Navier-Stokes equations with non-standard boundary conditions

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Abstract

Stokes and Navier-Stokes equations play a central role in fluid dynamics, engineering and applied mathematics. Based on the theory of semi-groups we carried out a systematic treatment of Stokes equations with Navier-type boundary conditions or, as called them in the literature, non-standard boundary conditions on the boundary of the fluid domain. These boundary conditions, while being perfectly motivated from the physical point of view, have been less studied than the most conventional Dirichlet boundary condition. The aim of the work is to resume the L^p theory that was developed for the Navier-Stokes equation with Dirichlet boundary condition for the Navier or Navier-type boundary conditions. Precisely, the following aspects will be presented : First, we set our problem and we explain the motivations of the different results. Next we define the Stokes operator with Navier-type boundary conditions and we give a useful remark which is a key tool in our work. Then, we give the analyticity of the Stokes semi-group with the corresponding boundary conditions. Finally we study the time dependent Stokes problem.

1. Problem

We are interested in the mathematical theoretical study of the **Stokes** Problem

$$\begin{cases} \frac{\partial \boldsymbol{u}}{\partial t} - \Delta \boldsymbol{u} + \nabla \pi = \boldsymbol{f}, \ \operatorname{div} \boldsymbol{u} = 0 & \text{in } \Omega \times (0,T), \\ \boldsymbol{u}(0) = \boldsymbol{u}_0 & \text{in } \quad \Omega, \end{cases} \qquad (1)$$

in a bounded domain Ω of $\mathbb{R}^3$ of class at least $C^{1,1}$.

Problem (1) can be used to describe the motion of a fluid occupying a domain Ω. The unkowns $\boldsymbol{u}$ and π stand respectively for the velocity field and the pressure of the fluid. The given data are the external force $\boldsymbol{f}$ and the initial velocity $\boldsymbol{u}_0$.

To study Problem (1) it is necessary to add appropriate boundary conditions. Since the pioneer work of Leray and Hopf, Stokes and Navier Stokes problems have often been studied with **Dirichlet** Boundary condition

$$\boldsymbol{u} = \boldsymbol{0} \quad \text{on } \Gamma \times (0,T),$$

also called **non-slip** boundary condition, formulated by **Stokes** in 1845. However, in some real life situations, it is necessary to impose other boundary conditions that describe better the behavior of the fluid on or near the boundary. **Navier** suggested in 1823 an alternative type of boundary conditions, called **Navier-slip** boundary condition, based on a proportionality between the tangential component of the normal dynamical tensor $\mathbb{D}\boldsymbol{u} = \frac{1}{2}(\nabla \boldsymbol{u} + \nabla \boldsymbol{u}^T)$ and the velocity

$$\boldsymbol{u} \cdot \boldsymbol{n} = 0, \quad 2\,\nu\,[\mathbb{D}\boldsymbol{u} \cdot \boldsymbol{n}]_{\boldsymbol{\tau}} + \alpha\, \boldsymbol{u}_{\boldsymbol{\tau}} = 0 \quad \text{on } \Gamma \times (0,T), \qquad (2)$$

where $\alpha \geq 0$ is the coefficient of friction. The Navier boundary condition (2) allows the fluid to slip on the wall and to measure the coefficient of friction. It is also known that these boundary conditions simulate better the flows near rough or perforated walls.

It is also known that in the case of flat boundary and when the coefficient of friction $\alpha = 0$, the Navier-boundary condition (2) is equivalent to the following **Navier-type boundary conditions**

$$\boldsymbol{u} \cdot \boldsymbol{n} = 0, \qquad \mathbf{curl}\, \boldsymbol{u} \times \boldsymbol{n} = \boldsymbol{0} \ \text{ on } \quad \Gamma \times (0,T). \qquad (3)$$

In our work we deal with the **Stokes problem with Navier-type boundary conditions** (3).

2. Stokes operator

Consider the following closed subspace of $\boldsymbol{L}^p(\Omega)$

$$\boldsymbol{L}^p_{\sigma,\tau}(\Omega) = \Big\{ \boldsymbol{f} \in \boldsymbol{L}^p(\Omega);\ \operatorname{div} \boldsymbol{f} = 0 \text{ in } \Omega,\ \boldsymbol{f} \cdot \boldsymbol{n} = 0 \text{ on } \Gamma \Big\}.$$

The Stokes operator with Navier-type boundary conditions is a densely defined closed linear operator on $\boldsymbol{L}^p_{\sigma,\tau}(\Omega)$, defined by :

$$\forall\, \boldsymbol{u} \in \mathbf{D}(A_p), \quad A_p\, \boldsymbol{u} = -P \Delta \boldsymbol{u} \quad \text{in } \Omega.$$

We recall that $P : \boldsymbol{L}^p(\Omega) \longmapsto \boldsymbol{L}^p_{\sigma,\tau}(\Omega)$ is the is the Helmholtz projection defined by:

$$\forall\, \boldsymbol{f} \in \boldsymbol{L}^p(\Omega), \quad P\boldsymbol{f} = \boldsymbol{f} - \mathbf{grad}\,\pi \quad \text{in } \Omega,$$

where $\pi \in W^{1,p}(\Omega)/\mathbb{R}$ is the unique solution up to an additive constant of the following **weak Neumann problem**

$$\operatorname{div}(\mathbf{grad}\,\pi - \boldsymbol{f}) = 0 \quad \text{in } \Omega, \quad (\mathbf{grad}\,\pi - \boldsymbol{f}) \cdot \boldsymbol{n} = 0 \quad \text{on } \Gamma.$$

Thanks to the work of **Amrouche** and **Seloula** [4, 5] and using the **regularity of** Ω, when Ω **is of class** $C^{2,1}$, we can characterize the domain of A_p as follows

$$\mathbf{D}(A_p) = \big\{ \boldsymbol{u} \in \boldsymbol{W}^{2,p}(\Omega);\ \operatorname{div} \boldsymbol{u} = 0 \text{ in } \Omega,\ \boldsymbol{u} \cdot \boldsymbol{n} = 0,\ \mathbf{curl}\, \boldsymbol{u} \times \boldsymbol{n} = \boldsymbol{0} \text{ on } \Gamma \big\}.$$

3. Key observation

A key observation is that (see [1]), due to the boundary conditions (3), we have:

$$\forall\, \boldsymbol{u} \in \mathbf{D}(A_p), \quad A_p\, \boldsymbol{u} = -\Delta \boldsymbol{u} \text{ in } \Omega.$$

Thus our work is reduced to study the following time-dependent **Laplacian problem with Navier type boundary conditions**:

$$\begin{cases} \frac{\partial \boldsymbol{u}}{\partial t} - \Delta \boldsymbol{u} = \boldsymbol{f}, & \operatorname{div} \boldsymbol{u} = 0 & \text{in } \Omega \times (0,T), \\ \boldsymbol{u} \cdot \boldsymbol{n} = 0, & \mathbf{curl}\, \boldsymbol{u} \times \boldsymbol{n} = \boldsymbol{0} & \text{on } \Gamma \times (0,T), \\ & \boldsymbol{u}(0) = \boldsymbol{u}_0 & \text{in } \quad \Omega, \end{cases} \qquad (4)$$

with $L^q(0,T; \boldsymbol{L}^p_{\sigma,\tau}(\Omega))$ and $\boldsymbol{u}_0 \in \boldsymbol{L}^p_{\sigma,\tau}(\Omega)$ and $1 < p,\ q < \infty$.

Next, using the weak Neumann defined above we recover an non-constant pressure for a function $\boldsymbol{f}$ in $L^q(0,T; \boldsymbol{L}^p(\Omega))$.

4. Analyticity of the Stokes semi-group

We prove in [1] that the Stokes operator A_p with the Navier-type boundary conditions (3) generates a bounded analytic semi-group on $\boldsymbol{L}^p_{\sigma,\tau}(\Omega)$ for all $1 < p < \infty$. To do that, we use a classical approach based on the study of the following complex resolvent of the Stokes operator with the corresponding boundary conditions

$$\begin{cases} \lambda \boldsymbol{u} - \Delta \boldsymbol{u} = \boldsymbol{f}, & \operatorname{div} \boldsymbol{u} = 0 & \text{in } \Omega, \\ \boldsymbol{u} \cdot \boldsymbol{n} = 0, & \mathbf{curl}\, \boldsymbol{u} \times \boldsymbol{n} = \boldsymbol{0} & \text{on } \Gamma, \end{cases} \qquad (5)$$

where $\lambda \in \mathbf{C}^*$ such that $\mathrm{Re}\lambda \geq 0$ and $\boldsymbol{f} \in \boldsymbol{L}^p_{\sigma,\tau}(\Omega)$.

The Problem (5) has a unique solution $\boldsymbol{u} \in \boldsymbol{W}^{1,p}(\Omega)$ satisfying the estimate

$$\|\boldsymbol{u}\|_{\boldsymbol{L}^p(\Omega)} \leq \frac{C(\Omega,p)}{|\lambda|} \|\boldsymbol{f}\|_{\boldsymbol{L}^p(\Omega)},$$

where the constant $C(\Omega,p)$ is independent of λ and $\boldsymbol{f}$. Moreover if Ω is of class $C^{2,1}$ then $\boldsymbol{u} \in \boldsymbol{W}^{2,p}(\Omega)$.

As a result, using the work of **Barbu** in [6], we deduce the analyticity result.

5. Homogeneous time-dependent Stokes problem

Let $\boldsymbol{u}_0 \in \boldsymbol{L}^p_{\sigma,\tau}(\Omega)$ and $\boldsymbol{f} = 0$. As for the heat equation, under the previous assumptions, the analyticity of the Stokes semi-group allows us to obtain a unique solution to Problem (4) that is regular for $t > 0$. More precisely, the solution $\boldsymbol{u}(t)$ satisfies

$$\boldsymbol{u} \in C([0,\ +\infty[,\ \boldsymbol{L}^p_{\sigma,\tau}(\Omega)) \cap C(]0,\ +\infty[,\ \mathbf{D}(A_p)),$$

$$\boldsymbol{u} \in C^k(]0,\ +\infty[,\ \mathbf{D}(A_p^\ell)), \quad \forall\, k \in \mathbb{N},\ \forall\, \ell \in \mathbb{N}^*.$$

Moreover we have the following estimates

$$\|\boldsymbol{u}(t)\|_{\boldsymbol{L}^p(\Omega)} \leq C(\Omega,p)\, \|\boldsymbol{u}_0\|_{\boldsymbol{L}^p(\Omega)},$$

$$\Big\| \frac{\partial \boldsymbol{u}(t)}{\partial t} \Big\|_{\boldsymbol{L}^p(\Omega)} \leq \frac{C(\Omega,p)}{t} \|\boldsymbol{u}_0\|_{\boldsymbol{L}^p(\Omega)}.$$

In addition, as described in see [2], the solution $\boldsymbol{u}(t)$ is called **weak solution** for the Stokes problem and satisfies in particular :

$$\boldsymbol{u} \in L^r(0,T_0;\ \boldsymbol{W}^{1,p}(\Omega)), \ \frac{\partial \boldsymbol{u}}{\partial t} \in L^r(0,T_0;\ [\boldsymbol{H}_0^{p'}(\mathrm{div},\Omega)]'), \quad (6)$$

for all $1 \leq r < 2$ and for all $T < \infty$.

We can also verify, thanks to [3] that for $p = 2$ the solution $\boldsymbol{u}(t)$ satisfies (6) for $r = 2$ included and

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}\,t}\|\boldsymbol{u}(t)\|^2_{\boldsymbol{L}^2(\Omega)} + \int_\Omega |\mathbf{curl}\,\boldsymbol{u}(t)|^2\,\mathrm{d}\,x = 0.$$

We recall that a function $\boldsymbol{f} \in [\boldsymbol{H}_0^{p'}(\mathrm{div},\Omega)]'$ if and only if $\boldsymbol{f} = \boldsymbol{\psi} + \mathbf{grad}\chi$, for some $\boldsymbol{\psi} \in \boldsymbol{L}^p(\Omega)$ and $\chi \in L^p(\Omega)$.

Remark: We have more classes of solutions which are not listed here. Namely, **strong** and **very weak** solutions (see [2]).

6. Maximal $L^p - L^q$ regularity for the Stokes Problem

Using the **boundedness of the pure imaginary powers** of the Stokes operator in $\boldsymbol{L}^p_{\sigma,\tau}(\Omega)$ (see [2]) and proceeding as **Giga** and **Sohr** in [7] we prove the existence of a unique solution to the inhomogeneous Stokes Problem (1) together with (3) satisfying the **maximal** $L^p - L^q$ **regularity** described below.

We suppose that Ω **is of class** $C^{2,1}$, $1 < p,q < \infty$, $\boldsymbol{f} \in L^q(0,T;\ \boldsymbol{L}^p(\Omega))$ and $\boldsymbol{u}_0 = \boldsymbol{0}$. Under these assumptions the Stokes Problem (1)) together with (3) has a unique solution $(\boldsymbol{u},\pi)$ such that

$$\boldsymbol{u} \in L^q(0,T_0;\ \boldsymbol{W}^{2,p}(\Omega)), \quad \frac{\partial \boldsymbol{u}}{\partial t} \in L^q(0,T;\ \boldsymbol{L}^p(\Omega))$$

and

$$\pi \in L^q(0,T;\ W^{1,p}(\Omega)/\mathbb{R}),$$

with $T_0 \leq T \text{ if } T < \infty$ and $T_0 < T \text{ if } T = \infty$.

Moreover we have the estimate

$$\int_0^T \Big\| \frac{\partial \boldsymbol{u}}{\partial t} \Big\|^q_{\boldsymbol{L}^p(\Omega)} \mathrm{d}\,t + \int_0^T \|\Delta \boldsymbol{u}(t)\|^q_{\boldsymbol{L}^p(\Omega)} \mathrm{d}\,t + \int_0^T \|\pi(t)\|^q_{W^{1,p}(\Omega)/\mathbb{R}}\,\mathrm{d} \leq C(p,q,\Omega) \int_0^T \|\boldsymbol{f}(t)\|^q_{\boldsymbol{L}^p(\Omega)} \mathrm{d}\,t.$$

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