The HLLC Riemann Solver

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Abstract:

This lecture is about a method to solve approximately the Riemann problem for the Euler equations in order to derive a numerical flux for a conservative method:

The HLLC Riemann solver

REFERENCES:

E F Toro, M Spruce and W Speares.
Restoration of the contact surface in the HLL Riemann solver. Technical report CoA 9204. Department of Aerospace Science, College of Aeronautics, Cranfield Institute of Technology. UK. June, 1992.

E F Toro, M Spruce and W Speares.
Restoration of the contact surface in the Harten-Lax-van Leer Riemann solver. Shock Waves. Vol. 4, pages 25-34, 1994.

Consider the general Initial Boundary Value Problem (IBVP)

$$\left.\begin{array}{lll} \text{PDEs} & : & \mathbf{U}_t + \mathbf{F}(\mathbf{U})_x = \mathbf{0} \;,\; 0 \le x \le L \;, t > 0 \;, \\ \text{ICs} & : & \mathbf{U}(x,0) = \mathbf{U}^{(0)}(x) \;, \\ \text{BCs} & : & \mathbf{U}(0,t) = \mathbf{U}_{\mathrm{l}}(t) \;,\; \mathbf{U}(L,t) = \mathbf{U}_{\mathrm{r}}(t) \;, \end{array}\right\} \quad (1)$$

with appropriate boundary conditions, as solved by the explicit conservative scheme

$$\mathbf{U}_i^{n+1} = \mathbf{U}_i^n - \frac{\Delta t}{\Delta x}[\mathbf{F}_{i+\frac{1}{2}} - \mathbf{F}_{i-\frac{1}{2}}] \;. \quad (2)$$

The choice of numerical flux $\mathbf{F}_{i+\frac{1}{2}}$ determines the scheme. There two classes of fluxes:

- Upwind or Godunov-type fluxes (wave propagation information used explicitly) and
- Centred or non-upwind (wave propagation information NOT used explicitly).

Godunov's flux (Godunov 1959) is

$$\mathbf{F}_{i+\frac{1}{2}} = \mathbf{F}(\mathbf{U}_{i+\frac{1}{2}}(0)) \; , \tag{3}$$

in which $\mathbf{U}_{i+\frac{1}{2}}(0)$ is the exact similarity solution $\mathbf{U}_{i+\frac{1}{2}}(x/t)$ of the Riemann problem

$$\left.\begin{array}{l} \mathbf{U}_t + \mathbf{F}(\mathbf{U})_x = \mathbf{0} \; , \\ \mathbf{U}(x,0) = \begin{cases} \mathbf{U}_\mathrm{L} & \text{if} \quad x < 0 \; , \\ \\ \mathbf{U}_\mathrm{R} & \text{if} \quad x > 0 \; , \end{cases} \end{array}\right\} \tag{4}$$

evaluated at $x/t = 0$.

Example: 3D Euler equations.

$$\mathbf{U} = \begin{bmatrix} \rho \\ \rho u \\ \rho v \\ \rho w \\ E \end{bmatrix} , \quad \mathbf{F} = \begin{bmatrix} \rho u \\ \rho u^2 + p \\ \rho u v \\ \rho u w \\ u(E+p) \end{bmatrix} . \tag{5}$$

The piece–wise constant initial data, in terms of primitive variables, is

$$\mathbf{W}_L = \begin{bmatrix} \rho_L \\ u_L \\ v_L \\ w_L \\ p_L \end{bmatrix} , \quad \mathbf{W}_R = \begin{bmatrix} \rho_R \\ u_R \\ v_R \\ w_R \\ p_R \end{bmatrix} . \tag{6}$$

The Godunov flux $\mathbf{F}(\mathbf{U}_{i+\frac{1}{2}}(0))$ results from evaluation $\mathbf{U}_{i+\frac{1}{2}}(x/t)$ at $x/t = 0$, that is along the t–axis.

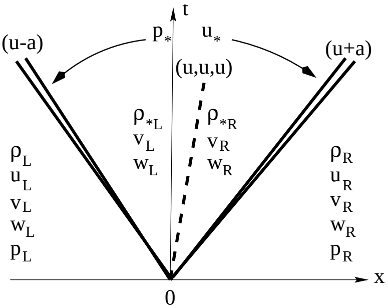


Fig. 1. Structure of the exact solution $\mathbf{U}_{i+\frac{1}{2}}(x/t)$ of the Riemann problem for the x–split three dimensional Euler equations. There are five wave families *associated* with the eigenvalues $u - a$, u (of multiplicity 3) and $u + a$.

Integral Relations

Consider the control volume $V = [x_L, x_R] \times [0, T]$ depicted in Fig. 2, with

$$x_L \leq TS_L \ , \quad x_R \geq TS_R \ , \tag{7}$$

S_L and S_R are the *fastest signal velocities* and T is a chosen time. The integral form of the conservation laws in (4) in V reads

$$\int_{x_L}^{x_R} \mathbf{U}(x, T)dx = \int_{x_L}^{x_R} \mathbf{U}(x, 0)dx + \int_0^T \mathbf{F}(\mathbf{U}(x_L, t))dt - \int_0^T \mathbf{F}(\mathbf{U}(x_R, t))dt \ . \tag{8}$$

Evaluation of the right–hand side of this expression gives

$$\int_{x_L}^{x_R} \mathbf{U}(x, T)dx = x_R\mathbf{U}_R - x_L\mathbf{U}_L + T(\mathbf{F}_L - \mathbf{F}_R) \ , \tag{9}$$

where $\mathbf{F}_L = \mathbf{F}(\mathbf{U}_L)$ and $\mathbf{F}_R = \mathbf{F}(\mathbf{U}_R)$.

We call (9) the *consistency condition*.

Now split left–hand side of (8) into three integrals, namely

$$\int_{x_L}^{x_R} \mathbf{U}(x, T)dx = \int_{x_L}^{TS_L} \mathbf{U}(x, T)dx + \int_{TS_L}^{TS_R} \mathbf{U}(x, T)dx + \int_{TS_R}^{x_R} \mathbf{U}(x, T)dx$$

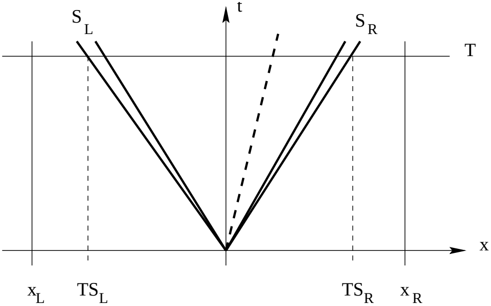


Fig. 2. Control volume $[x_L, x_R] \times [0, T]$ on x–t plane. S_L and S_R are the fastest signal velocities arising from the solution of the Riemann problem.

Evaluate the first and third terms on the right–hand side to obtain

$$\int_{x_L}^{x_R} \mathbf{U}(x,T)dx = \int_{TS_L}^{TS_R} \mathbf{U}(x,T)dx + (TS_L - x_L)\mathbf{U}_L + (x_R - TS_R)\mathbf{U}_R \; . \qquad (10)$$

Comparing (10) with (9) gives

$$\int_{TS_L}^{TS_R} \mathbf{U}(x,T)dx = T(S_R\mathbf{U}_R - S_L\mathbf{U}_L + \mathbf{F}_L - \mathbf{F}_R) \; . \qquad (11)$$

On division through by the length $T(S_R - S_L)$, which is the width of the wave system of the solution of the Riemann problem between the slowest and fastest signals at time T, we have

$$\frac{1}{T(S_R - S_L)} \int_{TS_L}^{TS_R} \mathbf{U}(x,T)dx = \frac{S_R\mathbf{U}_R - S_L\mathbf{U}_L + F_L - F_R}{S_R - S_L} \; . \qquad (12)$$

Thus, the integral average of the exact solution of the Riemann problem between the slowest and fastest signals at time T is a known constant, provided that the signal speeds S_L and S_R are known; such constant is the right–hand side of (12) and we denote it by

$$\mathbf{U}^{hll} = \frac{S_R\mathbf{U}_R - S_L\mathbf{U}_L + F_L - F_R}{S_R - S_L} \; . \tag{13}$$

We now apply the integral form of the conservation laws to the left portion of Fig. 10.2, that is the control volume $[x_L, 0] \times [0, T]$. We obtain

$$\int_{TS_L}^{0} \mathbf{U}(x, T)dx = -TS_L\mathbf{U}_L + T(\mathbf{F}_L - \mathbf{F}_{0L}) \; , \tag{14}$$

where $\mathbf{F}_{0L}$ is the flux $\mathbf{F}(\mathbf{U})$ along the t–axis. Solving for $\mathbf{F}_{0L}$ we find

$$\mathbf{F}_{0L} = \mathbf{F}_L - S_L\mathbf{U}_L - \frac{1}{T}\int_{TS_L}^{0} \mathbf{U}(x, T)dx \; . \tag{15}$$

Evaluation of the integral form of the conservation laws on the control volume $[0, x_R] \times [0, T]$ yields

$$\mathbf{F}_{0R} = \mathbf{F}_R - S_R \mathbf{U}_R + \frac{1}{T} \int_0^{TS_R} \mathbf{U}(x, T) dx \; . \qquad (16)$$

The reader can easily verify that the equality

$$\mathbf{F}_{0L} = \mathbf{F}_{0R}$$

results in the *consistency condition* (9). All relations so far are exact, as we are assuming the exact solution of the Riemann problem.

The Harten-Lax-van Leer (HLL) Approximate Riemann Solver (1983).

$$\tilde{\mathbf{U}}(x,t) = \begin{cases} \mathbf{U}_L & \text{if} \quad \frac{x}{t} \le S_L \ , \\ \mathbf{U}^{hll} & \text{if} \quad S_L \le \frac{x}{t} \le S_R \ , \\ \mathbf{U}_R & \text{if} \quad \frac{x}{t} \ge S_R \ , \end{cases} \tag{17}$$

Fig. 3 shows the two-wave structure of this approximate Riemann solver.

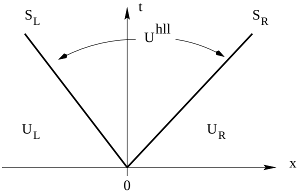


Fig. 3. Two-wave model. Approximate HLL Riemann solver. Solution in the *Star Region* consists of a single state $\mathbf{U}^{hll}$ separated from data states by two waves of speeds S_L and S_R.

The HLL flux $\mathbf{F}^{hll}$ for the subsonic case $S_L \leq 0 \leq S_R$ is found by inserting $\mathbf{U}^{hll}$ in (13) into (15) or (16) to obtain

$$\mathbf{F}^{hll} = \mathbf{F}_L + S_L(\mathbf{U}^{hll} - \mathbf{U}_L) , \quad (18)$$

or

$$\mathbf{F}^{hll} = \mathbf{F}_R + S_R(\mathbf{U}^{hll} - \mathbf{U}_R) . \quad (19)$$

Use of (13) in (18) or (19) gives the HLL flux

$$\mathbf{F}^{hll} = \frac{S_R \mathbf{F}_L - S_L \mathbf{F}_R + S_L S_R (\mathbf{U}_R - \mathbf{U}_L)}{S_R - S_L} \quad (20)$$

for the subsonic case $S_L \leq 0 \leq S_R$.

The corresponding HLL intercell flux for the approximate Godunov method is then given by

$$\mathbf{F}_{i+\frac{1}{2}}^{hll} = \begin{cases} \mathbf{F}_L & \text{if} \quad 0 \leq S_L \,, \\ \dfrac{S_R \mathbf{F}_L - S_L \mathbf{F}_R + S_L S_R (\mathbf{U}_R - \mathbf{U}_L)}{S_R - S_L} \,, & \text{if} \quad S_L \leq 0 \leq S_R \,, \\ \mathbf{F}_R & \text{if} \quad 0 \geq S_R \,. \end{cases} \tag{21}$$

- Given the speeds S_L and S_R we have an approximate intercell flux (21) to be used in the conservative formula (2) to produce an approximate Godunov method.
- A shortcoming of the HLL scheme, with its two-wave model, is exposed by contact discontinuities, shear waves and material interfaces, or any type of *intermediate waves*.

The HLLC Approximate Riemann Solver (Toro et al, 1992).

- The HLLC scheme is a modification of the HLL scheme whereby the missing contact and shear waves in the Euler equations are restored.
- HLLC for the Euler equations has a three-wave model

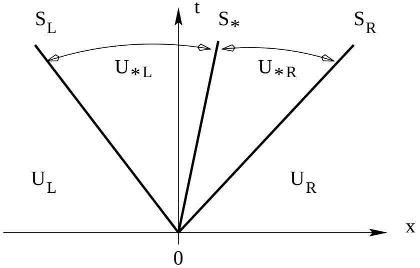


Fig. 4. HLLC approximate Riemann solver. Solution in the *Star Region* consists of two constant states separated from each other by a middle wave of speed S_*.

Useful Relations. Consider Fig. 2.

- Evaluation of the integral form of the conservation laws in the control volume reproduces the result of equation (12), even if variations of the integrand across the wave of speed S_* are allowed.
- Note that the consistency condition (9) effectively becomes the condition (12).
- By splitting the left–hand side of integral (12) into two terms we obtain

$$\frac{1}{T(S_R - S_L)} \int_{TS_L}^{TS_R} \mathbf{U}(x, T)dx = \mathbf{U}_{*L} + \mathbf{U}_{*R} \; , \tag{22}$$

where the following integral averages are introduced

$$
\left.
\begin{aligned}
\mathbf{U}_{*L} &= \frac{1}{T(S_* - S_L)} \int_{TS_L}^{TS_*} \mathbf{U}(x,T)dx \; , \\
\mathbf{U}_{*R} &= \frac{1}{T(S_R - S_*)} \int_{TS_*}^{TS_R} \mathbf{U}(x,T)dx \; .
\end{aligned}
\right\} \qquad (23)
$$

Use of (23) into (22) and use of (12), make condition (9)

$$
\left(\frac{S* - S_L}{S_R - S_L} \right) \mathbf{U}_{*L} + \left(\frac{S_R - S_*}{S_R - S_L} \right) \mathbf{U}_{*R} = \mathbf{U}^{hll} \; , \qquad (24)
$$

The HLLC approximate Riemann solver is given as follows

$$
\tilde{\mathbf{U}}(x,t) = \begin{cases}
\mathbf{U}_L & , \text{ if } \quad \frac{x}{t} \leq S_L \; , \\
\mathbf{U}_{*L} & , \text{ if } \quad S_L \leq \frac{x}{t} \leq S_* \; , \\
\mathbf{U}_{*R} & , \text{ if } \quad S_* \leq \frac{x}{t} \leq S_R \; , \\
\mathbf{U}_R & , \text{ if } \quad \frac{x}{t} \geq S_R \; .
\end{cases} \qquad (25)
$$

Now we seek a corresponding HLLC numerical flux of the form

$$\mathbf{F}^{hllc}_{i+\frac{1}{2}} = \begin{cases} \mathbf{F}_L & , \text{ if } \quad 0 \le S_L \ , \\ \mathbf{F}_{*L} & , \text{ if } \quad S_L \le 0 \le S_* \ , \\ \mathbf{F}_{*R} & , \text{ if } \quad S_* \le 0 \le S_R \ , \\ \mathbf{F}_R & , \text{ if } \quad 0 \ge S_R \ , \end{cases} \tag{26}$$

with the intermediate fluxes $\mathbf{F}_{*L}$ and $\mathbf{F}_{*R}$ still to be determined, see Fig. 4. By integrating over appropriate control volumes we obtain

$$\mathbf{F}_{*L} = \mathbf{F}_L + S_L(\mathbf{U}_{*L} - \mathbf{U}_L) \ , \tag{27}$$

$$\mathbf{F}_{*R} = \mathbf{F}_{*L} + S_*(\mathbf{U}_{*R} - \mathbf{U}_{*L}) \ , \tag{28}$$

$$\mathbf{F}_{*R} = \mathbf{F}_R + S_R(\mathbf{U}_{*R} - \mathbf{U}_R) \ . \tag{29}$$

These are three equations for the four unknowns vectors $\mathbf{U}_{*L}$, $\mathbf{F}_{*L}$, $\mathbf{U}_{*R}$, $\mathbf{F}_{*R}$.

We seek the solution for the two unknown intermediate fluxes $\mathbf{F}_{*L}$ and $\mathbf{F}_{*R}$. There are more unknowns than equations and some extra conditions need to be imposed, in order to solve the algebraic problem. We impose

$$\left.\begin{array}{ccccc} p_{*L} & = & p_{*R} & = & p_* \ , \\ u_{*L} & = & u_{*R} & = & u_* \ , \end{array}\right\} \text{for pressure and normal velocity} \quad (30)$$

$$\left.\begin{array}{cccccccc} v_{*L} & = & v_L & , & v_{*R} & = & v_R \ , \\ w_{*L} & = & w_L & , & w_{*R} & = & w_R \ . \end{array}\right\} \text{for tangential velocities} \quad (31)$$

Conditions (30), (31) are identically satisfied by the exact solution. In addition we impose

$$S_* = u_* \quad (32)$$

and thus if an estimate for S_* is known, the normal velocity component u_* in the *Star Region* is known.

Now equations (27) and (29) can be re–arranged as

$$S_L \mathbf{U}_{*L} - \mathbf{F}_{*L} = S_L \mathbf{U}_L - \mathbf{F}_L \ , \tag{33}$$

$$S_R \mathbf{U}_{*R} - \mathbf{F}_{*R} = S_R \mathbf{U}_R - \mathbf{F}_R \ , \tag{34}$$

where the right–hand sides of (33) and (34) are known constant vectors (data). We also note the useful relation

$$\mathbf{F}(\mathbf{U}) = u\mathbf{U} + p\mathbf{D} \ , \quad \mathbf{D} = [0, 1, 0, 0, u]^T \ . \tag{35}$$

Assuming S_L and S_R to be known and performing algebraic manipulations of the first and second components of equations (33)–(34) one obtains

$$p_{*L} = p_L + \rho_L(S_L - u_L)(S_* - u_L) \ , \quad p_{*R} = p_R + \rho_R(S_R - u_R)(S_* - u_R) \ . \tag{36}$$

From (30) $p_{*L} = p_{*R}$, which from (36) gives

$$S_* = \frac{p_R - p_L + \rho_L u_L(S_L - u_L) - \rho_R u_R(S_R - u_R)}{\rho_L(S_L - u_L) - \rho_R(S_R - u_R)} \ . \tag{37}$$

Manipulation of (33) and (34) and using p_{*L} and p_{*R} from (36) gives

$$\mathbf{F}_{*K} = \mathbf{F}_K + S_K(\mathbf{U}_{*K} - \mathbf{U}_K) \ , \tag{38}$$

for K=L and K=R, with the intermediate states given as

$$\mathbf{U}_{*K} = \rho_K \left(\frac{S_K - u_K}{S_K - S_*} \right) \begin{bmatrix} 1 \\ S_* \\ v_K \\ w_K \\ \frac{E_K}{\rho_K} + (S_* - u_K)\left[S_* + \frac{p_K}{\rho_K(S_K - u_K)} \right] \end{bmatrix} . \tag{39}$$

The final choice of the HLLC flux is made according to (26).

Variation 1 of HLLC.

From equations (33) and (34) we may write the following solutions for the state vectors $\mathbf{U}_{*L}$ and $\mathbf{U}_{*R}$

$$\mathbf{U}_{*K} = \frac{S_K \mathbf{U}_K - \mathbf{F}_K + p_{*K} \mathbf{D}_*}{S_L - S_*} \ , \quad \mathbf{D}_* = [0, 1, 0, 0, S_*] \ , \qquad (40)$$

with p_{*L} and p_{*R} as given by (36). Substitution of p_{*K} from (36) into (40) followed by use of (27) and (29) gives direct expressions for the intermediate fluxes as

$$\mathbf{F}_{*K} = \frac{S_*(S_K \mathbf{U}_K - \mathbf{F}_K) + S_K(p_K + \rho_L(S_K - u_K)(S_* - u_K))D_*}{S_K - S_*} \ , \qquad (41)$$

with the final choice of the HLLC flux made again according to (26).

Variation 2 of HLLC.

A different HLLC flux is obtained by assuming a single mean pressure value in the *Star Region*, and given by the arithmetic average of the pressures in (36), namely

$$P_{LR} = \frac{1}{2}[p_L + p_R + \rho_L(S_L - u_L)(S_* - u_L) + \rho_R(S_R - u_R)(S_* - u_R)] \ . \tag{42}$$

Then the intermediate state vectors are given by

$$\mathbf{U}_{*K} = \frac{S_K \mathbf{U}_K - \mathbf{F}_K + P_{LR}\mathbf{D}_*}{S_K - S_*} \ . \tag{43}$$

Substitution of these into (27) and (29) gives the fluxes $\mathbf{F}_{*L}$ and $\mathbf{F}_{*R}$ as

$$\mathbf{F}_{*K} = \frac{S_*(S_K \mathbf{U}_K - \mathbf{F}_K) + S_K P_{LR}\mathbf{D}_*}{S_K - S_*} \ . \tag{44}$$

Again the final choice of HLLC flux is made according to (26).

Remarks.

- The original HLLC formulation (38)–(39) enforces the condition $p_{*L} = p_{*R}$, which is satisfied by the exact solution.
- In the alternative HLLC formulation (41) we relax such condition, being more consistent with the pressure approximations (36).
- There is limited practical experience with the alternative HLLC formulations (41) and (44).
- *General equation of state.* All manipulations, assuming that wave speed estimates for S_L and S_R are available, are valid for any equation of state; this only enters when prescribing estimates for S_L and S_R.

Multidimensional multicomponent flow.
Consider the advection of a *chemical species* of concentrations q_l by the normal flow speed u. Then we can write the following advection equation

$$\partial_t q_l + u \partial_x q_l = 0 \text{ , for } l = 1, \dots, m \text{ .}$$

Note that these equations are written in non–conservative form. However, by combining these with the continuity equation we obtain a conservative form of these equations, namely

$$(\rho q_l)_t + (\rho u q_l)_x = 0 \text{ , for } l = 1, \dots, m \text{ .}$$

The eigenvalues of the enlarged system are as before, with the exception of $\lambda_2 = u$, which now, in three space dimensions, has multiplicity $m + 3$.

These conservation equations can then be added as new components to the conservation equations in (1) or (4), with the enlarged vectors of conserved variables and fluxes given as

$$\mathbf{U} = \begin{bmatrix} \rho \\ \rho u \\ \rho v \\ \rho w \\ E \\ \rho q_1 \\ \dots \\ \rho q_l \\ \dots \\ \rho q_m \end{bmatrix}, \quad \mathbf{F} = \begin{bmatrix} \rho u \\ \rho u^2 + p \\ \rho uv \\ \rho uw \\ u(E+p) \\ \rho u q_1 \\ \dots \\ \rho u q_l \\ \dots \\ \rho u q_m \end{bmatrix}. \tag{45}$$

The HLLC flux accommodates these new equations in a very natural way, and nothing special needs to be done. If the HLLC flux (38) is used, with **F** as in (45), then the intermediate state vectors are given by

$$\mathbf{U}_{*K} = \rho_K \left(\frac{S_K - u_K}{S_K - S_*} \right) \begin{bmatrix} 1 \\ S_* \\ v_K \\ w_K \\ \frac{E_K}{\rho_K} + (S_* - u_K) \left[S_* + \frac{p_K}{\rho_K (S_K - u_K)} \right] \\ (q_1)_K \\ \cdots \\ (q_l)_K \\ \cdots \\ (q_m)_K \end{bmatrix} . \tag{46}$$

for $K = L$ and $K = R$.

Wave Speed Estimates

We need estimates S_L, S_* and S_R. Davis (1988) suggested

$$S_L = u_L - a_L \ , \quad S_R = u_R + a_R \ , \tag{47}$$

$$S_L = \min\left\{u_L - a_L, u_R - a_R\right\} \ , \quad S_R = \max\left\{u_L + a_L, u_R + a_R\right\} \ . \tag{48}$$

Both Davis (1988) and Einfeldt (1988), proposed

$$S_L = \tilde{u} - \tilde{a} \ , \quad S_R = \tilde{u} + \tilde{a} \ , \tag{49}$$

$\tilde{u}$ and $\tilde{a}$ are the Roe–average particle and sound speeds respectively

$$\tilde{u} = \frac{\sqrt{\rho_L} u_L + \sqrt{\rho_R} u_R}{\sqrt{\rho_L} + \sqrt{\rho_R}} \ , \quad \tilde{a} = \left[(\gamma - 1)(\tilde{H} - \frac{1}{2}\tilde{u}^2)\right]^{1/2} \ , \tag{50}$$

with the enthalpy $H = (E + p)/\rho$ approximated as

$$\tilde{H} = \frac{\sqrt{\rho_L} H_L + \sqrt{\rho_R} H_R}{\sqrt{\rho_L} + \sqrt{\rho_R}} \ . \tag{51}$$

Einfeldt (1988) proposed the estimates

$$S_L = \bar{u} - \bar{d} \ , \quad S_R = \bar{u} + \bar{d} \ , \tag{52}$$

for his HLLE solver, where

$$\bar{d}^2 = \frac{\sqrt{\rho_L} a_L^2 + \sqrt{\rho_R} a_R^2}{\sqrt{\rho_L} + \sqrt{\rho_R}} + \eta_2 (u_R - u_L)^2 \tag{53}$$

and

$$\eta_2 = \frac{1}{2} \frac{\sqrt{\rho_L}\sqrt{\rho_R}}{(\sqrt{\rho_L} + \sqrt{\rho_R})^2} \ . \tag{54}$$

These wave speed estimates are reported to lead to effective and robust Godunov–type schemes.

One-wave model.

Consider a **one-wave model** with single speed $S^+ > 0$.

- **Rusanov:** By choosing $S_L = -S^+$ and $S_R = S^+$ in the HLL flux (20) one obtains a Rusanov flux (1961)

$$\mathbf{F}_{i+1/2} = \frac{1}{2}(\mathbf{F}_L + \mathbf{F}_R) - \frac{1}{2}S^+(\mathbf{U}_R - \mathbf{U}_L)\ . \qquad (55)$$

- **Lax-Friedrichs:** Another possibility is $S^+ = S^n_{max}$, the wave speed for imposing the CFL condition, which satisfies

$$S^n_{max} = \frac{C_{cfl}\Delta x}{\Delta t}\ , \qquad (56)$$

where C_{cfl} is the CFL coefficient. For $C_{cfl} = 1$, $S^+ = \frac{\Delta x}{\Delta t}$, which gives the Lax–Friedrichs numerical flux

$$\mathbf{F}_{i+1/2} = \frac{1}{2}(\mathbf{F}_L + \mathbf{F}_R) - \frac{1}{2}\frac{\Delta x}{\Delta t}(\mathbf{U}_R - \mathbf{U}_L)\ . \qquad (57)$$

Pressure–Based Wave Speed Estimates

Toro et al. (1994) suggested to first find an estimate p_* for the pressure in the *Star Region* and then take

$$S_L = u_L - a_L q_L \ , \quad S_R = u_R + a_R q_R \ , \tag{58}$$

$$q_K = \begin{cases} 1 & \text{if} \quad p_* \leq p_K \\ \left[1 + \frac{\gamma+1}{2\gamma}(p_*/p_K - 1)\right]^{1/2} & \text{if} \quad p_* > p_K \ . \end{cases} \tag{59}$$

- This choice discriminates between shocks and rarefactions.
- If the K wave is a rarefaction then the speed S_K is the speed of the head of the rarefaction, the fastest signal.
- If the K wave is a shock wave then the speed is an approximation of the shock speed.

A simple, acoustic type approximation for pressure is (Toro, 1991)

$$p_* = max(0, p_{pvrs}) \ , \quad p_{pvrs} = \frac{1}{2}(p_L + p_R) - \frac{1}{2}(u_R - u_L)\bar{\rho}\bar{a} \ , \quad (60)$$

where

$$\bar{\rho} = \frac{1}{2}(\rho_L + \rho_R) \ , \quad \bar{a} = \frac{1}{2}(a_L + a_R) \ . \quad (61)$$

Another choice is furnished by the Two–Rarefaction Riemann solver, namely

$$p_* = p_{tr} = \left[\frac{a_L + a_R - \frac{\gamma-1}{2}(u_R - u_L)}{a_L/p_L^z + a_R/p_R^z} \right]^{1/z} \ , \quad (62)$$

where

$$P_{LR} = \left(\frac{p_L}{p_R} \right)^z \ ; \quad z = \frac{\gamma - 1}{2\gamma} \ . \quad (63)$$

The Two–Shock Riemann solver gives

$$p_* = p_{ts} = \frac{g_L(p_0)p_L + g_R(p_0)p_R - \Delta u}{g_L(p_0) + g_R(p_0)} \,, \tag{64}$$

where

$$g_K(p) = \left[\frac{A_K}{p + B_K}\right]^{1/2} \,, \quad p_0 = max(0, p_{pvrs}) \,, \tag{65}$$

for $K = L$ and $K = R$.

Summary of HLLC Fluxes

- *Step I: pressure estimate p_*.*
- *Step II: wave speed estimates:*

$$S_L = u_L - a_L q_L \ , \quad S_R = u_R + a_R q_R \ , \qquad (66)$$

with

$$q_K = \begin{cases} 1 & \text{if} \quad p_* \leq p_K \\ \left[1 + \dfrac{\gamma+1}{2\gamma}(p_*/p_K - 1)\right]^{1/2} & \text{if} \quad p_* > p_K \ . \end{cases} \qquad (67)$$

and

$$S_* = \frac{p_R - p_L + \rho_L u_L (S_L - u_L) - \rho_R u_R (S_R - u_R)}{\rho_L (S_L - u_L) - \rho_R (S_R - u_R)} \ . \qquad (68)$$

- *Step III: HLLC flux.* Compute the HLLC flux, according to

$$\mathbf{F}^{hllc}_{i+\frac{1}{2}} = \begin{cases} \mathbf{F}_L & \text{if} \quad 0 \leq S_L \ , \\ \mathbf{F}_{*L} & \text{if} \quad S_L \leq 0 \leq S_* \ , \\ \mathbf{F}_{*R} & \text{if} \quad S_* \leq 0 \leq S_R \ , \\ \mathbf{F}_R & \text{if} \quad 0 \geq S_R \ , \end{cases} \qquad (69)$$

$$\mathbf{F}_{*K} = \mathbf{F}_K + S_K(\mathbf{U}_{*K} - \mathbf{U}_K) \tag{70}$$

and

$$\mathbf{U}_{*K} = \rho_K \left(\frac{S_K - u_K}{S_K - S_*} \right) \begin{bmatrix} 1 \\ S_* \\ v_K \\ w_K \\ \frac{E_K}{\rho_K} + (S_* - u_K) \left[S_* + \frac{p_K}{\rho_K (S_K - u_K)} \right] \end{bmatrix} . \tag{71}$$

There are two variants of the HLLC flux in the third step, as seen below.

- *Step III: HLLC flux, Variant 1.* Compute the numerical fluxes as

$$\mathbf{F}_{*K} = \frac{S_*(S_K\mathbf{U}_K - \mathbf{F}_K) + S_K(p_K + \rho_L(S_K - u_K)(S_* - u_K))\mathbf{D}_*}{S_K - S_*} \; ,$$

$$\mathbf{D}_* = [0, 1, 0, 0, S_*]^T \; , \tag{72}$$

and the final HLLC flux chosen according to (69).

- *Step III: HLLC flux, Variant 2.* Compute the numerical fluxes as

$$\mathbf{F}_{*K} = \frac{S_*(S_K\mathbf{U}_K - \mathbf{F}_K) + S_K P_{LR}\mathbf{D}_*}{S_K - S_*} \; , \tag{73}$$

with $\mathbf{D}_*$ as in (72) and

$$P_{LR} = \frac{1}{2}[p_L + p_R + \rho_L(S_L - u_L)(S_* - u_L) + \rho_R(S_R - u_R)(S_* - u_R)] \; . \tag{74}$$

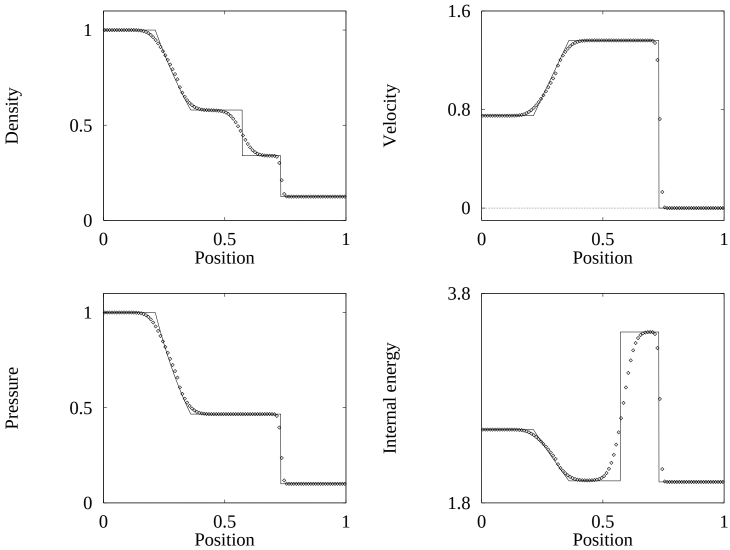
The final HLLC flux is chosen according to (69).

Numerical Results

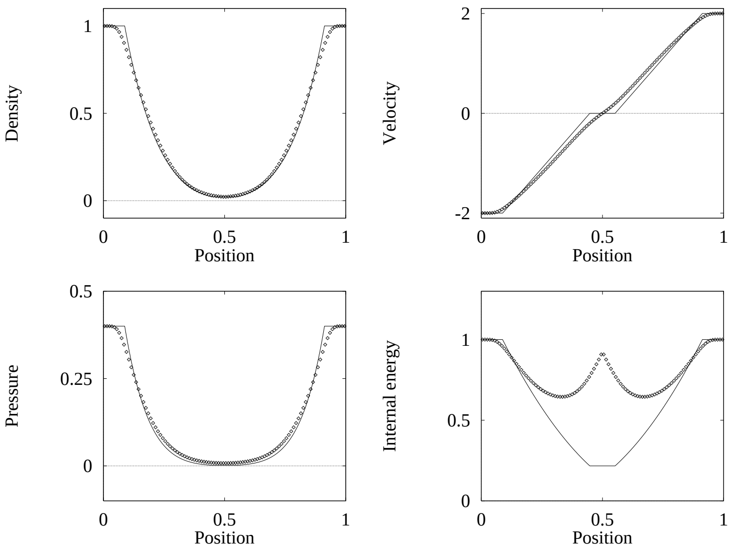
Test problems:

Test	ρ_L	u_L	p_L	ρ_R	u_R	p_R
1	1.0	0.75	1.0	0.125	0.0	0.1
2	1.0	-2.0	0.4	1.0	2.0	0.4
3	1.0	0.0	1000.0	1.0	0.0	0.01
4	5.99924	19.5975	460.894	5.99242	-6.19633	46.0950
5	1.0	-19.59745	1000.0	1.0	-19.59745	0.01
6	1.4	0.0	1.0	1.0	0.0	1.0
7	1.4	0.1	1.0	1.0	0.1	1.0

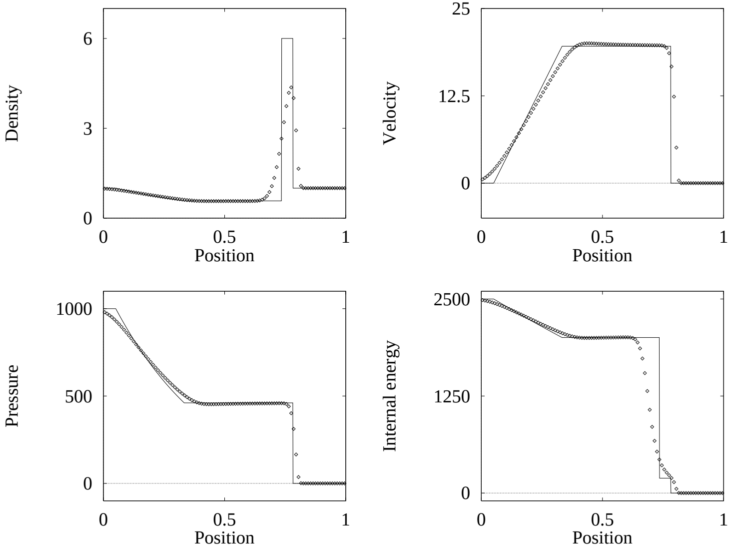
Table 1. Data for seven test problems with exact solution



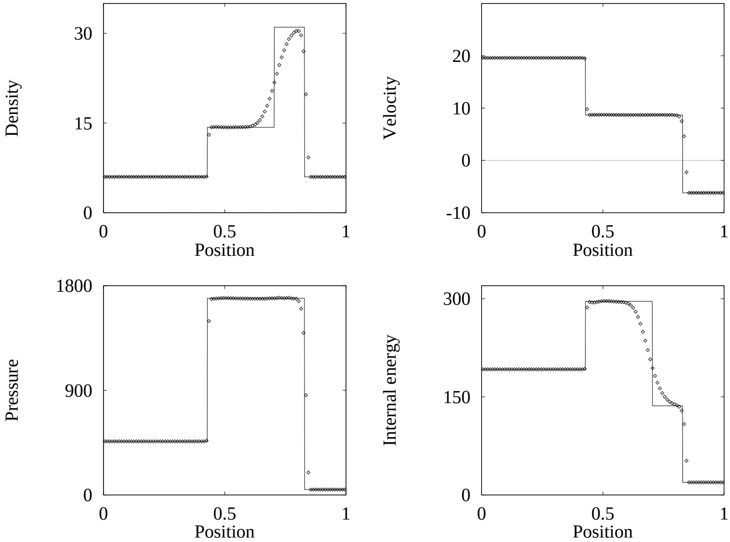
Godunov's method with HLLC Riemann solver applied to Test 1, with $x_0 = 0.3$. Numerical (symbol) and exact (line) solutions are compared at time 0.2.



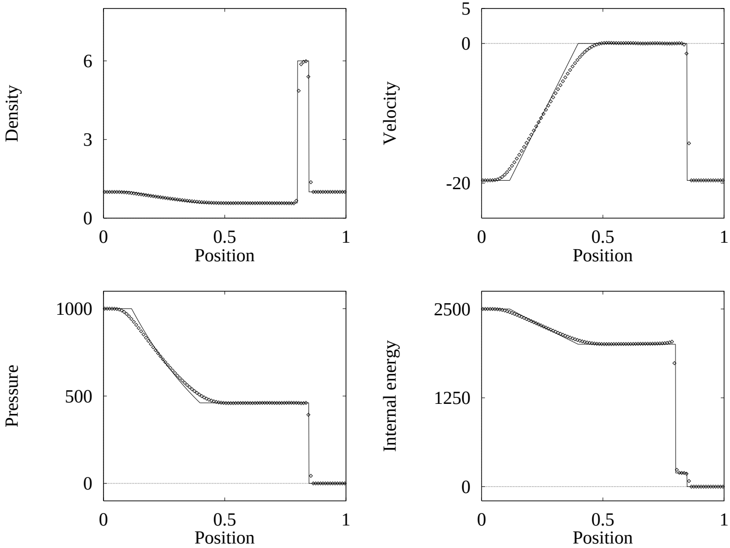
Godunov's method with HLLC Riemann solver applied to Test 2, with $x_0 = 0.5$. Numerical (symbol) and exact (line) solutions are compared at time 0.15.



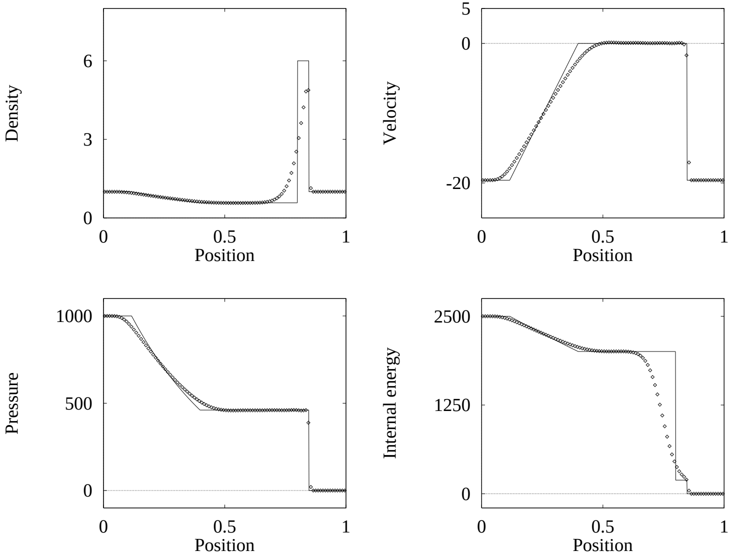
Godunov's method with HLLC Riemann solver applied to Test 3, with $x_0 = 0.5$. Numerical (symbol) and exact (line) solutions are compared at time 0.012.



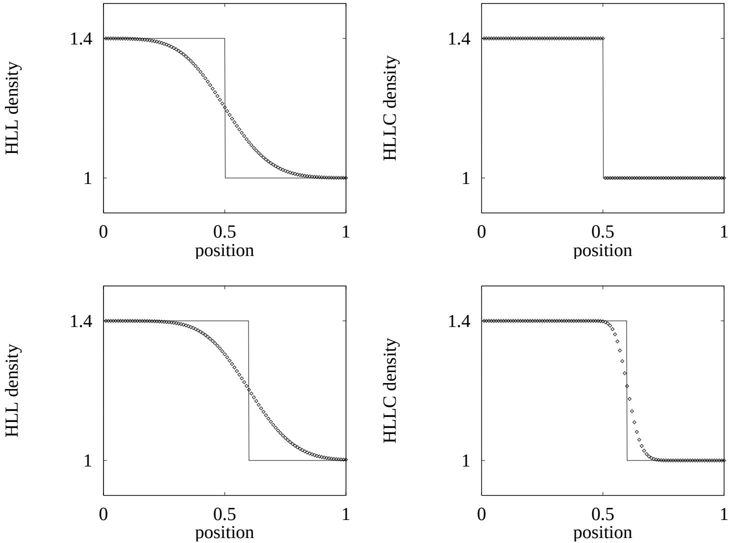
Godunov's method with HLLC Riemann solver applied to Test 4, with $x_0 = 0.4$. Numerical (symbol) and exact (line) solutions are compared at time 0.035.



Godunov's method with HLLC Riemann solver applied to Test 5, with $x_0 = 0.8$. Numerical (symbol) and exact (line) solutions are compared at time 0.012.



Godunov's method with HLL Riemann solver applied to Test 5, with $x_0 = 0.8$. Numerical (symbol) and exact (line) solutions are compared at time 0.012.



Godunov's method with HLL (left) and HLLC (right) Riemann solvers applied to Tests 6 and 7. Numerical (symbol) and exact (line) solutions are compared at time 2.0.

Closing Remarks:

- We have presented HLLC for the Euler equations.
- For the 2D shallow water equations see Toro E F Shock capturing methods for free-surface shallow flows. Wiley and Sons, 2001.
- For Turbulent flow applications (implicit version of HLLC), see Batten, Leschziner and Goldberg (1997).
- For extensions to MHD equations see Gurski (2004), Li (2005), Mignone et al. (2006++).
- For application to two-phase flow see Tokareva and Toro, JCP (2010).
- For extensions see Takahiro (2005) and Bouchut (2007), Mignone (2005).