{u}_0|^2 + \varrho_0 e(\varrho_0,\vartheta_0)\right) \ \mathrm{d}x \tag{3.20}$$

3.5 Relaxing regularity of test functions

Using a density argument we can extend the class of test functions r, Θ, $\mathbf{U}$ *appearing in the relative entropy inequality (3.15).*

- *Sufficient conditions for the left hand side of the relative entropy inequality to be well defined are, for example,*

$$0 < \underline{\varrho} \leq r \leq \overline{\varrho} < \infty, \quad 0 < \underline{\vartheta} \leq \Theta \leq \overline{\vartheta} < \infty, \tag{3.21}$$

$$\mathbf{U} \in L^\infty(0,T;L^6(\Omega,R^3)). \tag{3.22}$$

- *A short inspection of the right hand side (3.15) implies that the integrals are well-defined if, for example*

$$\partial_t \mathbf{U} \in L^\infty(0,T;L^6(\Omega;R^3)), \quad \nabla_x \mathbf{U} \in L^\infty(0,T;L^\infty(\Omega,R^{3\times 3})), \tag{3.23}$$

$$\partial_t \Theta \in L^\infty(0,T;L^4(\Omega)), \qquad \nabla_x \Theta \in L^\infty(0,T;L^\infty(\Omega;R^3)), \tag{3.24}$$

$$\partial_t r \in L^\infty(0,T;L^3(\Omega)), \quad \nabla_x r \in L^\infty(0,T;L^6(\Omega)). \tag{3.25}$$

- *Finally,*

$$\mathbf{U}|_{\partial\Omega} = 0. \tag{3.26}$$

Consequently, the relative entropy inequality (3.15), are valid even if we replace the hypotheses on smoothness and integrability of the test functions $(r, \Theta, \mathbf{U})$ *by weaker hypotheses, namely (3.21-3.26). In particular,* r, ϑ, $\mathbf{U}$ *may be another (strong) solution emanating from the same initial data* ϱ_0, ϑ_0 $\mathbf{u}_0$.

4 Weak strong uniqueness

4.1 Main result

We say that $\{\tilde{\varrho}, \tilde{\vartheta}, \tilde{\mathbf{u}}\}$ *is a classical (strong) solution to the Navier-Stokes-Fourier system in* $(0,T)\times\Omega$ *if*

$$\tilde{\varrho} \in C^1([0,T]\times\overline{\Omega}),\ \tilde{\vartheta},\ \partial_t\tilde{\vartheta},\ \nabla^2\tilde{\vartheta} \in C([0,T]\times\overline{\Omega}),\ \tilde{\mathbf{u}},\ \partial_t\tilde{\mathbf{u}},\ \nabla^2\tilde{\mathbf{u}} \in C([0,T]\times\overline{\Omega}; R^3), \quad (4.1)$$

$$\tilde{\varrho}(t,x) \geq \underline{\varrho} > 0,\ \tilde{\vartheta}(t,x) \geq \underline{\vartheta} > 0 \textit{ for all } (t,x),$$

and $\tilde{\varrho}$, $\tilde{\vartheta}$, $\tilde{\mathbf{u}}$ *satisfy equations (1.1 - 1.7).*

Theorem 4.1 *Let* $\Omega \subset R^3$ *be a bounded Lipschitz domain. Suppose that the thermodynamic functions p, e, s satisfy hypotheses (1.9 - 1.15), and that the transport coefficients* μ, η, *and* κ *obey (1.16), (1.17). Let* $(\varrho, \vartheta, \mathbf{u})$ *be a weak solution of the Navier-Stokes-Fourier system in* $(0,T)\times\Omega$ *in the sense specified in Section 1, and let* $(\tilde{\varrho}, \tilde{\vartheta}, \tilde{\mathbf{u}})$ *be a strong solution emanating from the same initial data. Then*

$$\varrho \equiv \tilde{\varrho},\ \vartheta \equiv \tilde{\vartheta},\ \mathbf{u} \equiv \tilde{\mathbf{u}}.$$

We will show weak-strong uniqueness by using the relative entropy inequality (3.15) with test functions $r = \tilde{\varrho}$, $\Theta = \tilde{\vartheta}$, *and* $\mathbf{U} = \tilde{\mathbf{u}}$.

The idea is to apply a Gronwall type argument to deduce the desired result. Here, the hypothesis of thermodynamic stability formulated in (5.3) and incorporation in (5.2) will play a crucial role.

The first step will therefore be to rewrite in this particular case inequality (3.15) in such a way that the Gronwall lemma can be applied.

4.2 Relative entropy inequality with a strong solution as a test function

Lemma 4.1

$$\int_\Omega \left(\frac{1}{2}\varrho|\mathbf{u}-\tilde{\mathbf{u}}|^2 + \mathcal{E}(\varrho,\vartheta|\tilde{\rho},\tilde{\vartheta})\right)(\tau,\cdot)\ \mathrm{d}x \tag{4.2}$$

$$+\int_0^\tau\int_\Omega \left(\frac{\tilde{\vartheta}}{\vartheta}\mathbb{S}(\vartheta,\nabla\mathbf{u}):\nabla\mathbf{u} - \mathbb{S}(\vartheta,\nabla\mathbf{u}):\nabla\tilde{\mathbf{u}} + \mathbb{S}(\tilde{\vartheta},\nabla\tilde{\mathbf{u}}):\nabla(\tilde{\mathbf{u}}-\mathbf{u}) + \frac{\vartheta-\tilde{\vartheta}}{\tilde{\vartheta}}\mathbb{S}(\tilde{\vartheta},\nabla\tilde{\mathbf{u}}):\nabla\tilde{\mathbf{u}}\right)\ \mathrm{d}x\mathrm{d}t$$

$$-\int_0^\tau\int_\Omega \left(\frac{\tilde{\vartheta}}{\vartheta}\frac{\mathbf{q}(\vartheta,\nabla_x\vartheta)}{\vartheta}\cdot\nabla_x\vartheta - \frac{\mathbf{q}(\vartheta,\nabla\vartheta)}{\vartheta}\cdot\nabla\tilde{\vartheta} + \frac{\mathbf{q}(\tilde{\vartheta},\nabla\tilde{\vartheta})}{\tilde{\vartheta}}\cdot\nabla(\tilde{\vartheta}-\vartheta) + \frac{\vartheta-\tilde{\vartheta}}{\tilde{\vartheta}}\frac{\mathbf{q}(\tilde{\vartheta},\nabla\tilde{\vartheta})}{\tilde{\vartheta}}\cdot\nabla\tilde{\vartheta}\right)\ \mathrm{d}x\mathrm{d}t$$

$$\le \mathcal{R}(\varrho,\vartheta,\mathbf{u}|\tilde{\varrho},\tilde{\vartheta},\tilde{\mathbf{u}})$$

where

$$\mathcal{R}(\varrho,\vartheta,\mathbf{u}|\tilde{\varrho},\tilde{\vartheta},\tilde{\mathbf{u}}) = \int_0^\tau\int_\Omega \Big((\varrho-\tilde{\varrho})\partial_t\tilde{\mathbf{u}} + (\varrho\mathbf{u}-\tilde{\varrho}\tilde{\mathbf{u}})\cdot\nabla_x\tilde{\mathbf{u}}\Big)\cdot(\tilde{\mathbf{u}}-\mathbf{u})\ \mathrm{d}x\mathrm{d}t$$

$$+\int_0^\tau\int_\Omega(\varrho-\tilde{\varrho})(\tilde{\mathbf{u}}-\mathbf{u})\cdot\frac{\nabla p(\tilde{\rho},\tilde{\vartheta})}{\tilde{\rho}}\ \mathrm{d}x\mathrm{d}t$$

$$-\int_0^\tau\int_\Omega\Big(\mathcal{S}(\varrho,\vartheta)-(\varrho-\tilde{\varrho})\partial_\varrho\mathcal{S}(\tilde{\varrho},\tilde{\vartheta})-(\vartheta-\tilde{\vartheta})\partial_\vartheta\mathcal{S}(\tilde{\varrho},\tilde{\vartheta})-\mathcal{S}(\tilde{\varrho},\tilde{\vartheta})\Big)\Big(\partial_t\tilde{\vartheta}+\tilde{\mathbf{u}}\cdot\nabla\tilde{\vartheta}\Big)\ \mathrm{d}x\mathrm{d}t$$

$$-\int_0^\tau\int_\Omega\Big(p(\varrho,\vartheta)-(\varrho-\tilde{\varrho})\partial_\varrho p(\tilde{\varrho},\tilde{\vartheta})-(\vartheta-\tilde{\vartheta})\partial_\vartheta p(\tilde{\varrho},\tilde{\vartheta})-p(\tilde{\varrho},\tilde{\vartheta})\Big)\mathrm{div}\tilde{\mathbf{u}}\ \mathrm{d}x\mathrm{d}t$$

$$+\int_0^\tau\int_\Omega\varrho\Big(s(\tilde{\varrho},\tilde{\vartheta})-s(\varrho,\vartheta)\Big)(\mathbf{u}-\tilde{\mathbf{u}})\cdot\nabla_x\tilde{\vartheta}\ \mathrm{d}x\mathrm{d}t,$$

with

$$\mathcal{S}(\varrho,\vartheta) = \varrho s(\varrho,\vartheta). \tag{4.3}$$

4.3 First step to get (4.2)

$$\int_\Omega \left(\frac{1}{2}\varrho|\mathbf{u}-\tilde{\mathbf{u}}|^2+\mathcal{E}(\varrho,\vartheta|\tilde\rho,\tilde\vartheta)\right)(\tau,\cdot)\ \mathrm{d}x \tag{4.4}$$

$$+\int_0^\tau\int_\Omega\left(\tilde\vartheta\frac{\mathbb{S}(\vartheta,\nabla\mathbf{u})}{\vartheta}:\nabla\mathbf{u}-\mathbb{S}(\vartheta,\nabla\mathbf{u}):\nabla\tilde{\mathbf{u}}\right)\ \mathrm{d}x\mathrm{d}t$$

$$-\int_0^\tau\int_\Omega\left(\frac{\tilde\vartheta}{\vartheta}\frac{\mathbf{q}(\vartheta,\nabla_x\vartheta)}{\vartheta}\cdot\nabla_x\vartheta-\frac{\mathbf{q}(\vartheta,\nabla\vartheta)}{\vartheta}\cdot\nabla\tilde\vartheta\right)\ \mathrm{d}x\mathrm{d}t$$

$$\leq\int_0^\tau\int_\Omega\varrho\Big(\partial_t\tilde{\mathbf{u}}+\mathbf{u}\cdot\nabla_x\tilde{\mathbf{u}}\Big)\cdot(\tilde{\mathbf{u}}-\mathbf{u})\ \mathrm{d}x\mathrm{d}t$$

$$+\int_0^\tau\int_\Omega\varrho\Big(s(\tilde\rho,\tilde\vartheta)-s(\varrho,\vartheta)\Big)(\partial_t\tilde\vartheta+\tilde{\mathbf{u}}\cdot\nabla\tilde\vartheta)\ \mathrm{d}x\mathrm{d}t+\int_0^\tau\int_\Omega\left(1-\frac{\varrho}{\tilde\rho}\right)\Big(\partial_t p(\tilde\rho,\tilde\vartheta)+\tilde{\mathbf{u}}\cdot\nabla_x p(\tilde\rho,\tilde\vartheta)\Big)\ \mathrm{d}x\mathrm{d}t$$

$$+\int_0^\tau\int_\Omega\Big(p(\tilde\rho,\tilde\vartheta)-p(\varrho,\vartheta)\Big)\mathrm{div}\tilde{\mathbf{u}}\ \mathrm{d}x\mathrm{d}t+\int_0^\tau\int_\Omega\varrho(\tilde{\mathbf{u}}-\mathbf{u})\cdot\frac{\nabla p(\tilde\rho,\tilde\vartheta)}{\tilde\rho}\ \mathrm{d}x\mathrm{d}t$$

$$+\int_0^\tau\int_\Omega\varrho\Big(s(\tilde\varrho,\tilde\vartheta)-s(\varrho,\vartheta)\Big)(\mathbf{u}-\tilde{\mathbf{u}})\cdot\nabla_x\tilde\vartheta\ \mathrm{d}x\mathrm{d}t.$$

We denote

$$A=\Big(\tilde\rho(\partial_t\tilde{\mathbf{u}}+\tilde{\mathbf{u}}\cdot\nabla\tilde{\mathbf{u}})+\nabla p(\tilde\rho,\tilde\vartheta)\Big)\cdot(\mathbf{u}-\tilde{\mathbf{u}})+\mathbb{S}(\tilde\vartheta,\nabla\tilde{\mathbf{u}}):\nabla(\mathbf{u}-\tilde{\mathbf{u}})$$

and notice that

$$\int_\Omega A dx=0.$$

Adding $\int_0^\tau\int_\Omega \mathrm{A}\mathrm{d}x\mathrm{d}t$ *to the right hand side of (4.4) we obtain*

$$\int_\Omega \left(\frac{1}{2}\varrho|\mathbf{u}-\tilde{\mathbf{u}}|^2+\mathcal{E}(\varrho,\vartheta|\tilde{\rho},\tilde{\vartheta})\right)(\tau,\cdot)\ \mathrm{d}x \tag{4.5}$$

$$+\int_0^\tau\int_\Omega\left(\tilde{\vartheta}\frac{\mathbb{S}(\vartheta,\nabla\mathbf{u})}{\vartheta}:\nabla\mathbf{u}-\mathbb{S}(\vartheta,\nabla\mathbf{u}):\nabla\tilde{\mathbf{u}}+\mathbb{S}(\tilde{\vartheta},\nabla\tilde{\mathbf{u}}):\nabla(\tilde{\mathbf{u}}-\mathbf{u})\right)\ \mathrm{d}x\mathrm{d}t$$

$$-\int_0^\tau\int_\Omega\left(\frac{\tilde{\vartheta}}{\vartheta}\frac{\mathbf{q}(\vartheta,\nabla_x\vartheta)}{\vartheta}\cdot\nabla_x\vartheta-\frac{\mathbf{q}(\vartheta,\nabla\vartheta)}{\vartheta}\cdot\nabla\tilde{\vartheta}\right)\ \mathrm{d}x\mathrm{d}t$$

$$\leq\int_0^\tau\int_\Omega\Big((\varrho-\tilde{\varrho})\partial_t\tilde{\mathbf{u}}+(\varrho\mathbf{u}-\tilde{\varrho}\tilde{\mathbf{u}})\cdot\nabla_x\tilde{\mathbf{u}}\Big)\cdot(\tilde{\mathbf{u}}-\mathbf{u})\ \mathrm{d}x\mathrm{d}t$$

$$+\int_0^\tau\int_\Omega\varrho\Big(s(\tilde{\rho},\tilde{\vartheta})-s(\varrho,\vartheta)\Big)(\partial_t\tilde{\vartheta}+\tilde{\mathbf{u}}\cdot\nabla\tilde{\vartheta})\ \mathrm{d}x\mathrm{d}t+\int_0^\tau\int_\Omega\left(1-\frac{\varrho}{\tilde{\rho}}\right)\Big(\partial_t p(\tilde{\rho},\tilde{\vartheta})+\tilde{\mathbf{u}}\cdot\nabla_x p(\tilde{\rho},\tilde{\vartheta})\Big)\ \mathrm{d}x\mathrm{d}t$$

$$+\int_0^\tau\int_\Omega\Big(p(\tilde{\rho},\tilde{\vartheta})-p(\varrho,\vartheta)\Big)\mathrm{div}\tilde{\mathbf{u}}\ \mathrm{d}x\mathrm{d}t+\int_0^\tau\int_\Omega(\varrho-\tilde{\varrho})(\tilde{\mathbf{u}}-\mathbf{u})\cdot\frac{\nabla p(\tilde{\rho},\tilde{\vartheta})}{\tilde{\rho}}\ \mathrm{d}x\mathrm{d}t$$

$$+\int_0^\tau\int_\Omega\varrho\Big(s(\tilde{\varrho},\tilde{\vartheta})-s(\varrho,\vartheta)\Big)(\mathbf{u}-\tilde{\mathbf{u}})\cdot\nabla_x\tilde{\vartheta}\ \mathrm{d}x\mathrm{d}t.$$

4.4 Second step in the proof of (4.2)

We denote

$$B=(\vartheta-\tilde{\vartheta})\Big(\tilde{\rho}(\partial_t s(\tilde{\rho},\tilde{\vartheta})+\tilde{\mathbf{u}}\cdot\nabla s(\tilde{\rho},\tilde{\vartheta}))-\frac{\mathbb{S}(\tilde{\vartheta},\nabla\tilde{\mathbf{u}}):\nabla\tilde{\mathbf{u}}}{\tilde{\vartheta}}+\frac{\mathbf{q}(\tilde{\vartheta},\nabla\tilde{\vartheta})\cdot\nabla\tilde{\vartheta}}{\tilde{\vartheta}^2}\Big)+\frac{\mathbf{q}(\tilde{\vartheta},\nabla\tilde{\vartheta})\cdot\nabla(\tilde{\vartheta}-\vartheta)}{\tilde{\vartheta}}.$$

and notice

$$\int_0^\tau\int_\Omega B\mathrm{d}x\mathrm{d}t=0,$$

Adding $\int_0^\tau\int_\Omega B\mathrm{d}x\mathrm{d}t$ *to the right hand side of (4.5)*

$$\int_\Omega \left(\frac{1}{2}\varrho|\mathbf{u}-\tilde{\mathbf{u}}|^2+\mathcal{E}(\varrho,\vartheta|\tilde{\rho},\tilde{\vartheta})\right)(\tau,\cdot)\ \mathrm{d}x \tag{4.6}$$

$$+\int_0^\tau\int_\Omega\left(\frac{\tilde{\vartheta}}{\vartheta}\mathbb{S}(\vartheta,\nabla\mathbf{u}):\nabla\mathbf{u}-\mathbb{S}(\vartheta,\nabla\mathbf{u}):\nabla\tilde{\mathbf{u}}+\mathbb{S}(\tilde{\vartheta},\nabla\tilde{\mathbf{u}}):\nabla(\tilde{\mathbf{u}}-\mathbf{u})+\frac{\vartheta-\tilde{\vartheta}}{\tilde{\vartheta}}\mathbb{S}(\tilde{\vartheta},\nabla\tilde{\mathbf{u}}):\nabla\tilde{\mathbf{u}}\right)\ \mathrm{d}x\mathrm{d}t$$

$$-\int_0^\tau\int_\Omega\left(\frac{\tilde{\vartheta}}{\vartheta}\frac{\mathbf{q}(\vartheta,\nabla_x\vartheta)}{\vartheta}\cdot\nabla_x\vartheta-\frac{\mathbf{q}(\vartheta,\nabla\vartheta)}{\vartheta}\cdot\nabla\tilde{\vartheta}+\frac{\mathbf{q}(\tilde{\vartheta},\nabla\tilde{\vartheta})}{\tilde{\vartheta}}\cdot\nabla(\tilde{\vartheta}-\vartheta)+\frac{\vartheta-\tilde{\vartheta}}{\tilde{\vartheta}}\frac{\mathbf{q}(\tilde{\vartheta},\nabla\tilde{\vartheta})}{\tilde{\vartheta}}\cdot\nabla\tilde{\vartheta}\right)\ \mathrm{d}x\mathrm{d}t$$

$$\leq\int_0^\tau\int_\Omega\Big((\varrho-\tilde{\varrho})\partial_t\tilde{\mathbf{u}}+(\varrho\mathbf{u}-\tilde{\varrho}\tilde{\mathbf{u}})\cdot\nabla_x\tilde{\mathbf{u}}\Big)\cdot(\tilde{\mathbf{u}}-\mathbf{u})\ \mathrm{d}x\mathrm{d}t$$

$$+\int_0^\tau\int_\Omega\varrho\Big(s(\tilde{\rho},\tilde{\vartheta})-s(\varrho,\vartheta)\Big)\Big(\partial_t\tilde{\vartheta}+\tilde{\mathbf{u}}\cdot\nabla\tilde{\vartheta}\Big)\ \mathrm{d}x\mathrm{d}t$$

$$+\int_0^\tau\int_\Omega\Big[\left(1-\frac{\varrho}{\tilde{\rho}}\right)\Big(\partial_t p(\tilde{\rho},\tilde{\vartheta})+\tilde{\mathbf{u}}\cdot\nabla_x p(\tilde{\rho},\tilde{\vartheta})\Big)+\tilde{\rho}\Big(\partial_t s(\tilde{\rho},\tilde{\vartheta})+\tilde{\mathbf{u}}\cdot\nabla s(\tilde{\rho},\tilde{\vartheta})\Big)(\vartheta-\tilde{\vartheta})\Big]\ \mathrm{d}x\mathrm{d}t$$

$$+\int_0^\tau\int_\Omega\Big(p(\tilde{\rho},\tilde{\vartheta})-p(\varrho,\vartheta)\Big)\mathrm{div}\tilde{\mathbf{u}}\ \mathrm{d}x\mathrm{d}t+\int_0^\tau\int_\Omega(\varrho-\tilde{\varrho})(\tilde{\mathbf{u}}-\mathbf{u})\cdot\frac{\nabla p(\tilde{\rho},\tilde{\vartheta})}{\tilde{\rho}}\ \mathrm{d}x\mathrm{d}t$$

$$+\int_0^\tau\int_\Omega\varrho\Big(s(\tilde{\varrho},\tilde{\vartheta})-s(\varrho,\vartheta)\Big)(\mathbf{u}-\tilde{\mathbf{u}})\cdot\nabla_x\tilde{\vartheta}\ \mathrm{d}x\mathrm{d}t.$$

4.5 Using Gibbs relation

Recall that

$$\frac{1}{\rho}\partial_\vartheta p(\rho,\vartheta) = -\rho\partial_\rho s(\rho,\vartheta)$$

and

$$\partial_\varrho \mathcal{S}(\tilde\varrho,\tilde\vartheta) = s(\tilde\varrho,\tilde\vartheta) + \tilde\varrho\partial_\varrho s(\tilde\varrho,\tilde\vartheta), \quad \partial_\vartheta \mathcal{S}(\tilde\varrho,\tilde\vartheta) = \tilde\varrho\partial_\vartheta s(\tilde\varrho,\tilde\vartheta).$$

we get

$$\left(1-\frac{\varrho}{\tilde\rho}\right)\Big(\partial_t p(\tilde\rho,\tilde\vartheta) + \tilde{\mathbf{u}}\cdot\nabla_x p(\tilde\rho,\tilde\vartheta)\Big) + (\vartheta-\tilde\vartheta)\tilde\rho\Big(\partial_t s(\tilde\rho,\tilde\vartheta) + \tilde{\mathbf{u}}\cdot\nabla s(\tilde\rho,\tilde\vartheta)\Big) \tag{4.7}$$

$$= \Big(\left(1-\frac{\varrho}{\tilde\rho}\right)\partial_\rho p(\tilde\rho,\tilde\vartheta) + \tilde\rho(\vartheta-\tilde\vartheta)\partial_\rho s(\tilde\rho,\tilde\vartheta)\Big)\Big(\partial_t\tilde\rho + \tilde{\mathbf{u}}\cdot\nabla\tilde\rho\Big)$$

$$+\Big(\left(1-\frac{\varrho}{\tilde\rho}\right)\partial_\vartheta p(\tilde\rho,\tilde\vartheta) + \tilde\rho(\vartheta-\tilde\vartheta)\partial_\vartheta s(\tilde\rho,\tilde\vartheta)\Big)\Big(\partial_t\tilde\vartheta + \tilde{\mathbf{u}}\cdot\nabla\tilde\vartheta\Big).$$

$$\Big(\left(1-\frac{\varrho}{\tilde\rho}\right)\partial_\rho p(\tilde\rho,\tilde\vartheta) + \tilde\rho(\vartheta-\tilde\vartheta)\partial_\rho s(\tilde\rho,\tilde\vartheta)\Big)\Big(\partial_t\tilde\rho + \tilde{\mathbf{u}}\cdot\nabla\tilde\rho\Big) \tag{4.8}$$

$$= \Big((\varrho-\tilde\rho)\partial_\rho p(\tilde\rho,\tilde\vartheta) + (\vartheta-\tilde\vartheta)\partial_\vartheta p(\tilde\rho,\tilde\vartheta)\Big)div\tilde{\mathbf{u}}$$

---

and

$$\Big(\left(1-\frac{\varrho}{\tilde\rho}\right)\partial_\vartheta p(\tilde\rho,\tilde\vartheta) + \tilde\rho(\vartheta-\tilde\vartheta)\partial_\vartheta s(\tilde\rho,\tilde\vartheta)\Big)\Big(\partial_t\tilde\vartheta + \tilde{\mathbf{u}}\cdot\nabla\tilde\vartheta\Big) \tag{4.9}$$

$$= \tilde\varrho\Big((\varrho-\tilde\rho)\partial_\rho s(\tilde\rho,\tilde\vartheta) + (\vartheta-\tilde\vartheta)\partial_\vartheta s(\tilde\rho,\tilde\vartheta)\Big)\Big(\partial_t\tilde\vartheta + \tilde{\mathbf{u}}\cdot\nabla\tilde\vartheta\Big)$$

$$= \Big(\tilde\varrho s(\tilde\varrho,\tilde\vartheta) - \varrho s(\tilde\varrho,\tilde\vartheta) + (\varrho-\tilde\varrho)\partial_\varrho\mathcal{S}(\tilde\varrho,\tilde\vartheta) + (\vartheta-\tilde\vartheta)\partial_\varrho\mathcal{S}(\tilde\varrho,\tilde\vartheta)\Big)\Big(\partial_t\tilde\vartheta + \tilde{\mathbf{u}}\cdot\nabla\tilde\vartheta\Big).$$

Formula (4.2) is proved.

4.6 Estimates

4.6.1 Viscous terms

$$\int_0^\tau \int_\Omega \left(\frac{\tilde\vartheta}{\vartheta} \mathbb{S}(\vartheta, \nabla \mathbf{u}) : \nabla \mathbf{u} - \mathbb{S}(\vartheta, \nabla \mathbf{u}) : \nabla \tilde{\mathbf{u}} + \mathbb{S}(\tilde\vartheta, \nabla \tilde{\mathbf{u}}) : \nabla (\tilde{\mathbf{u}} - \mathbf{u}) + \frac{\vartheta - \tilde\vartheta}{\tilde\vartheta} \mathbb{S}(\tilde\vartheta, \nabla \tilde{\mathbf{u}}) : \nabla \tilde{\mathbf{u}} \right) \, \mathrm{d}x\mathrm{d}t \tag{4.10}$$

$$\geq \alpha \|\mathbf{u} - \tilde{\mathbf{u}}\|^2_{L^2(0,T;W^{1,2}(\Omega;R^3))} - c \int_0^\tau \int_\Omega \Big(\varrho (\mathbf{u} - \tilde{\mathbf{u}})^2 + \mathcal{E}(\varrho, \vartheta \mid \tilde\varrho, \tilde\vartheta) \Big) \mathrm{d}x\mathrm{d}t,$$

where α and c are convenient positive constants.

4.6.2 Heat conductivity

$$-\int_0^\tau \int_\Omega \left(\frac{\tilde\vartheta}{\vartheta} \frac{\mathbf{q}(\vartheta, \nabla \vartheta)}{\vartheta} \cdot \nabla \vartheta - \frac{\mathbf{q}(\vartheta, \nabla \vartheta)}{\vartheta} \cdot \nabla \tilde\vartheta + \frac{\mathbf{q}(\tilde\vartheta, \nabla \tilde\vartheta)}{\tilde\vartheta} \cdot \nabla (\tilde\vartheta - \vartheta) + \frac{\vartheta - \tilde\vartheta}{\tilde\vartheta} \frac{\mathbf{q}(\tilde\vartheta, \nabla \tilde\vartheta)}{\tilde\vartheta} : \nabla \tilde\vartheta \right) \, \mathrm{d}x\mathrm{d}t \tag{4.11}$$

$$\geq \alpha \| \sqrt{\kappa(\vartheta)} \nabla_x (\log \vartheta - \log \tilde\vartheta) \|^2_{L^2((0,\tau)\times\Omega;R^3)} - c \int_0^\tau \int_\Omega \mathcal{E}(\varrho, \vartheta \mid \tilde\varrho, \tilde\vartheta) \mathrm{d}x\mathrm{d}t.$$

where α and c are convenient positive constants.

4.6.3 Right hand side

$$\Big| \mathcal{R}(\varrho, \vartheta, \mathbf{u} | \tilde\varrho, \tilde\vartheta, \tilde{\mathbf{u}}) | \leq c \int_0^\tau \int_\Omega \Big(\frac{1}{2} \varrho (\mathbf{u} - \tilde{\mathbf{u}})^2 + \mathcal{E}(\varrho, \vartheta \mid \tilde\varrho, \tilde\vartheta) \Big) \mathrm{d}x\mathrm{d}t + \delta \Big\| \mathbf{u} - \tilde{\mathbf{u}} \Big\|^2_{W^{1,2}(\Omega)}$$

4.7 Conclusion

$$\int_\Omega \left(\frac{1}{2}\varrho|\mathbf{u}-\tilde{\mathbf{u}}|^2 + \mathcal{E}(\varrho,\vartheta|\tilde{\rho},\tilde{\vartheta})\right)(\tau,\cdot)\ \mathrm{d}x \tag{4.12}$$
$$+\alpha\left(\left\|\sqrt{\kappa(\vartheta)}\Big(\nabla_x\log\vartheta-\nabla_x\log\tilde{\vartheta}\Big)\right\|^2_{L^2((0,\tau)\times\Omega)} + \|\mathbf{u}-\tilde{\mathbf{u}}\|^2_{L^2(0,T;W^{1,2}(\Omega;R^3))}\right)$$
$$\leq c\int_0^\tau\int_\Omega\Big(\frac{1}{2}\varrho(\mathbf{u}-\tilde{\mathbf{u}})^2+\mathcal{E}(\varrho,\vartheta\mid\tilde{\varrho},\tilde{\vartheta})\Big)\mathrm{d}x\mathrm{d}t$$

for a. a. $\tau\in(0,T)$.

We conclude by applying the Gronwall lemma to this inequality.

5 Existence of renormalized weak solutions

5.1 Existence theorem

Theorem 5.1 *Let $\Omega \subset R^3$ be a bounded Lipschitz domain. Suppose that the thermodynamic functions p, e, s satisfy hypotheses (1.9 - 1.15), and that the transport coefficients μ, η, and κ obey (1.16), (1.17). Finally assume that the initial data (1.18) verify*

$$\int_\Omega \Big(\frac{1}{2}\varrho_0 \mathbf{u}_0^2 + \varrho_0 e(\varrho_0, \vartheta_0) + \varrho_0 |s(\varrho_0, \vartheta_0)|\Big) \,\mathrm{d}x < \infty. \tag{5.1}$$

Then the complete Navier-Stokes-Fourier system (1.1–1.7) admits at least one renormalized weak solution.

Theorem 5.2 *Let $\Omega \subset R^3$ be a bounded Lipschitz domain. Suppose that the thermodynamic functions p, e, s satisfy hypotheses (1.9 - 1.15), and that the transport coefficients μ, η, and κ obey (1.16), (1.17). Finally assume that the initial data (1.18) verify (5.1). Let $(\varrho_n, \vartheta_n, \mathbf{u}_n)$ be a sequence of weak solutions to the complete Navier-Stokes-Fourier system (1.1–1.7). Then there exists a subsequence (denoted again $(\varrho_n, \vartheta_n, \mathbf{u}_n)$) such that*

$$\varrho_n \rightharpoonup * \varrho \text{ in } L^\infty(0,T; L^\gamma(\Omega)),$$

$$\vartheta_n \rightharpoonup \vartheta \text{ in } L^2(0,T; W^{1,2}(\Omega)),$$

$$\mathbf{u}_n \rightharpoonup \mathbf{u} \text{ in } L^2(0,T; W_0^{1,2}(\Omega; R^3)),$$

and the trio $(\varrho, \vartheta, \mathbf{u})$ is again a weak solution of the complete Navier-Stokes-Fourier system (1.1–1.7).

5.2 Dissipation inequality, estimates

Any weak solution satisfies the so called dissipation inequality:

$$\int_\Omega \left(\frac{1}{2} \varrho |\mathbf{u}|^2 + H_{\overline{\vartheta}}(\varrho, \vartheta) \right) (\tau, \cdot) \mathrm{d}x \tag{5.2}$$

$$+ \int_0^\tau \int_\Omega \frac{\overline{\vartheta}}{\vartheta} \left(\mathbb{S}(\vartheta, \nabla_x \mathbf{u}) : \nabla_x \mathbf{u} - \frac{\mathbf{q}(\vartheta, \nabla_x \vartheta) \cdot \nabla_x \vartheta}{\vartheta} \right) \mathrm{d}x\mathrm{d}t \le$$

$$\int_\Omega \left(\frac{1}{2} \varrho_0 |\mathbf{u}_0|^2 + H_{\overline{\vartheta}}(\varrho_0, \vartheta_0) \right) \mathrm{d}x$$

for a.a. $\tau \in (0, T)$*, where*

$$H_{\overline{\vartheta}}(\varrho, \vartheta) = \varrho e(\varrho, \vartheta) - \overline{\vartheta} \varrho s(\varrho, \vartheta).$$

Indeed, the dissipation inequality (5.2) is obtained from the sum of identity (1.22) and the entropy balance (1.21) multiplied by $\overline{\vartheta}$*.*

We suppose that the fluid verifies the thermodynamic stability conditions,

$$\frac{\partial p(\varrho, \vartheta)}{\partial \varrho} > 0, \ \frac{\partial e(\varrho, \vartheta)}{\partial \vartheta} > 0 \textit{ for all } \varrho, \vartheta > 0. \tag{5.3}$$

We easily verify by using (1.8), that

$$\frac{\partial H_{\overline{\vartheta}}}{\partial \vartheta}(\varrho, \vartheta) = \varrho \frac{\vartheta - \overline{\vartheta}}{\vartheta} \frac{\partial e}{\partial \vartheta}(\varrho, \vartheta) \textit{ and } \frac{\partial^2 H_{\overline{\vartheta}}}{\partial \varrho^2}(\varrho, \overline{\vartheta}) = \frac{1}{\varrho} \frac{\partial p}{\partial \varrho}(\varrho, \overline{\vartheta}). \tag{5.4}$$

Thus, the thermodynamic stability in terms of the function $H_{\overline{\vartheta}}$*, can be reformulated as follows:*

$$\varrho \mapsto H_{\overline{\vartheta}}(\varrho, \overline{\vartheta}) \textit{ is strictly convex,} \tag{5.5}$$

while

$$\vartheta \mapsto H_{\overline{\vartheta}}(\varrho, \vartheta) \textit{ attains its global minimum at } \vartheta = \overline{\vartheta}. \tag{5.6}$$

5.3 Weak compactness of the set of weak solutions

1) Helmholtz function (ballistic free energy) and estimates due to the dissipation inequality

2) Improved estimates of density. Tools: Testing of momentum equation by Bogovskii operator - threshold $\gamma = 3/2$

3) Strong convergence temperature. Tools: Div-curl lemma, Theory of parametrized Young measures

4) Effective viscous flux identity. Tools: Testing of momentum equation by "$\nabla_x \Delta^{-1}$*": compensated compactness I (applied to a commutator including density momentum and a Riesz type operator), compensated compactness II (applied to a commutator including temperature dependent viscosity and symmetrized gradients of velocity)- treshold* $\gamma = 3/2$

5) Limiting density ϱ *is a solution of the renormalized continuity equation. Tool: DiPerna-Lions transport theory that is applicable provided* ϱ *is squared integrable - treshold* $\gamma = 9/5$

6) Boundedness of oscillations defect measure (that is a particular number characterizing the approximating sequence of densities).

7) Boundedness of oscillations implies that the limiting density is again renormalized solution to the continuity equation

8) Evaluation of the propagation of oscillations in the density sequence by using the renormalized continuity equation. Strong convergence of the density sequence.

6 Weak compactness for the Navier-Stokes-Poisson system

6.1 Navier-Stokes-Poisson system

$$\partial_t \varrho + \mathrm{div}_x(\varrho \mathbf{u}) = 0 \ \textit{in} \ Q_T, \tag{6.1}$$

$$\partial_t(\varrho \mathbf{u}) + \mathrm{div}_x(\varrho \mathbf{u} \otimes \mathbf{u}) + \nabla_x p(\varrho) = \mathrm{div}_x \mathbb{S}(\nabla_x \mathbf{u}) \ \textit{in} \ Q_T, \tag{6.2}$$

$$\mathbf{u}|_{(0,T)\times\partial\Omega} = 0, \tag{6.3}$$

$$\varrho(0) = \varrho_0, \quad \mathbf{u}(0) = \mathbf{u}_0. \tag{6.4}$$

$$p(0) = 0, \ p'(\varrho) > 0 \ \textit{for all} \ \varrho > 0, \ \lim_{\varrho \to \infty} \frac{p'(\varrho)}{\varrho^{\gamma - 1}} = a > 0 \tag{6.5}$$

for a certain $\gamma > 3/2$.

6.2 Weak formulation

We say that ϱ, $\mathbf{u}$ *represent a bounded energy* weak solution *to problem (6.1 - 6.4) if:*

- *functional spaces*

$$\varrho \geq 0,\ \ \varrho \in L^\infty(0,T;L^\gamma(\Omega)) \text{ for a certain } \gamma > 3/2,\ \ \mathbf{u} \in L^2(0,T;W_0^{1,2}(\Omega;R^3)),$$

$$p(\varrho) \in L^1((0,T)\times\Omega);$$

- *equation of continuity (6.1) is satisfied in the weak sense ,*

$$\Big[\int_\Omega \varrho\varphi(t)\ \mathrm{d}x\Big]_0^\tau = \int_0^T \int_\Omega \varrho\partial_t\varphi + \varrho\mathbf{u}\cdot\nabla_x\varphi\ \mathrm{d}x\ \mathrm{d}t \tag{6.6}$$

  for all $\tau \in [0,T]$*, for any test function* $\varphi \in C_c^\infty([0,T]\times\overline{\Omega})$ *;*

- *momentum equation (6.2), together with the no-slip boundary condition (6.3), is satisfied in the weak sense,*

$$\Big[\int_\Omega \varrho\mathbf{u}\varphi(t)\ \mathrm{d}x\Big]_0^\tau = \int_0^T \int_\Omega \Big(\varrho\mathbf{u}\cdot\partial_t\varphi + \varrho(\mathbf{u}\otimes\mathbf{u}) : \nabla_x\varphi + p(\varrho)\mathrm{div}_x\varphi\Big)\ \mathrm{d}x\ \mathrm{d}t \tag{6.7}$$

$$= \int_0^T \int_\Omega \mathbb{S}(\nabla_x\mathbf{u}) : \nabla_x\varphi\ \mathrm{d}x\ \mathrm{d}t - \int_\Omega \varrho_0\mathbf{u}_0\cdot\varphi(0,\cdot)\ \mathrm{d}x;$$

  for all $\tau \in [0,T]$*, for any test function* $\varphi \in C_c^\infty([0,T]\times\Omega)$

- *energy inequality*

$$\int_\Omega \Big(\frac{1}{2}\varrho|\mathbf{u}|^2 + H(\varrho)\Big)(\tau,\cdot)\ \mathrm{d}x + \int_0^\tau \int_\Omega \mathbb{S}(\nabla_x\mathbf{u}) : \nabla_x\mathbf{u}\ \mathrm{d}x\ \mathrm{d}t \leq \tag{6.8}$$

$$\int_\Omega \Big(\frac{1}{2}\varrho_0|\mathbf{u}_0|^2 + H(\varrho_0)\Big)\ \mathrm{d}x \text{ for a.a. } \tau \in (0,T),$$

  where

$$H(\varrho) \equiv \varrho\int_1^\varrho \frac{p(z)}{z^2}\ \mathrm{d}z.$$

6.3 Equations on the level n

$$\partial_t \varrho_n + \mathrm{div}_x(\varrho_n \mathbf{u}_n) = 0 \ \textit{in} \ Q_T, \tag{6.9}$$

$$\partial_t b(\varrho_n) + \mathrm{div}_x\Big(b(\varrho_n)\mathbf{u}_n)\Big) = \Big(b(\varrho_n) - \varrho_n b'(\varrho_n)\Big)\mathrm{div}\mathbf{u}_n \quad \textit{in} \ Q_T, \tag{6.10}$$

$$\partial_t(\varrho_n \mathbf{u}_n) + \mathrm{div}_x(\varrho_n \mathbf{u}_n \otimes \mathbf{u}_n) + \nabla_x p(\varrho_n) = \mathrm{div}_x \mathbb{S}(\nabla_x \mathbf{u}_n) \ \textit{in} \ Q_T, \tag{6.11}$$

$$\int_\Omega \left(\frac{1}{2}\varrho_n|\mathbf{u}_n|^2 + H(\varrho_n)\right)(\tau,\cdot)\ \mathrm{d}x + \int_0^\tau \int_\Omega \mathbb{S}(\nabla_x \mathbf{u}_n) : \nabla_x \mathbf{u}_n \ \mathrm{d}x \ \mathrm{d}t \leq \tag{6.12}$$
$$\int_\Omega \left(\frac{1}{2}\varrho_0|\mathbf{u}_0|^2 + H(\varrho_0)\right)\ \mathrm{d}x \ \textit{for a.a.}\ \tau \in (0,T), \quad H(\varrho) \equiv \varrho \int_1^\varrho \frac{p(z)}{z^2}\ \mathrm{d}z.$$

6.4 Limiting equations

$$\partial_t \varrho + \mathrm{div}_x(\varrho \mathbf{u}) = 0 \ \textit{in} \ Q_T, \tag{6.13}$$

$$\partial_t \overline{b(\varrho)} + \mathrm{div}_x\Big(\overline{b(\varrho)}\mathbf{u})\Big) = \overline{\Big(b(\varrho) - \varrho b'(\varrho)\Big)\mathrm{div}\mathbf{u}} \quad \textit{in} \ Q_T, \tag{6.14}$$

$$\partial_t(\varrho \mathbf{u}) + \mathrm{div}_x(\varrho \mathbf{u} \otimes \mathbf{u}) + \nabla_x \overline{p(\varrho)} = \mathrm{div}_x \mathbb{S}(\nabla_x \mathbf{u}) \ \textit{in} \ Q_T. \tag{6.15}$$

6.5 Effective viscous flux

Let

$$T_k(\varrho) = \min(\varrho, k), \quad k > 0.$$

Then

$$\overline{p(\varrho)T_k(\varrho)} - \overline{p(\varrho)}\ \overline{T_k(\varrho)} = (\frac{4}{3}\mu + \eta)\Big(\overline{T_k(\varrho)\mathrm{div}\mathbf{u}} - \overline{T_k(\varrho)}\ \mathrm{div}\mathbf{u}\Big),$$

where

$$\overline{p(\varrho)T_k(\varrho)} - \overline{p(\varrho)}\ \overline{T_k(\varrho)} \geq 0$$

6.6 Oscillations defect measure

$$\mathrm{osc}_{\gamma+1}[\varrho_n \rightharpoonup \varrho] \equiv \sup_{k>0} \limsup_{n\to\infty} \int_0^T \int_\Omega \Big|T_k(\varrho_n) - T_k(\varrho)\Big|^{\gamma+1} \mathrm{d}x\mathrm{d}t < \infty$$

6.7 Renormalized continuity equation

Let $(\varrho_n, \mathbf{u}_n)$ *satisfy weak formulation of (6.9), (6.10) and*

$$\varrho_n \rightharpoonup \varrho \text{ in } L^1((0,T)\times\Omega),$$

$$\mathbf{u}_n \rightharpoonup \mathbf{u} \text{ in } L^2(0,T;W^{1,2}(\Omega)),$$

$$\mathrm{osc}_q[\varrho_n \rightharpoonup \varrho]((0,T)\times\Omega) < \infty$$

with some $q > 2$*. Then* $(\varrho, \mathbf{u})$ *satisfies as well the renormalized continuity equation in the weak sense.*