The rotating Navier-Stokes-Fourier-Poisson system on thin domains

Bernard Ducomet[1], Matteo Caggio[2], Šárka Nečasová[2], and Milan Pokorný[3]

1) CEA, DAM, DIF, F-91297 Arpajon, France

2) Institute of Mathematics of the Academy of Sciences of the Czech Republic

3) Charles University in Prague, Faculty of Mathematics and Physics

1 Introduction

We consider the compressible Navier–Stokes–Fourier–Poisson system describing the motion of a viscous heat conducting rotating fluid confined to a straight layer $\Omega_\epsilon = \omega \times (0,\epsilon)$, where ω is a 2-D domain. We show that the weak solutions in the 3D domain converge to the strong solution of the 2-D Navier–Stokes–Fourier–Poisson system on ω as $\epsilon \to 0$ on the time interval, where the strong solution exists. We consider two different regimes, self-gravitation and external gravitation effect, in dependence on the asymptotic behaviour of the Froude number.

2 Primitive system

The motion of a viscous heat conducting rotating fluid is describe by the following system of equations written in its dimensionless form, with the Froude number $[\mathrm{Fr}^2] = \epsilon^\beta$ and all the other fluid dynamics numbers equal to one:

$$\partial_t \varrho + \mathrm{div}_\epsilon (\varrho \mathbf{u}) = 0, \tag{2.1}$$

$$\partial_t (\varrho \mathbf{u}) + \mathrm{div}_\epsilon (\varrho \mathbf{u} \otimes \mathbf{u}) + \varrho \mathbf{u} \times \omega + \nabla_\epsilon p(\varrho,\vartheta) = \mathrm{div}_\epsilon S(\vartheta, \nabla_x \mathbf{u}) + \epsilon^{-\beta} \varrho \nabla_\epsilon \phi + \varrho \nabla_\epsilon |x \times \omega|^2, \tag{2.2}$$

$$\partial_t (\varrho s(\varrho,\vartheta)) + \mathrm{div}_\epsilon (\varrho s(\varrho,\vartheta)\mathbf{u}) + \mathrm{div}_\epsilon \left(\frac{\mathbf{q}(\vartheta, \nabla_\epsilon \vartheta)}{\vartheta} \right) = \frac{1}{\vartheta} \left(S(\vartheta, \nabla_\epsilon \mathbf{u}) : \nabla_\epsilon \mathbf{u} - \frac{\mathbf{q} \cdot \nabla_\epsilon \vartheta}{\vartheta} \right), \tag{2.3}$$

where,

$$\nabla_\epsilon = \left(\nabla_h, \frac{1}{\epsilon} \partial_{x_3} \right), \quad \mathrm{div}_\epsilon = \left(\mathrm{div}_h, \frac{1}{\epsilon} \partial_{x_3} \right).$$

Above, ϱ and $\mathbf{u}$ are the density and the velocity of the fluid, functions of time and spatial coordinates (t,x); p and s the pressure and the specific entropy, functions of the density and temperature ϑ; Coriolis force $\varrho \mathbf{u} \times \omega$ and centrifugal force $\varrho \nabla_\epsilon |x \times \omega|^2$ are taken into account, where ω is the angular velocity; $\mathbf{q}$ is the heat flux expressed by the Fourier's law

$$\mathbf{q} = -\kappa(\vartheta) \nabla_x \vartheta, \tag{2.4}$$

with $\kappa(\vartheta)$ diffusivity coefficient function of the temperature; and the stress tensor S is given by the following relation:

$$S(\vartheta, \nabla_x \mathbf{u}) = \mu(\vartheta) \left(\nabla_x \mathbf{u} + \nabla_x^t \mathbf{u} - \frac{2}{3} \mathrm{div}_x \mathbf{u} \mathbf{I} \right) + \eta(\vartheta) \mathrm{div}_x \mathbf{u} \mathbf{I}, \tag{2.5}$$

where the shear viscosity coefficients $\mu(\vartheta)$ and $\eta(\vartheta)$ are functions of the temperature. The potential ϕ obeys the Poisson's equation

$$-\triangle \phi = 4\pi G(\eta \varrho + (1-\eta) g), \tag{2.6}$$

where G is the Newton constant, η a positive prameter, g a given function modelling the external gravitational effect, and

$$\nabla_\epsilon \phi(t,x) = \epsilon G \int_\Omega \eta \varrho(t,\xi) \frac{(x_1 - \xi_1, x_2 - \xi_2, \epsilon(x_3 - \xi_3))}{\left(|x_h - \xi_h|^2 + \epsilon^2 |x_3 - \xi_3|^2 \right)^{3/2}} d\xi$$

$$+ G \int_{\mathbb{R}^3} (1-\eta) g(y) \frac{(x_1 - y_1, x_2 - y_2, \epsilon(x_3 - y_3))}{\left(|x_h - y_h|^2 + \epsilon^2 |x_3 - y_3|^2 \right)^{3/2}} dy := \mathbf{E}. \tag{2.7}$$

The boundary conditions are given by the following relations:

$$\mathbf{u}|_{\partial\omega \times (0,\epsilon)} = 0, \ \mathbf{u} \cdot \mathbf{n}|_{\omega \times \{0,\epsilon\}} = 0, \ [S \cdot \mathbf{n}] \times \mathbf{n}|_{\omega \times \{0,\epsilon\}} = 0, \ \nabla \vartheta \cdot \mathbf{n}|_{\omega \times \{0,\epsilon\}} = 0 \tag{2.8}$$

3 Target system

We perform the singular limit $\epsilon \to 0$ for $\beta = \eta = 1$ (self-gravitation) and $\beta = \eta = 0$ (external gravitational effect) showing the following convergence

$$[\varrho, \mathbf{u}] \to [r, \mathbf{V}],$$

where $\mathbf{V} = [\mathbf{w}, 0]$ and the pair $[r(t,x_h), \mathbf{w}(t,x_h), \Theta(t,x_h)]$ solves the 2D rotating Navier-Stokes-Fourier-Poisson system on $(0,T) \times \omega$:

$$\partial_t r + \mathrm{div}_h (r\mathbf{w}) = 0, \tag{3.1}$$

$$r \partial_t \mathbf{w} + r\mathbf{w} \cdot \nabla_h \mathbf{w} + \nabla_h p(r,\Theta) + r(\mathbf{w} \times \omega) = \mathrm{div}_h S(\Theta, \nabla_h \mathbf{w}) + r \nabla_h \phi_h + r \nabla_h |x \times \omega|^2, \tag{3.2}$$

$$r \partial_t s + r\mathbf{w} \cdot \nabla_h s + \mathrm{div}_h \left(\frac{\mathbf{q}_h(\Theta, \nabla_h \Theta)}{\Theta} \right) = \frac{1}{\Theta} \left(S_h(\Theta, \nabla_h \mathbf{w}) : \nabla_h \mathbf{w} - \frac{\mathbf{q}_h(\Theta, \nabla_h \Theta) \cdot \nabla_h \Theta}{\Theta} \right). \tag{3.3}$$

Above,

$$\phi_h(t,x_h) = G \int_\omega \frac{r(t,y_h)}{|x_h - y_h|} dy_h \quad \text{for} \quad \beta = \eta = 1, \tag{3.4}$$

$$\phi_h(t,x_h) = G \int_{\mathbb{R}^3} \frac{g(y)}{\sqrt{|x_h - y_h|^2 + y_3^2}} dy \quad \text{for} \quad \beta = \eta = 0. \tag{3.5}$$

4 Convergence

4.1 Weak solutions of the primitive system

Definition 4.1. *We say that $(\varrho, \mathbf{u}, \vartheta)$ is a weak solution of problem (2.1–2.3) if*

$$\varrho \geq 0, \ \vartheta > 0 \ \textit{for a.a.} \ (t,x) \in (0,T) \times \Omega,$$

$$\varrho \in C_{weak}([0,T]; L^\gamma(\Omega)), \ \varrho \mathbf{u} \in C_{weak}([0,T]; L^{\frac{2\gamma}{\gamma+1}}(\Omega;\mathbb{R}^3)),$$
$$\mathbf{u} \in L^2(0,T; W^{1,2}_{0,n}(\Omega;\mathbb{R}^3)),$$
$$\vartheta \in L^\infty(0,T; L^4(\Omega)) \cap L^2(0,T; W^{1,2}(\Omega));$$

ϱ, $\mathbf{u}$, ϑ satisfy (2.1 - 2.3) in the sense of distributions; the following total energy balance holds

$$\int_\Omega \left(\frac{1}{2} \varrho |\mathbf{u}|^2 + \varrho e(\varrho,\vartheta) \right)(\tau, \cdot)\, dx = \int_\Omega \left(\frac{1}{2\varrho_0} |\varrho_0 \mathbf{u}_0|^2 + \varrho_0 e_0(\varrho_0, \vartheta_0) \right) dx$$
$$+ \int_0^\tau \int_\Omega \left(\epsilon^{-\beta} \varrho \mathbf{E} \cdot \mathbf{u} + \varrho \nabla_x |x \times \omega|^2 \cdot \mathbf{u} \right) \tag{4.1}$$

where e is the internal energy, function of the density and temperature; and the integral representation of the gravitational force is given by (2.7).

Proposition 4.1. *Let E_0 and S_0 be positive constant. Suppose the thermodynamic functions p, s and e and the transport coefficients μ, η and κ satisfy some proper constitutive relations (for details see [1]). Suppose the initial data satisfy*

$$\int_\Omega \left(\frac{1}{2} \varrho |\mathbf{u}|^2 + \varrho e(\varrho,\vartheta) \right)(0, \cdot)\, dx := \int_\Omega \left(\frac{1}{2\varrho_0} |\varrho_0 \mathbf{u}_0|^2 + \varrho_0 e_0(\varrho_0, \vartheta_0) \right) dx \leq E_0, \tag{4.2}$$

$$\int_\Omega \varrho s(\varrho,\vartheta)(0,\cdot)\, dx := \int_\Omega \varrho_0 s(\varrho_0, \vartheta_0)\, dx \leq S_0, \tag{4.3}$$

Then problem (2.1–2.3) with boundary conditions (2.8) admits at least one weak solution in the sense of Definition 4.1.

4.2 Strong solution of the target system

Proposition 4.2. *Suppose that the functions $p \in C^2((0,\infty)^2)$, μ, η, $\kappa \in C^1(0,\infty)$, the initial conditions $(r_0, \mathbf{V}_0, \Theta_0)$ satisfy some proper constitutive relations (for details see [1]) and*

$$r_0 \in W^{2,2}(\omega), \inf_\omega r_0 > 0, \mathbf{V}_0 \in W^{3,2}(\omega;\mathbb{R}^2) \cap W^{1,2}_0(\omega;\mathbb{R}^2), \Theta_0 \in W^{3,2}(\omega), \inf_\omega \Theta_0 > 0. \tag{4.4}$$

Then, there exists a positive constant T_* such that $(r, \mathbf{V}, \Theta)$ is the unique classical solution to problem (3.1–3.3) with the boundary conditions

$$\mathbf{V}|_{\partial\omega} = \vec{0}, \quad \frac{\partial \Theta}{\partial \vec{n}}|_{\partial\omega} = 0, \tag{4.5}$$

for any $T < T_*$ such that,

$$r \in C([0,T]; W^{3,2}(\omega)) \cap C^1([0,T]; W^{2,2}(\omega)), \tag{4.6}$$
$$\mathbf{V} \in C([0,T]; W^{3,2}(\omega,\mathbb{R}^2)) \cap C^1([0,T]; W^{1,2}(\omega,\mathbb{R}^2)), \tag{4.7}$$
$$\Theta \in C([0,T]; W^{3,2}(\omega)) \cap C^1([0,T]; W^{1,2}(\omega)). \tag{4.8}$$

4.3 Convergence result

We introduce the relative entropy functional

$$\mathcal{I}(\varrho, \mathbf{u}, \vartheta \mid r, \mathbf{V}, \Theta) = \int_\Omega \left(\frac{1}{2} \varrho |\mathbf{u} - \mathbf{V}|^2 + \mathcal{E}(\varrho, \vartheta \mid r, \Theta) \right)(\tau, \cdot) dx, \tag{4.9}$$

where for

$$H_\Theta(\varrho,\vartheta) = \varrho e(\varrho,\vartheta) - \Theta \varrho s(\varrho,\vartheta)$$

we define

$$\mathcal{E}(\varrho, \vartheta \mid r, \Theta) = H_\Theta(\varrho,\vartheta) - \partial_\varrho H_\Theta(r,\Theta)(\varrho - r) - H_\Theta(r,\Theta).$$

Our main result reads

Theorem 4.1. *Suppose the thermodynamic functions p, s and e and the transport coefficients μ and η satisfy some proper constitutive relations (for details see [1]). Let $(r_0, \mathbf{V}_0, \Theta_0)$ satisfy assumptions of Proposition 4.2 and let $T_* > 0$ be the time interval of existence of the strong solution to problem (3.1–3.3). Let $(\varrho, \mathbf{u}, \vartheta)$ be a sequence of weak solutions to the 3-D compressible Navier–Stokes–Fourier–Poisson system (2.1–2.3) emanating from the initial data $(\varrho_0, \mathbf{u}_0, \vartheta_0)$.*

Suppose that

$$\mathcal{I}(\varrho_0, \mathbf{u}_0, \vartheta_0 \mid r_0, \mathbf{V}_0, \Theta_0) \to 0. \tag{4.10}$$

Then

$$\mathcal{I}(\varrho, \mathbf{u}, \vartheta \mid r, \mathbf{V}, \Theta) \to 0, \tag{4.11}$$

where the triple $(r, \mathbf{V}, \Theta)$ satisfies the 2-D compressible Navier–Stokes–Fourier–Poisson system (3.1–3.3) with the boundary conditions (4.5) on the time interval $[0,T]$ for any $0 < T < T_*$.

References

[1] Bernard Ducomet, Matteo Caggio, Šárka Nečasová, and Milan Pokorný *The rotating Navier-Stokes-Fourier-Poisson system on thin domains*, arXiv:1606.01054v1.