Global well-posedness for ghost effect system

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Abstract

We prove the global existence of weak solutions of the so-called ghost effect system which has been derived recently in [2]. Our result extends the result by Levermore et al. treating the question of local existence [2] and improve the result by Bresch et al. in [1] for low Mach number System.

1 Introduction

The asymptotic analysis of the Boltzmann system for small Knudsen numbers shows that there is an important class of problems where both the Euler equations and the Navier-Stokes equations fail to describe the behavior of a gas in the continuum limit. In other words, if one lives in the world in the continuum limit, something that cannot be perceived in the world produces a finite effect on the behavior of a gas. Thus it is called the *ghost effect*. The ghost effect system derived in [2] describes the evolution of the density $\rho(t,x)$, velocity $u(t,x)$, and pressure $p(t,x)$ as function of time $t \in \mathbb{R}^+$ and position x over a torus domain $\mathbb{T}^d$ $(d = 2,3)$,

$$\begin{aligned} \partial_t \rho + \operatorname{div}(\rho u) &= 0, \\ \partial_t(\rho u) + \operatorname{div}(\rho u \otimes u) + \nabla \pi - 2 \operatorname{div}(\rho D(u)) &= \operatorname{div}(K), \\ \operatorname{div} u &= -2\kappa \Delta \log \rho, \end{aligned} \tag{1}$$

where κ is the heat conductivity coefficient and K is a capillary force given by

$$K = \frac{c}{(2m+1)} \rho^{2m+1} \nabla\nabla \log \rho + \frac{c\,m}{(m+1)(2m+1)^2} \Delta \rho^{2m+1}\, \mathbb{I} + \frac{c}{(m+1)} \rho^{m+1} \Delta \rho^m\, \mathbb{I}.$$

with c is a constant which can take both negative and nonnegative value and $\mathbb{I}$ is the identity matrix. Note that, we can prove

$$\begin{aligned} \operatorname{div}(K) &= c\,\rho \nabla\Big(\sqrt{\rho^{2m-1}} \Delta\big(\int_0^\rho \sqrt{s^{2m-1}} ds\big)\Big) \\ &= \frac{c}{m(m+1)} [\operatorname{div}(\rho^{m+1} \nabla\nabla \rho^m) + m \nabla(\rho^{m+1} \Delta \rho^m)]. \end{aligned}$$

Particular cases of System (1).

- ▶ $c = 0 \Rightarrow$ low Mach number System proposed by Lions.
- ▶ $\kappa = c = 0 \Rightarrow$ Non-homogeneous incompressible Navier-Stokes eqs.

2 Reformulation of the system

By introducing a new variable w as follow

$$\mathrm{w} = u + 2\kappa \nabla \log \rho, \quad 0 < \kappa < 1,$$

System (1) recasts in the following system for the new unknown (ρ, w, π)

$$\begin{aligned} &\partial_t \rho + \operatorname{div}(\rho \mathrm{w}) - 2\kappa \Delta \rho = 0, \\ &\partial_t(\rho \mathrm{w}) + \operatorname{div}(\rho u \otimes \mathrm{w}) + \nabla \pi_1 - 2(1-\kappa) \operatorname{div}(\rho D(u)) \\ &-2\kappa \operatorname{div}(\rho A(u)) = c \rho \nabla\big(\sqrt{\rho^{2m-1}} \Delta\big(\textstyle\int_0^\rho \sqrt{s^{2m-1}} ds\big)\big), \\ &\operatorname{div} \mathrm{w} = 0, \end{aligned} \tag{2}$$

where we have denoted by

$$A(u) = \frac{1}{2}(\nabla u - \nabla^t u).$$

3 Main result

The main result in this work is announced in the following theorem:

Theorem

1- Assume $c \geq 0$. Let $0 < \kappa < 1$ and $0 \leq m \leq 1/2$. Moreover suppose that the initial data (ρ^0, w^0) satisfies

$$\rho^0 \in H^1(\Omega), \quad 0 < r \leq \rho^0 \leq R < \infty, \quad \mathrm{w}^0 \in H, \tag{3}$$

where

$$H = \{z \in L^2(\Omega);\ \operatorname{div} z = 0\}$$

then there exists at least one global weak sol. (ρ, w) of System (2) with

$$||\rho||_{L^2(0,T;H^1(\Omega))} + ||\mathrm{w}||_{L^\infty(0,T;H)} + ||\mathrm{w}||_{L^2(0,T;V)} \leq C,$$

$$||\rho||_{L^\infty(0,T;H^1(\Omega)) \cap L^2(0,T;H^2(\Omega))} + ||D(u)||_{L^2(0,T;L^2(\Omega))} \leq C,$$

where

$$V = \{z \in W^{1,2}(\Omega);\ \operatorname{div} z = 0\}.$$

2- Assume $c < 0$ and let $0 < \kappa < 1$. Moreover suppose that the initial data (ρ^0, w^0) satisfies (3) and the following condition holds

$$r < R \leq \Big(\frac{8\kappa\sqrt{\kappa(1-\kappa)}(2m+1)}{|c|} r\Big)^{1/(2m+1)}, \tag{4}$$

then there exists at least one global weak sol. (ρ, w) of System (2).

4 Strategy of the proof

New functional inequality: To prove Theorem 1, we need here the following functional inequality:

Lemma 1. For $\rho \in H^2(\Omega)$ and $0 \leq m \leq 1/2$, there exists a constant $c_0 > 0$ such that

$$\begin{aligned} \mathcal{I} &= \int_\Omega \rho^{m+1} \nabla\nabla \rho^m : \nabla\nabla \log \rho\, dx + m \int_\Omega \rho^{m+1} \Delta \rho^m \Delta \log \rho\, dx \\ &\geq 4\, c_0 \frac{m(m+1)}{(2m+1)^2} \int_\Omega (\Delta \rho^{\frac{2m+1}{2}})^2\, dx. \end{aligned} \tag{5}$$

The constant c_0 should be satisfy an upper bound, namely

$$0 < c_0 \leq 1 - \frac{(d-1)^2}{d(d+2)} \frac{(1-2m)}{1+2m}.$$

4.1 A priori estimates

Maximum principle for the continuity equation give

$$0 < r \leq \rho \leq R < \infty$$

$$||\rho||_{L^\infty(0,T;L^2(\Omega))} + ||\rho||_{L^2(0,T;H^1(\Omega))} \leq C$$

Proposition 1.

Case when $c > 0$. Let (ρ, w) smooth sol. of (2), then if $0 \leq m \leq 1/2$, there exists $c_0 > 0$ such that (ρ, w) satisfies

$$\begin{aligned} &\frac{d}{dt} \int_\Omega \rho\Big(\frac{|\mathrm{w}|^2}{2} + (1-\kappa)\kappa \frac{|2\nabla \log \rho|^2}{2}\Big) dx \\ &+ c\frac{m^2}{2} \frac{d}{dt} \int_\Omega \rho^{2m+1} |\nabla \log \rho|^2 dx + 2(1-\kappa) \int_\Omega \rho |D(u)|^2\, dx \\ &+ 2\kappa \int_\Omega \rho |A(u)|^2 dx + \frac{8\, c\, \kappa\, c_0}{(2m+1)^2} \int_\Omega \Big(\Delta \rho^{\frac{2m+1}{2}}\Big)^2 dx \leq 0. \end{aligned} \tag{6}$$

Case when $c < 0$. Let (ρ, w) smooth sol. of (2) and $c < 0$, then there exists $\beta > 0$ such that

$$\begin{aligned} &\frac{d}{dt} \int_\Omega \rho\Big(\frac{|\mathrm{w}|^2}{2} + (1-\kappa)\kappa \frac{|2\nabla \log \rho|^2}{2}\Big) dx \\ &+ 2(1-\kappa) \int_\Omega \rho |D(u) - \frac{1}{d} \operatorname{div} u\, \mathbb{I}|^2\, dx + 2\kappa \int_\Omega \rho |A(\mathrm{w})|^2 dx \\ &- \frac{|c|}{(2m+1)} \frac{\beta}{2} R^{2m+1} \int_\Omega |\nabla \mathrm{w}|^2\, dx + \frac{8(1-\kappa)\kappa^2}{d} \int_\Omega \rho |\Delta \log \rho|^2\, dx \\ &- \frac{|c|}{2\beta\,(2m+1)} R^{2m+1} \int_\Omega |\Delta \log \rho|^2\, dx \leq 0. \end{aligned} \tag{7}$$

Proof. Multiplying $(2)_2$ by w and integrating by parts

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt} \int_\Omega \rho |\mathrm{w}|^2 dx + 2(1-\kappa) \int_\Omega \rho |D(u)|^2 dx \\ &+ 4(1-\kappa)\kappa \int_\Omega \rho \nabla u : \nabla^2 \log \rho\, dx + 2\kappa \int_\Omega \rho |A(u)|^2 dx \\ &- c \int_\Omega \rho \nabla\Big(\sqrt{\rho^{2m-1}} \Delta\big(\int_0^\rho \sqrt{s^{2m-1}} ds\big)\Big) \cdot \mathrm{w}\, dx = 0. \end{aligned} \tag{8}$$

Remark that, testing $(1)_1$ by $-\Delta\sqrt{\rho}/\sqrt{\rho}$ we get

$$\frac{d}{dt} \int_\Omega \rho \frac{|\nabla \log \rho|^2}{2} dx - \int_\Omega \rho \nabla u : \nabla^2 \log \rho\, dx = 0.$$

Moreover we have

$$\int_\Omega \rho |D(u)|^2\, dx = \int_\Omega \rho |D(u) - \frac{1}{d} \operatorname{div} u\, \mathbb{I}|^2\, dx + \int_\Omega \frac{\rho}{d} |\operatorname{div} u|^2\, dx,$$

Case when $c > 0$.

$$\begin{aligned} &- c \int_\Omega \rho \nabla\Big(\sqrt{\rho^{2m-1}} \Delta\big(\int_0^\rho \sqrt{s^{2m-1}} ds\big)\Big) \cdot \mathrm{w}\, dx \\ &= -c \int_\Omega \rho \nabla\Big(\sqrt{\rho^{2m-1}} \Delta\big(\int_0^\rho \sqrt{s^{2m-1}} ds\big)\Big) \cdot u\, dx \\ &- 2\,\kappa\, c \int_\Omega \rho \nabla\Big(\sqrt{\rho^{2m-1}} \Delta\big(\int_0^\rho \sqrt{s^{2m-1}} ds\big)\Big) \cdot \nabla \log \rho\, dx \\ &= c\frac{m^2}{2} \frac{d}{dt} \int_\Omega \rho^{2m-1} |\nabla \rho|^2 dx + \frac{8\, c\, \kappa\, c_0}{(2m+1)^2} \int_\Omega \Big(\Delta \rho^{\frac{2m+1}{2}}\Big)^2 dx \end{aligned}$$

Case when $c < 0$.

$$\begin{aligned} &- c \int_\Omega \rho \nabla\Big(\sqrt{\rho^{2m-1}} \Delta\big(\int_0^\rho \sqrt{s^{2m-1}} ds\big)\Big) \cdot \mathrm{w}\, dx \\ &= \frac{c}{(2m+1)} \int_\Omega \rho^{2m+1} \nabla^2 \log \rho : \nabla \mathrm{w}\, dx \\ &\leq \frac{|c|}{(2m+1)} R^{2m+1} \Big(\frac{1}{2\beta} \|\nabla^2 \log \rho\|^2_{L^2(\Omega)} + \frac{\beta}{2} \|\nabla \mathrm{w}\|^2_{L^2(\Omega)}\Big) \end{aligned}$$

4.2 Construction of solution

The procedure of construction of solution follow the idea developped recently in [1] for low Mach number system. Precisely, we construct the approximate solution using an augmented approximate system. We propose to study the following system

$$\partial_t \rho + \operatorname{div}(\rho \mathrm{w}) - 2\kappa \Delta \rho = 0, \tag{9}$$

$$\begin{aligned} &\partial_t(\rho \mathrm{w}) + \operatorname{div}((\rho \mathrm{w} - 2\kappa \nabla \rho) \otimes \mathrm{w}) - 2(1-\kappa) \operatorname{div}(\rho D(\mathrm{w})) \\ &-2\kappa \operatorname{div}(\rho A(\mathrm{w})) + \nabla \pi_1 + \epsilon[\Delta^2 \mathrm{w} - \operatorname{div}((1 + |\nabla \mathrm{w}|^2) \nabla \mathrm{w})] \\ &\quad = -2\kappa(1-\kappa) \operatorname{div}(\rho \nabla v) + \frac{c}{2(2m+1)} \operatorname{div}(\rho^{2m+1} \nabla v), \end{aligned} \tag{10}$$

$$\partial_t(\rho v) + \operatorname{div}((\rho \mathrm{w} - 2\kappa \nabla \rho) \otimes v) - 2\kappa \operatorname{div}(\rho \nabla v) = -2 \operatorname{div}(\rho \nabla^t \mathrm{w}), \tag{11}$$

$$\operatorname{div} \mathrm{w} = 0.$$

The Steps:

▶ Construction of solution with $\epsilon > 0$.

- For given w, one can find a solution ρ sol. of the continuity equation (9) using the standard theory of parabolic equations.
- Having obtained ρ, we can construct an approximate solution (w, v) using a fixed point argument at the level of the Galerkin approximate system. Precisely, we rewrite equations (10)-(11) as a fixed point problem.

$$(\mathrm{w}(t), v(t)) = \mathcal{F}[\mathrm{w}, v](t) \tag{12}$$

▶ Identification $v = 2\nabla \log \rho$

▶ Passage to the limit $\epsilon \to 0 \Rightarrow \exists$ of sol (ρ, w) of System (2).

5 Forthcoming Research

Actually a generalization of Inequality (5) is proved and its application to the model of compressible Navier-Stokes-Korteweg underway in collaboration with D. Bresch and A. Vasseur.

References

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