On some Equivalent Theorems: Poincaré, Korn, De Rham, Necas, Lions and Bogovskii

Chérif Amrouche
Laboratoire de Mathématiques et de leurs Applications
CNRS UMR 5142
Université de Pau et des Pays de l'Adour

joint work with
Philippe G. Ciarlet, City University of Hong-Kong
Cristinel Mardare, Univ. Pierre et Marie Curie (Paris 6)

Prague, August 31, 2016

Outline

I. Recall of Some Classical Theorems

In this work, we assume that Ω is a bounded open connected of $\mathbb{R}^N$, $N \geq 2$, with Lipschitz-continuous boundary.

The notation ${}_{X'}<,>_X$ denotes a duality pairing between a topological space X and its dual X'.

We shall use bold characters for the vector fields or the vector spaces and the non-bold characters for the scalars.

The letter C denotes a constant that not necessarily the same at its various occurrences.
We set

$$\boldsymbol{V}(\Omega) = \{\boldsymbol{\varphi} \in \boldsymbol{H}_0^1(\Omega);\ \mathrm{div}\ \boldsymbol{\varphi} = 0 \quad \text{in } \Omega\}.$$

I.1 The Classical J.L. Lions Lemma

We know that

$$f \in L^2(\Omega) \Longrightarrow f \in H^{-1}(\Omega) \quad \text{and} \quad \nabla f \in \boldsymbol{H}^{-1}(\Omega).$$

The classical Lions Lemma asserts that the reciprocal holds:

Theorem 1 (Classical Lions Lemma)

We have the following property

$$f \in H^{-1}(\Omega) \quad \text{and} \quad \nabla f \in \boldsymbol{H}^{-1}(\Omega) \Longrightarrow f \in L^2(\Omega).$$

- G. Duvaut – J.L. Lions (1972) where Ω is $\mathcal{C}^\infty$ (see also L. Tartar (1978))

I.2 Necas Inequalities

Theorem 2 (Necas Inequality)

i) For any $f \in L^2(\Omega)$, we have

$$\|f\|_{L^2(\Omega)} \leq C(\|f\|_{H^{-1}(\Omega)} + \|\nabla f\|_{\boldsymbol{H}^{-1}(\Omega)}) \quad (1)$$

ii) More generally, for any integer m and any $1 < p < \infty$, we have for all $f \in W^{m,p}(\Omega)$,

$$\|f\|_{W^{m,p}(\Omega)} \leq C(\|f\|_{W^{m-1,p}(\Omega)} + \|\nabla f\|_{\boldsymbol{W}^{m-1,p}(\Omega)}) \quad (2)$$

- J. Necas (1965)

I.3 Korn's Inequality, Duvaut-Lions (1972)

Setting

$$e_{ij}(\boldsymbol{v}) = \frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}.$$

Theorem 3 (Korn's Inequality)

$$\forall \boldsymbol{v} \in \boldsymbol{H}^1(\Omega), \quad \|\boldsymbol{v}\|_{\boldsymbol{H}^1(\Omega)} \leq C(\|\boldsymbol{v}\|_{\boldsymbol{L}^2(\Omega)} + \|e(\boldsymbol{v})\|_{\boldsymbol{L}^2(\Omega)}) \qquad (3)$$

Proof. To prove this inequality, it suffices to prove that

$$(\boldsymbol{v} \in \boldsymbol{L}^2(\Omega) \quad \text{and} \quad e(\boldsymbol{v}) \in \boldsymbol{L}^2(\Omega)) \Longrightarrow \nabla \boldsymbol{v} \in \boldsymbol{L}^2(\Omega)$$

and then to use the Banach Theorem. Let us consider $\boldsymbol{v} \in \boldsymbol{L}^2(\Omega)$ and $e(\boldsymbol{v}) \in \boldsymbol{L}^2(\Omega)$. Thanks to the following relation

$$\frac{\partial}{\partial x_j}\left(\frac{\partial v_k}{\partial x_i}\right) = \frac{\partial e_{ik}}{\partial x_j} + \frac{\partial e_{ij}}{\partial x_k} - \frac{\partial e_{jk}}{\partial x_i}.$$

we deduce that

$$\nabla \frac{\partial v_k}{\partial x_i} \in \boldsymbol{H}^{-1}(\Omega).$$

Because $\frac{\partial v_k}{\partial x_i} \in H^{-1}(\Omega)$, the classical Lions Lemma implies that $\frac{\partial v_k}{\partial x_i} \in L^2(\Omega)$.

Remark. There exist different proofs, that do not use Lions Lemma, of the Korn inequality:

- J. Gobert (1962): Proof uses Calderon-Zygmund singular integrals
- V.A Kondrat'ev – O.A. Oleinik (1988): Proof uses integral inequalities with $(\mathrm{dist}(\cdot,\Gamma))^2$ as a weight and hypoellipticity of Δ.

I.4 De Rham's Theorem

It is clear that if $q \in L^2(\Omega)$, then $\nabla q \in \boldsymbol{H}^{-1}(\Omega)$ and

$$\forall \boldsymbol{v} \in \boldsymbol{V}(\Omega), \quad < \nabla q \, , \boldsymbol{v} > = 0.$$

We will prove later that the divergence operator

$$\text{div} \ : \ \boldsymbol{H}_0^1(\Omega)/\boldsymbol{V}(\Omega) \mapsto L_0^2(\Omega)$$

is an isomorphism. So, by duality, we deduce that

Theorem 4 (De Rham in $\boldsymbol{H}^{-1}(\Omega)$, First Version)

The operator

$$\textbf{grad} \ : \ L^2(\Omega)/\mathbb{R} \mapsto \boldsymbol{V}(\Omega)^\circ \tag{4}$$

where

$$\boldsymbol{V}(\Omega)^\circ = \{\boldsymbol{f} \in \boldsymbol{H}^{-1}(\Omega);\, \forall \boldsymbol{v} \in \boldsymbol{V}(\Omega), < \boldsymbol{f} \, , \boldsymbol{v} > = 0\}.$$

is an isomorphism.
That means that for any $\boldsymbol{f} \in \boldsymbol{H}^{-1}(\Omega)$ satisfying

$$\forall \boldsymbol{v} \in \boldsymbol{V}(\Omega), \quad < \boldsymbol{f} \, , \boldsymbol{v} > = 0,$$

there exists $\pi \in L^2(\Omega)$, unique up to an additive constant, such that

$$f = \nabla \pi.$$

Theorem 5 (Stokes Equations)

For any $\boldsymbol{f} \in \boldsymbol{H}^{-1}(\Omega)$, there exists a unique solution

$$(\boldsymbol{u}, \pi) \in \boldsymbol{H}_0^1(\Omega) \times L_0^2(\Omega)$$

to the Stokes equations

$$-\Delta \boldsymbol{u} + \nabla \pi = \boldsymbol{f} \quad \text{and} \quad \operatorname{div} \boldsymbol{u} = 0 \quad \text{in } \Omega. \tag{5}$$

Proof. Using Lax-Milgram Lemma, there exists a unique $\boldsymbol{u} \in \boldsymbol{V}(\Omega)$ satisfying the following variational formulation:

$$\text{Find } \boldsymbol{u} \in \boldsymbol{V}(\Omega) \text{ such that}$$

$$\forall \boldsymbol{v} \in \boldsymbol{V}(\Omega), \quad \int_\Omega \nabla \boldsymbol{u} : \nabla \boldsymbol{v} = < \boldsymbol{f}, \boldsymbol{v} > \tag{6}$$

Consequently we have

$$\forall \boldsymbol{v} \in \boldsymbol{V}(\Omega), \quad < -\Delta \boldsymbol{u} - \boldsymbol{f}, \boldsymbol{v} > = 0.$$

Finally, by De Rham's Theorem, there exists $\pi \in L^2(\Omega)$, unique up an additive constant, such that

$$-\Delta \boldsymbol{u} + \nabla \pi = \boldsymbol{f} \quad \text{in } \Omega.$$

We can improve the previous De Rham Theorem (see Amrouche-Girault, Czech. Math. Journ., 1994).
Setting

$$\mathcal{V}(\Omega) = \{\boldsymbol{v} \in \boldsymbol{\mathcal{D}}(\Omega);\, \operatorname{div} \boldsymbol{v} = 0 \quad \text{in } \Omega\}\,,$$

we have

Theorem 6. (De Rham in $\boldsymbol{H}^{-1}(\Omega)$ Second Version)

Let $\boldsymbol{f} \in \boldsymbol{H}^{-1}(\Omega)$ satisfying

$$\forall \boldsymbol{v} \in \mathcal{V}(\Omega), \quad < \boldsymbol{f}\,, \boldsymbol{v} > = 0,$$

Then there exists $\pi \in L^2(\Omega)$, unique up to an additive constant, such that

$$f = \nabla \pi.$$

Proof. i) Firstly, we prove the algebraical and topological following indentity:

$$\{u \in L^2_{\text{loc}}(\Omega);\ \nabla u \in \boldsymbol{H}^{-1}(\Omega)\} = L^2(\Omega).$$

ii) Therefore, it suffices to proof the existence of π in $L^2_{\text{loc}}(\Omega)$. To this end, consider an increasing sequence $(\Omega_k)_k$ of Lipschitz-continuous, connected open sets such that

$$\overline{\Omega_k} \subset \Omega, \quad \cup_k \Omega_k = \Omega.$$

Take any function $\boldsymbol{v} \in \boldsymbol{V}(\Omega_k)$ and let us extend it by zero outside Ω_k. Then the extended function, still denoted by $\boldsymbol{v}$ belongs to $\boldsymbol{V}(\Omega)$. Then, for all sufficiently small $\varepsilon > 0$, we have:

$$\rho_\varepsilon \star \boldsymbol{v} \in \boldsymbol{\mathcal{D}}(\Omega), \quad \text{div}\,(\rho_\varepsilon \star \boldsymbol{v}) = 0.$$

As $\rho_\varepsilon \star \boldsymbol{v} \in \mathcal{V}(\Omega)$, the assumption on $\boldsymbol{f}$ yields:

$$< \boldsymbol{f},\ \boldsymbol{v} >= \lim_{\varepsilon \to 0} < \boldsymbol{f},\ \rho_\varepsilon \star \boldsymbol{v} > = 0.$$

De Rham's Theorem first version applied in Ω_k to $\boldsymbol{f}_{|\Omega_k}$ implies that there exists $\pi_k \in L^2(\Omega_k)$ such that $f_{|\Omega_k} = \nabla \pi_k$. And since $\pi_{k+1} - \pi_k$ is constant in Ω_k, this constant can be choose so that $\pi_{k+1} = \pi_k$ in Ω_k, and hence $\boldsymbol{f} = \nabla \pi$ with $\pi \in L^2(\omega)$ for any proper subset ω of Ω, i.e $\pi \in L^2_{\text{loc}}(\Omega)$.

Remark We have proved that the following properties are equivalent:

(i) **The operator** div **is onto**: div $: \boldsymbol{H}_0^1(\Omega)/\boldsymbol{V}(\Omega) \mapsto L_0^2(\Omega)$ is an isomorphism.

(ii) **First Version of De Rham's Theorem**

(iii) **Second Version of De Rham's Theorem**

II. First Equivalence Theorem: De Rham, Lions, Necas

Theorem 7 (Lions-Necas-De Rham and Divergence operator)

The followings are equivalent:
(i) **The operator** div **is onto**: div $: \boldsymbol{H}_0^1(\Omega)/\boldsymbol{V}(\Omega) \mapsto L_0^2(\Omega)$ is an isomorphism.
(ii) **First Version of De Rham's Theorem**: For any $\boldsymbol{f} \in \boldsymbol{H}^{-1}(\Omega)$ satisfying

$$\forall \boldsymbol{v} \in \boldsymbol{V}(\Omega), \quad < \boldsymbol{f}\,, \boldsymbol{v} > = 0,$$

there exists $\pi \in L^2(\Omega)$, unique up to an additive constant, such that $f = \nabla \pi$.
(iii) **Second Version of De Rham's Theorem**: Let $\boldsymbol{f} \in \boldsymbol{H}^{-1}(\Omega)$ satisfying

$$\forall \boldsymbol{v} \in \mathcal{V}(\Omega), \quad < \boldsymbol{f}\,, \boldsymbol{v} > = 0.$$

Then there exists $\pi \in L^2(\Omega)$, unique up to an additive constant, such that $f = \nabla \pi$.
(iv) **Classical Lions Lemma**: $L^2(\Omega) = \{q \in H^{-1}(\Omega);\, \nabla q \in \boldsymbol{H}^{-1}(\Omega)\}$.
(v) **Necas Inequality**:

$$\forall q \in L^2(\Omega), \quad \|q\|_{L^2(\Omega)} \leq C(\|q\|_{H^{-1}(\Omega)} + \|\nabla q\|_{\boldsymbol{H}^{-1}(\Omega)}) \qquad (7)$$

(vi) **General Lions Lemma**: $L^2(\Omega) = \{q \in \mathcal{D}'(\Omega);\, \nabla q \in \boldsymbol{H}^{-1}(\Omega)\}$.

Sketch of Proof of Theorem 1 . According to the previous remark, we know that the the properties (i), (ii) and (iii) are equivalent.

1. Implication (i) $\Longrightarrow$ (iv). We deduce from point (i) that the operator

$$\mathrm{div} : \boldsymbol{V}(\Omega)^{\perp} \mapsto L^2_0(\Omega) \tag{8}$$

is an isomorphism and by duality that

$$\mathbf{grad} \ : L^2(\Omega)/\mathbb{R} \mapsto \boldsymbol{V}(\Omega)^{\circ} \tag{9}$$

is also an isomorphism, where the polar set $\boldsymbol{V}(\Omega)^{\circ}$ is defined by

$$\boldsymbol{V}(\Omega)^{\circ} = \{\boldsymbol{f} \in \boldsymbol{H}^{-1}(\Omega);\, \forall \boldsymbol{v} \in \boldsymbol{V}(\Omega),\, < \boldsymbol{f}\, , \boldsymbol{v} > = 0\}.$$

Let now

$$q \in H^{-1}(\Omega) \quad \text{such that} \quad \nabla q \in \boldsymbol{H}^{-1}(\Omega).$$

Using the density of $\mathcal{D}(\Omega)$ in

$$\boldsymbol{H}_0^1(\mathrm{div};\,\Omega) = \{\boldsymbol{v} \in \boldsymbol{H}_0^1(\Omega);\, \mathrm{div}\,\boldsymbol{v} \in H_0^1(\Omega)\},$$

we deduce that for all $\boldsymbol{v} \in \boldsymbol{V}(\Omega)$:

$$_{\boldsymbol{H}^{-1}(\Omega)} < \nabla q\,,\boldsymbol{v} >_{\boldsymbol{H}_0^1(\Omega)} = -\,_{H^{-1}(\Omega)} < q\,,\mathrm{div}\,\boldsymbol{v} >_{H_0^1(\Omega)} = 0.$$

So $\nabla q \in \boldsymbol{V}(\Omega)^{\circ}$ and then

$$\nabla q = \nabla p \quad \text{with some } \ p \text{ in } \ L^2(\Omega),$$

which proves the point ii), since

$$q = p + C \in L^2(\Omega), \quad \text{for some constant } C.$$

2. Implication (iv) $\Longrightarrow$ (v). The space

$$E(\Omega) = \{f \in H^{-1}(\Omega);\, \nabla f \in \boldsymbol{H}^{-1}(\Omega)\},$$

endowed with the graph norm is complete. The canonical injection

$$\mathrm{Id} : L^2(\Omega) \mapsto E(\Omega)$$

is one-to-one, (clearly) continuous and onto by the classical Lions Lemma. Therefore, by Banach open mapping Theorem, the inverse Id^{-1} is also continuous and then we deduce the Necas inequality

$$\forall f \in L^2(\Omega), \quad \|f\|_{L^2(\Omega)} \leq C(\|f\|_{H^{-1}(\Omega)} + \|\nabla f\|_{\boldsymbol{H}^{-1}(\Omega)}) \quad (10)$$

3. Implication (v) $\Longrightarrow$ (i). Recall firstly the Peetre-Tartar Theorem. Let E_1, E_2, E_3 be three Banach spaces, $A \in \mathcal{L}(E_1;\, E_2)$ and $B \in \mathcal{L}(E_1;\, E_3)$ a compact operator such that

$$\forall u \in E_1, \quad \|u\|_{E_1} \simeq \|Au\|_{E_2} + \|Bu\|_{E_3}.$$

Then the mapping

$$A : E_{1/\mathrm{Ker}\ A} \mapsto \mathcal{R}(A)$$

is an isomorphism and $\mathcal{R}(A)$ is a closed subspace of E_2. Observe that the Necas inequality implies that

$$\forall f \in L^2(\Omega), \quad \|f\|_{L^2(\Omega)} \ \simeq \ \|f\|_{H^{-1}(\Omega)} + \|\nabla f\|_{\boldsymbol{H}^{-1}(\Omega)} \tag{11}$$

Because the injection of $L^2(\Omega)$ in $H^{-1}(\Omega)$ is compact, we can apply the Peetre-Tartar Theorem, with

$$A = \nabla,\ B = Id,\ E_1 = L^2(\Omega),\ E_2 = \boldsymbol{H}^{-1}(\Omega),\ E_3 = H^{-1}(\Omega),$$

to deduce that the operator

$$\mathbf{grad}\ : L^2(\Omega)/\mathbb{R} \mapsto \boldsymbol{V}(\Omega)^{\circ} \tag{12}$$

is an isomorphism and by duality

$$\mathrm{div} : \boldsymbol{V}(\Omega)^{\perp} \mapsto L^2_0(\Omega) \tag{13}$$

is also an isomorphism, which proves the point i).

4. Implication (iv) $\iff$ (vi). It is clear that the general Lions Lemma implies the classical Lions Lemma. Conversely, suppose that the classical Lions Lemma holds. Let $q \in \mathcal{D}'(\Omega)$ such that $\nabla q \in \boldsymbol{H}^{-1}(\Omega)$. Then

$$\forall \boldsymbol{v} \in \mathcal{V}(\Omega), \quad < \nabla q \, , \boldsymbol{v} > = 0.$$

Because the classical Lions Lemma is equivalent to the second version of De Rham's Theorem, there exists $\pi \in L^2(\Omega)$ such that $\nabla q = \nabla \pi$. Since $q = \pi + C$, we deduce that $q \in L^2(\Omega)$.

II. The divergence operator

Theorem 8

Let Ω be a bounded, connected open subset of $\mathbb{R}^n$, with $n \geq 2$ and with a Lipschitz-continuous boundary.
i) The following operator

$$\text{div} \quad : \boldsymbol{H}_0^1(\Omega)/\boldsymbol{V}(\Omega) \mapsto L_0^2(\Omega) \tag{14}$$

is an isomorphism.
ii) Moreover the operator

$$\text{div} \quad : \boldsymbol{\mathcal{D}}(\Omega) \mapsto \mathcal{D}(\Omega) \cap L_0^2(\Omega) \tag{15}$$

is onto

Proof. This result is due to Bogovskii (1979) with a very short proof. The following more complete proof is inspired by the proof given in the book of P. Galdi (1994).
To get (14), we will prove that for any $f \in L_0^2(\Omega)$, there exists a vector field $\boldsymbol{u} \in \boldsymbol{H}_0^1(\Omega)$ such that div $\boldsymbol{u} = f$ and satisfying the inequality

$$\|\boldsymbol{u}\|_{\boldsymbol{H}^1(\Omega)} \leq C\|f\|_{L^2(\Omega)}. \tag{16}$$

First step. We suppose that Ω is starlike with respect to some open ball B contained it in. In this case, the vector $\boldsymbol{u} := \boldsymbol{R}f$ is constructed explicitly.

i) Construction of $\boldsymbol{R}$. Suppose $f \in \mathcal{D}(\Omega)$ and let $\tilde{f} \in \mathcal{D}(\mathbb{R}^n)$ its extension by 0 outside of Ω, with $n \geq 2$. Let θ a fixed function satisfying

$$\theta \in \mathcal{D}(\mathbb{R}^n), \quad 0 \leq \theta \leq 1, \quad \text{with } \operatorname{supp} \theta \subset B \quad \text{and} \quad \int_{\mathbb{R}^n} \theta = 1.$$

Consider the function $t \mapsto t^n \tilde{f}(y + t(x - y))$, with $t \in \mathbb{R}$ and $(x, y) \in \mathbb{R}^n \times \mathbb{R}^n$. Then

$$\frac{d}{dt}(t^n \tilde{f})(y + t(x - y)) = t^{n-1} \nabla_x \cdot ((x - y)\tilde{f}(y + t(x - y)).$$

Multiplying this relation by $\theta(y)$ and integrating with respect y in Ω and with respect t on $]1, \infty[$. Then,

$$f(x) = -\int_{\Omega} \int_1^{\infty} \theta(y) t^{n-1} \nabla_x \cdot ((x - y)\tilde{f}(y + t(x - y))dt\, dy.$$

The following vector field

$$x \in \Omega, \quad \boldsymbol{u}(x) = -\int_{\mathbb{R}^n} (x-y)\theta(y) \int_1^\infty t^{n-1} \tilde{f}(y + t(x-y)) dt\, dy. \tag{17}$$

satisfies, as we will prove later,

$$\operatorname{div} \boldsymbol{u} = f \quad \text{in } \Omega.$$

Setting successively $z = y + t(x-y)$, $t = (r-1)/r$, $s = 1 - r$ and after changing z into y and s into t, we obtain

$$x \in \Omega, \quad \boldsymbol{u}(x) = \int_{\mathbb{R}^n} (x-y) f(y) \int_1^\infty t^{n-1} \theta(y + t(x-y)) dt\, dy. \tag{18}$$

Setting finally $r = t|x-y|$, we have also

$$\boldsymbol{u}(x) := \boldsymbol{R}f(x) = \int_\Omega \frac{x-y}{|x-y|^n} f(y) \int_{|x-y|}^\infty \theta(y + r\frac{x-y}{|x-y|}) r^{n-1} dr\, dy. \tag{19}$$

We verify then that supp $\boldsymbol{R}f$ is compact and $\boldsymbol{R}f$ is $\mathcal{C}^\infty(\Omega)$.

ii) Proof of div $\boldsymbol{R}f = f$. We observe first that

$$\boldsymbol{u}(x) = \lim_{\varepsilon \to 0} \int_{|x-y| \geq \varepsilon} \tilde{f}(y) \boldsymbol{K}(x, y)\, dy,$$

where

$$\boldsymbol{K}(x, y) = (x-y) \int_1^{\infty} t^{n-1} \theta(y+t(x-y)) dt = \frac{x-y}{|x-y|^n} \int_{|x-y|}^{\infty} \theta(y+r \frac{x-y}{|x-y|}) r^{n-1} dr.$$

Then

$$\frac{\partial u_i}{\partial x_j}(x) = \lim_{\varepsilon \to 0} [\int_{|x-y| \geq \varepsilon} \tilde{f}(y) \frac{\partial K_i}{\partial x_j}(x, y)\, dy + \int_{|x-y| = \varepsilon} \tilde{f}(y) \frac{x_j - y_j}{|x-y|} K_i(x, y)\, d\sigma_y] \tag{20}$$

and

$$\lim_{\varepsilon \to 0} \int_{|x-y|=\varepsilon} \tilde{f}(y) \frac{x_j - y_j}{|x-y|} K_i(x,y)\, d\sigma_y = \tilde{f}(x) \int_\Omega \frac{(x_j - y_j)(x_i - y_i)}{|x-y|^2} \theta(y)\, d\sigma_y \quad (21)$$

Besides, we have

$$\begin{aligned} \frac{\partial K_i}{\partial x_j}(x,y) &= \delta_{ij} \int_1^\infty t^{n-1} \theta(y + t(x-y))dt + (x_i - y_i) \int_1^\infty t^n \frac{\partial \theta}{\partial x_j}(y + t(x-y))dt \\ &= \frac{\delta_{ij}}{|x-y|^n} \int_0^\infty \theta(x + r\frac{x-y}{|x-y|})(|x-y| + r)^{n-1} dr \\ &+ \frac{x_i - y_i}{|x-y|^n} \int_0^\infty \frac{\partial \theta}{\partial x_j}(x + r\frac{x-y}{|x-y|})(|x-y| + r)^n dr. \end{aligned} \quad (22)$$

From this relations, we deduce that

$$\begin{aligned} \nabla_x \cdot \boldsymbol{K}(x,y) &= n \int_1^{\infty} t^{n-1} \theta(y + t(x-y)) + \int_1^{\infty} t^n \frac{\partial \theta}{\partial t}(y + t(x-y)) \\ &= -\theta(x) \end{aligned}$$

and

$$\mathrm{div}\, \boldsymbol{u}(x) = -\theta(x) \int_\Omega f(y) dy + f(x) = f(x)$$

because by assumption the integral of f is equal to 0.

iii) We will prove now that

$$\boldsymbol{R}: \; L^2_0(\Omega) \mapsto \boldsymbol{H}^1_0(\Omega)$$

is continuous, that means that

$$\|\nabla \boldsymbol{R} f\|_{\boldsymbol{L}^2(\Omega)} \leq C \|f\|_{L^2(\Omega)}. \tag{23}$$

Thanks to the Newton's binôme formula, we have the following decomposition:

$$\frac{\partial K_i}{\partial x_j}(x,y) = K_{ij}(x, x-y) + G_{ij}(x,y),$$

where

$$\begin{aligned} K_{ij}(x, x-y) &= \frac{\delta_{ij}}{|x-y|^n} \int_0^\infty \theta(x + r\frac{x-y}{|x-y|}) r^{n-1} dr + \\ &+ \frac{x_i - y_i}{|x-y|^{n+1}} \int_0^\infty \frac{\partial \theta}{\partial x_j}(x + r\frac{x-y}{|x-y|}) r^n dr \\ : &= \frac{k_{ij}(x, x-y)}{|x-y|^n} \end{aligned}$$

and G_{ij} satisfies the following estimate:

$$|G_{ij}(x,y)| \leq C\delta(\Omega)^{n-1}/|x-y|^{n-1},$$

with $C = C(\theta, n)$.

We deduce from (20)-(22) that

$$\begin{aligned}
\frac{\partial (Rf)_i}{\partial x_j}(x) &= \int_\Omega K_{ij}(x, x-y) f(y) dy + \int_\Omega G_{ij}(x,y) f(y) dy + \\
&+ f(x) \int_\Omega \frac{(x_j - y_j)(x_i - y_i)}{|x-y|^2} \theta(y)\, dy \\
:&= J_1 f(x) + J_2 f(x) + J_3 f(x).
\end{aligned}$$

It is clear that $|J_3 f(x)| \leq |f(x)|$ and then

$$\|J_3 f\|_{L^2(\Omega)} \leq \|f\|_{L^2(\Omega)}.$$

We have also

$$|J_2 f(x)| \leq C\delta(\Omega)^{n-1} \Big| \int_{\mathbb{R}^n} \frac{\tilde{f}(y)}{|x-y|^{n-1}} dy \Big|.$$

That means that

$$|J_2 f(x)| \leq C\delta(\Omega)^{n-1} |I_1 \tilde{f}(x)|,$$

where $I_1 \tilde{f}$ is the Riesz potential of order 1. Hence

$$\|J_2 f\|_{L^2(\Omega)} \leq C\delta(\Omega)^{n-1} \|I_1 \tilde{f}\|_{L^2(\mathbb{R}^n)}.$$

Recall now that for any $\varphi \in \mathcal{D}(\mathbb{R}^n)$,

$$\|\nabla I_1 \varphi\|_{\boldsymbol{L}^2(\mathbb{R}^n)} \leq C\|\varphi\|_{L^2(\mathbb{R}^n)}$$

and by duality, for any $\varphi \in \mathcal{D}(\mathbb{R}^n)$ such that $\int_{\mathbb{R}^n} \varphi = 0$,

$$\|I_1 \varphi\|_{L^2(\mathbb{R}^n)} \leq C\|\varphi\|_{W_0^{-1,2}(\mathbb{R}^n)}$$

where $W_0^{-1,2}(\mathbb{R}^n)$ is the dual space of

$$W_0^{1,2}(\mathbb{R}^n) = \{v \in \mathcal{D}'(\mathbb{R}^n);\ \frac{v}{\omega} \in L^2(\mathbb{R}^n)\ \nabla v \in \boldsymbol{L}^2(\mathbb{R}^n)\}$$

with

$\omega = 1+|x| \quad \text{if} \quad n \geq 3 \quad \text{and} \quad \omega = (1+|x|)\ln(2+|x|) \quad \text{if} \quad n = 2.$

Applying the previous inequality to $\tilde{f}$, because Ω is bounded, we get

$$\|I_1 \tilde{f}\|_{L^2(\mathbb{R}^n)} \leq C\|\tilde{f}\|_{W_0^{-1,2}(\mathbb{R}^n)} \leq C\|f\|_{L^2(\Omega)}.$$

Finally, concerning the estimate of $J_1 f$, we need the following lemma due to Calderon-Zygmund.

Lemma (Calderon-Zygmund)

Let be $K(x,y) = N(x, x-y)$, where N is homogeneous of degre $-n$ in y and

i) for any x,

$$\int_{|y|=1} N(x,y)dy = 0,$$

ii) there exist $q > 0$ and $C > 0$ such that

$$\forall x, \quad \int_{|y|=1} |N(x,y)|^q dy \leq C,$$

then the operator defined by

$$Kf(x) = \int_{\mathbb{R}^n} K(x,y)f(y)dy$$

is continuous from $L^p(\mathbb{R}^n)$ into $L^p(\mathbb{R}^n)$, for any p such that $q/(q-1) \leq p < \infty$.

Applying this lemma with $N(x,y) = K_{ij}(x,y)$. It is clear that $K_{ij}(x,y)$ is homogeneous of degre $-n$ in y and

$$\begin{aligned}\int_{|z|=1} K_{ij}(x,z)dz &= \int_{|z|=1}\int_0^{\infty} \big(\delta_{ij}\theta(x+rz)r^{n-1}drd\sigma_z + \\ &+ z_i\frac{\partial\theta}{\partial x_j}(x+rz)r^n\big)drd\sigma_z \\ &= \int_{\mathbb{R}^n}\big(\delta_{ij}\theta(x+y) + y_i\frac{\partial\theta}{\partial x_j}(x+y)\big)dy = 0\end{aligned}$$

Finally, the property ii) is satisfied by using the fact that supp $\theta \subset B$ and Ω is bounded, that finishes the proof of the continuity property (23) and the first step of the proof of Theorem 1.

Second step. We know that every bounded Lipschitz domain Ω is the union of a finite number of bounded domains, each of which is starlike with respect to an open ball. Thanks to a partition of unity subordinated to this covering, we extend the result of the step 1 to the case where Ω is a bounded Lipschitz domain.

Third step. We have prove that

$$\forall f \in \mathcal{D}(\Omega) \cap L^2_0(\Omega), \quad \exists \boldsymbol{u} \in \boldsymbol{\mathcal{D}}(\Omega) \quad \text{such that} \quad \text{div } \boldsymbol{u} = f$$

with the inequality

$$\|\boldsymbol{u}\|_{\boldsymbol{H}^1(\Omega)} \leq C\|f\|_{L^2(\Omega)}.$$

Let us consider now $f \in L^2_0(\Omega)$ only. And let $\psi \in \mathcal{D}(\Omega)$ fixed with $\int_\Omega \psi = 1$ and

$$f_k \in \mathcal{D}(\Omega) \quad \text{such that} \quad f_k \to f \quad \text{in } L^2(\Omega).$$

Setting

$$F_k = f_k - \psi \int_\Omega f_k$$

then

$$F_k \in \mathcal{D}(\Omega) \cap L^2_0(\Omega), \quad F_k \to f \quad \text{in } L^2(\Omega)$$

and there exists $\boldsymbol{u}_k \in \mathcal{D}(\Omega)^n$ satisfying

$$\text{div } \boldsymbol{u}_k = F_k,$$

with

$$\boldsymbol{u}_k \to \boldsymbol{u} \in \boldsymbol{H}_0^1(\Omega) \quad \text{and} \quad \text{div } \boldsymbol{u} = f.$$

Moreover

$$\|\boldsymbol{u}\|_{\boldsymbol{H}^1(\Omega)} \leq C\|f\|_{L^2(\Omega)}.$$

We are now in position to give some extensions.

Theorem 9.

For any integer $m \geq 1$, the following operator

$$\mathrm{div} \quad : \boldsymbol{H}_0^{m+1}(\Omega)/V_{m+1} \mapsto H_0^m(\Omega) \cap L_0^2(\Omega) \tag{24}$$

is an isomorphism, where

$$V_{m+1} = \left\{ \boldsymbol{v} \in \boldsymbol{H}_0^{m+1}(\Omega);\, \mathrm{div}\, \boldsymbol{v} = 0 \right\}.$$

Proof. We give here a sketch of the proof and we consider only the case $m = 1$, the same raisoning being avalaible for $m \geq 2$. We will use here the same arguments as in the point iii) of the first theorem concerning the divergence operator defined on $\boldsymbol{H}_0^1(\Omega)$ and we would like to prove the following estimate:

$$\forall f \in H_0^1(\Omega) \cap L_0^2(\Omega), \quad \| \frac{\partial^2 \boldsymbol{R} f}{\partial x_k \partial x_j} \|_{\boldsymbol{L}^2(\Omega)} \leq C \| f \|_{H^1(\Omega)}$$

for any $1 \leq j, k \leq n$.
Rewriting the relation (19) under the form

$$\boldsymbol{R} f(x) = \int_{\mathbb{R}^n} \frac{z}{|z|^n} \tilde{f}(x - z) \int_0^\infty \theta(x + s\frac{z}{|z|})(|z| + s)^{n-1} ds\, dy,$$

we obtain

$$\begin{aligned} \frac{\partial \boldsymbol{R} f}{\partial x_j}(x) &= \int_{\mathbb{R}^n} \frac{z}{|z|^n} \frac{\partial \tilde{f}}{\partial x_j}(x - z) \int_0^\infty \theta(x + s\frac{z}{|z|})(|z| + s)^{n-1} ds\, dy \\ &+ \int_{\mathbb{R}^n} \frac{z}{|z|^n} \tilde{f}(x - z) \int_0^\infty \frac{\partial \theta}{\partial x_j}(x + s\frac{z}{|z|})(|z| + s)^{n-1} ds\, dy \\ &=: \boldsymbol{g}_1(x) + \boldsymbol{g}_2(x). \end{aligned}$$

Estimate of $\|\frac{\partial \boldsymbol{g}_1}{\partial x_k}\|_{L^2(\Omega)^n}$

As below, we prove that

$$\|\frac{\partial \boldsymbol{g}_1}{\partial x_k}\|_{\boldsymbol{L}^2(\Omega)} \leq C\|\frac{\partial f}{\partial x_j}\|_{L^2(\Omega)}.$$

Estimate of $\|\frac{\partial \boldsymbol{g}_2}{\partial x_k}\|_{L^2(\Omega)^n}$

We remark that $\boldsymbol{g}_2$ is the same form as $\boldsymbol{R}f$, with the difference that θ is replaced by $\frac{\partial \theta}{\partial x_j}$. Note that we does'nt use the property $\int_B \theta = 1$ to find the estimate of the point iv). That means that with the same raisonning, we obtain

$$\|\frac{\partial \boldsymbol{g}_2}{\partial x_k}\|_{\boldsymbol{L}^2(\Omega)} \leq C\|f\|_{L^2(\Omega)}.$$

Hence we have established the result for $f \in \mathcal{D}(\Omega) \cap L^2_0(\Omega)$ and proceeding as in the step 3 of the proof of the first theorem concerning the divergence operator defined on $\boldsymbol{H}^1_0(\Omega)$, we extend this one to the case where $f \in H^1_0(\Omega) \cap L^2_0(\Omega)$.

Corollary 10. (De Rham in $\boldsymbol{H}^{-m-1}(\Omega)$ First Version)

Let m a positve integer and $\boldsymbol{f} \in \boldsymbol{H}^{-m-1}(\Omega)$ satisfying

$$\forall \boldsymbol{v} \in V_{m+1}, \quad < \boldsymbol{f} \, , \boldsymbol{v} >= 0.$$

Then there exists $\pi \in H^{-m}(\Omega)$, unique up to an additive constant, such that $f = \nabla \pi$.

Using then the densité of $\mathcal{V}(\Omega)$ in V_{m+1}, we can prove the following theorem:

Theorem 11. (De Rham in $\boldsymbol{H}^{-m-1}(\Omega)$, Second Version)

Let m a positve integer and $\boldsymbol{f} \in \boldsymbol{H}^{-m-1}(\Omega)$ satisfying the following property:

$$\forall \boldsymbol{v} \in \mathcal{V}(\Omega), \quad < \boldsymbol{f}\,, \boldsymbol{v} > = 0.$$

Then there exists $\pi \in H^{-m}(\Omega)$, unique up an additive constant, such that

$$\boldsymbol{f} = \nabla \pi.$$

As application, we can give a new proof of De Rham's theorem.

Theorem 12 (Original De Rham).

Let $\boldsymbol{f} \in \boldsymbol{\mathcal{D}}'(\Omega)$ satisfying the following property:

$$\forall \boldsymbol{v} \in \mathcal{V}(\Omega), \quad < \boldsymbol{f}\,, \boldsymbol{v} > = 0.$$

Then there exists $\pi \in \mathcal{D}'(\Omega)$, unique up an additive constant, such that

$$\boldsymbol{f} = \nabla \pi.$$

Proof. It is an immediate consequence of the fact that we have prove that the divergence operator

$$\text{div} \ : \ \boldsymbol{\mathcal{D}}(\Omega)/\mathcal{V}(\Omega) \mapsto \mathcal{D}(\Omega) \cap L^2_0(\Omega).$$

is bijective.

Theorem 13 (General Lions Lemma)

For any integer m and any $1 < p < \infty$,

$$f \in \mathcal{D}'(\Omega) \quad \text{and} \quad \nabla f \in \boldsymbol{W}^{-m-1,p}(\Omega) \Longrightarrow f \in W^{-m-1,p}(\Omega).$$

- G. Geymonat – P. Suquet (1986): $m = 0$
- W. Borchers – H. Sohr (1990): $m \geq 0$
- C. Amrouche – V. Girault (1994): for any integer m

IV. Second Equivalence Theorem: Korn, Lions, Poincaré

Recall first the **original Poincaré Lemma**:

Let Ω a bounded open **simply connected** of $\mathbb{R}^3$ and let

$$\boldsymbol{f} \in \mathcal{C}^1(\Omega) \quad \text{such that} \quad \mathbf{curl}\ \boldsymbol{f} = \mathbf{0}.$$

Then

$$\boldsymbol{f} = \mathbf{grad}\ \chi \quad \text{with} \quad \chi \in \mathcal{C}^2(\Omega).$$

Theorem 14 (Equivalence Lions-Poincaré).

i) Suppose that Ω is simply-connected. Then the general J.L. Lions lemma implies that the following **Weak version of Poincaré's lemma** holds: Assume that

$$\boldsymbol{f} \in \boldsymbol{H}^{-1}(\Omega) \quad \text{with} \quad \mathbf{curl}\, \boldsymbol{f} = \mathbf{0} \text{ in } \boldsymbol{H}^{-2}(\Omega). \tag{25}$$

Then there exists a scalar potential χ in $L^2(\Omega)$, uniquely determined up to the addition of a constant, such that

$$\boldsymbol{f} = \mathbf{grad}\, \chi \quad \text{in } \Omega \quad \text{and} \quad \|\chi\|_{L^2(\Omega)} \leq C \|\boldsymbol{f}\|_{\boldsymbol{H}^{-1}(\Omega)}. \tag{26}$$

ii) Conversely, the weak Poincaré Lemma on any simply-connected domain in $\mathbb{R}^N$ implies that J.L. Lions Lemma holds on any domain in $\mathbb{R}^N$.

Proof. i) To prove that the general J.L. Lions lemma implies the weak version of Poincaré's lemma, let

$$\boldsymbol{f} \in \boldsymbol{H}^{-1}(\Omega) \quad \text{be such that} \quad \mathbf{curl}\ \boldsymbol{f} = \mathbf{0} \text{ in } \Omega.$$

We know that there exists a unique

$$(\boldsymbol{u}, \pi) \in \boldsymbol{H}^1_0(\Omega) \times L^2_0(\Omega)$$

such that

$$-\Delta \boldsymbol{u} + \nabla p = \boldsymbol{f} \quad \text{and} \quad \text{div}\ \boldsymbol{u} = 0 \quad \text{in } \Omega. \tag{27}$$

Hence

$$\Delta(\mathbf{curl}\ \boldsymbol{u}) = \mathbf{0} \quad \text{in } \Omega$$

so that the hypoellipticity of the polyharmonic operator Δ implies that

$$\mathbf{curl}\ \boldsymbol{u} \in \mathcal{C}^\infty(\Omega).$$

Since div $\boldsymbol{u} = 0$, we deduce that

$$\Delta \boldsymbol{u} = -\mathbf{curl\ curl}\ \boldsymbol{u} \in \mathcal{C}^\infty(\Omega).$$

Now $\Delta \boldsymbol{u}$ is a smooth irrotational vector field and by the classical Poincaré theorem, there exists $q \in \mathcal{C}^\infty(\Omega)$ such that

$$\nabla q = \Delta \boldsymbol{u} = \nabla p - \boldsymbol{f} \quad \text{in } \boldsymbol{H}^{-1}(\Omega).$$

The distribution $\tilde{p}$ defied by $\tilde{p} = p - q$ satisfies

$$\nabla \tilde{p} = \boldsymbol{f} \in \boldsymbol{H}^{-1}(\Omega).$$

Consequently, $\tilde{p} \in L^2(\Omega)$ by the general J.L. Lions Lemma.

ii) The converse is immediate. Indeed, suppose that the weak version of Poincaré's Lemma holds and let

$$f \in (\mathcal{D}(\Omega))' \quad \text{such that } \nabla f \in \boldsymbol{H}^{-1}(\Omega).$$

Because $\mathbf{curl}\ (\nabla \boldsymbol{f}) = \mathbf{0}$ in Ω, there exists $\chi \in L^2(\Omega)$ satisfying

$$\nabla \boldsymbol{f} = \nabla \chi.$$

Then $f = \chi + C$ and $f \in L^2(\Omega)$.

V. The curl operator

Let us introduce the following space:

$$\mathcal{G} = \{\boldsymbol{v} \in \boldsymbol{\mathcal{D}}(\Omega);\ \mathbf{curl}\,\boldsymbol{v} = \mathbf{0}\},$$

where we suppose here that Ω is a Lipschitzian bounded domain of $\mathbb{R}^3$.

Theorem 11.

Let $\boldsymbol{f} \in \boldsymbol{\mathcal{D}}'(\Omega)$ satisfying the following property:

$$\forall \boldsymbol{v} \in \mathcal{G}, \quad < \boldsymbol{f}\,, \boldsymbol{v} > = 0. \tag{28}$$

Then there exists $\boldsymbol{\psi} \in \boldsymbol{\mathcal{D}}'(\Omega)$ with div $\boldsymbol{\psi} = 0$ and such that

$$\boldsymbol{f} = \mathbf{curl}\,\boldsymbol{\psi}.$$

Proof. Note that the condition (28) implies that

$$\operatorname{div} \boldsymbol{f} = 0,$$

which is a necessary condition for that

$$\boldsymbol{f} = \mathbf{curl}\, \boldsymbol{\psi}.$$

To prove the result is equivalent to prove that the following operator

$$\mathbf{curl} \ : \ \boldsymbol{\mathcal{D}}(\Omega)/\mathcal{G} \mapsto \mathcal{V}(\Omega) \perp \boldsymbol{K}_\tau(\Omega)$$

is bijective, where

$$\boldsymbol{K}_\tau(\Omega) = \{\boldsymbol{v} \in \boldsymbol{L}^2(\Omega);\ \operatorname{div} \boldsymbol{v} = 0,\ \mathbf{curl}\, \boldsymbol{v} = \mathbf{0} \ \text{ in } \Omega \quad \text{and} \quad \boldsymbol{v} \cdot \boldsymbol{n} = 0 \text{ on } \Gamma\}.$$

Recall that

$$\boldsymbol{K}_\tau(\Omega) = \{\mathbf{0}\} \quad \text{when } \ \Omega \text{ is simply connected.}$$

We will use here the same ideas as in the proof of Theorem 2.

Construction of the operator T.

Let $\boldsymbol{f} \in \mathcal{V}(\Omega)$ satisfying the following property:

$$\forall \boldsymbol{v} \in \boldsymbol{K}_\tau(\Omega), \quad \int_\Omega \boldsymbol{f} \cdot \boldsymbol{v} = 0.$$

Instead of the operator $\boldsymbol{R}$, we set

$$\boldsymbol{T}\boldsymbol{f}(x) := \int_\Omega \boldsymbol{f}(y) \times \frac{x-y}{|x-y|^3} \int_{|x-y|}^{\infty} \theta(y + r\frac{x-y}{|x-y|})r^{n-1} dr\, dy, \tag{29}$$

where Ω is starlike with respect to some open ball B contained it in and supp $\theta \subset B$. Then, we verify that

$$\boldsymbol{f} = \textbf{curl}\, \boldsymbol{T}\boldsymbol{f} \quad \text{and} \quad \boldsymbol{f} = \textbf{curl}\, \boldsymbol{T}\boldsymbol{f} \in \boldsymbol{\mathcal{D}}(\Omega).$$

And we finish the proof for general Lipschitzian bounded domain as in Theorem 2.

For Further Reading

C. Amrouche, P. G. Ciarlet, C. Mardare, On a lemma of Jacques-Louis Lions and its relation to other fundamental results, *Journal de Mathématiques Pures et Appliquées*, **104** (2015), 207– 226

C. Amrouche, V. Girault, Decomposition of vector spaces and application to the Stokes problem in arbitrary dimension, *Czech. Math. Jour.* **44**, (1994), 109 –140.

M.E. Bogovskii, Solution of the first boundary value problem for an equation of continuity of an incompressible medium. (Russian) *Dokl. Akad. Nauk SSSR* **248-5**, (1979), 1037–1040. (Translation) *Soviet Math. Dokl.* **20-5**, (1979), 1094–1098.

G. Duvaut, J.L. Lions, Les Inéquations en Mécanique et en Physique, Dunod, Paris, (1972).

P.G. Galdi, An introduction to the mathematical theory of the Navier-Stokes equations. Vol. I. Linearized steady problems. Springer Tracts in Natural Philosophy, 38. Springer-Verlag, New York, (1994).

J. Necas Equations aux Dérivées Partielles, *Presses de l'Université de Montréal*, Montréal, 1965.

Thank you for your attention!