Notes on the Stokes flows in the half space with a rapid decay rate in space and time

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Stokes system

- Stokes system

$$\begin{aligned} & u_t - \Delta u + \nabla p = 0, \quad \text{in } \mathbb{R}^n_+ \times (0,\infty), \\ & \operatorname{div} u = 0, \quad \text{in } \mathbb{R}^n_+ \times (0,\infty), \\ & u|_{t=0} = h, \qquad u|_{x_n=0} = 0. \end{aligned} \tag{1}$$

- $u = e^{-tA}h$

- Our goal: Finding the conditions of h satisfying following inequality

$$|e^{-tA}h(x)| \leq c(1 + |x| + \sqrt{t})^{-\alpha})$$

- In particular, we are interesting in rapid space and time decay rate of solution ($\alpha > n$).

Representation of Solution

- V. A. Solonnikov(1967)

$$(e^{-tA}h)_i(x) = \sum_{j=1}^{n} \int_{\mathbb{R}^n_+} G_{ij}(x,y,t)h_j(y)dy, \; i = 1, \cdots, n,$$

$$\begin{aligned} G_{ij}(x,y,t) &= \delta_{ij}\Big(\Gamma(x-y,t) - \Gamma(x-y^*,t)\Big) \\ &\quad + 4(1-\delta_{jn})D_{x_j}\int_0^{x_n}\int_{\mathbb{R}^{n-1}} D_{x_i}E(x-z)\Gamma(z-y^*,t)dz \\ &:= \delta_{ij}\Big(\Gamma(x-y,t) - \Gamma(x-y^*,t)\Big) + G^*_{ij}(x,y,t), \end{aligned}$$

Γ: Fundamental solution of heat equation,
E: Fundamental solution of Laplace equation
$y^* = (y_1, \cdot, y_{n-1}, -y_n)$,

- Ukai's formula: Useful to estimate related L^p- inequality $(1 < p < \infty)$ and time decay.
- Solonnikov's formula : Useful to estimate related L^∞ inequality and space decay.

Estimate of Kernel

- V. A. Solonnikov(1967)

$$|G^*_{ij}(x,y,t)|,\ |G_{ij}(x,y,t)| \le c(|x-y^*|^2+t)^{-\frac{n}{2}} e^{-\frac{cy_n^2}{t}}$$

Slow decay rate

- P. Maremonti(2004)

$$|e^{-tA}h(x,t)| \leq c(1+t^{\frac{1}{2}}+|x|)^{-\alpha}||(1+|x|)^{\alpha}h\|_{L^\infty(\mathbb{R}^n_+)}$$

$\alpha \in (0, n)$

$$|e^{-tA}h(x)| \leq c(1+\sqrt{t}+|x|)^{-n}\ln(2+\sqrt{t})\|(1+|x|)^{n}h\|_{L^\infty(\mathbb{R}^n_+)}.$$

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$\alpha \in (0, n)$

$$|e^{-tA}h(x)| \le c(1+\sqrt{t}+|x|)^{-n}\ln(2+\sqrt{t})\|(1+|x|)^{n}h\|_{L^\infty(\mathbb{R}^n_+)}.$$

How can obtain about rapid decay rate of solution? That is,

$$\alpha > n \quad ???$$

- We can not use the solonnikov's estimate for rapid decay of solution.
  $\Rightarrow$
  1) We need the understanding about kernel.
  2) We need the condition of initial data h except decay rate.

Theorem (1)

Let $\alpha > n$. Let $\operatorname{div} h = 0$ in $\mathbb{R}^n_+$ and $h_n|_{x_n=0} = 0$ with $(1+|x|)^\alpha h$, $(1+|x|)^\alpha R_j' h_j \in L^\infty(\mathbb{R}^n_+)$, $j = 1, \cdots, n-1$. Then,

$$|e^{-tA}h(x)| \le \begin{cases} (1+\sqrt{t}+|x|)^{-\alpha} \\ \times\Big(\|(1+|x|)^\alpha h\|_{L^\infty(\mathbb{R}^n_+)} + \|(1+|x|)^\alpha R_j' h_j\|_{L^\infty(\mathbb{R}^n_+)}\Big), \\ \text{if } n < \alpha < n+1, \\ (1+\sqrt{t}+|x|)^{-n-1} \ln(2+\sqrt{t}) \\ \times\Big(\|(1+|x|)^{n+1} h\|_{L^\infty(\mathbb{R}^n_+)} + \|(1+|x|)^{n+1} R_j' h_j\|_{L^\infty(\mathbb{R}^n_+)}\Big) \\ (1+\sqrt{t}+|x|)^{-n-1} \\ \times\Big(\|(1+|x|)^\alpha h\|_{L^\infty(\mathbb{R}^n_+)} + \|(1+|x|)^\alpha R_j' h_j\|_{L^\infty(\mathbb{R}^n_+)}\Big), \\ \text{if } \alpha > n+1. \end{cases}$$

Here $R' = (R_1, \cdots, R_{n-1})$ denotes $n-1$ dimensional Riesz

Our Result (2)

Theorem (2)

Let $\alpha > n$. Let $\mathrm{div}\, h = 0$ in $\mathbb{R}^n_+$ and $h_n|_{x_n=0} = 0$ with $(1+|x|)^\alpha h,\ (1+|x|)^\alpha R'_j h_j \in L^\infty(\mathbb{R}^n_+)$, $j = 1, \cdots, n-1$. Then,

$$
|\nabla_x e^{-tA} h(x)| \leq \begin{cases}
t^{-\frac{1}{2}}(1+\sqrt{t}+|x|)^{-\alpha} \\
\times\Big(\|(1+|x|)^\alpha h\|_{L^\infty(\mathbb{R}^n_+)} + \|(1+|x|)^\alpha R'_j h_j\|_{L^\infty(\mathbb{R}^n_+)} \Big), \\
\text{if } n < \alpha < n+1, \\
t^{-\frac{1}{2}}(1+\sqrt{t}+|x|)^{-n-1} \ln(2+\sqrt{t}) \\
\times\Big(\|(1+|x|)^{n+1} h\|_{L^\infty(\mathbb{R}^n_+)} + \|(1+|x|)^{n+1} R'_j h_j\|_{L^\infty(\mathbb{R}^n_+)} \\
t^{-\frac{1}{2}}(1+\sqrt{t}+|x|)^{-n-1} \\
\times\Big(\|(1+|x|)^\alpha h\|_{L^\infty(\mathbb{R}^n_+)} + \|(1+|x|)^\alpha R'_j h_j\|_{L^\infty(\mathbb{R}^n_+)} \Big), \\
\text{if } \alpha > n+1.
\end{cases}
$$

Examples

Let $n < \alpha < n+1$ and $(1+|x|)^\alpha h \in L^\infty(\mathbb{R}^n_+)$.

- 1) $(1+x_n)^{1+\alpha}\nabla' h \in L^\infty(\mathbb{R}^n_+)$,
  2) $\int_{\mathbb{R}^{n-1}} h(y', y_n) dy' = 0$ for all $y_n > 0$,
  3) $\int_{\mathbb{R}^{n-1}} y' h(y', y_n) dy' = 0$ for all $y_n > 0$
  4) $h_j(x', x_n)$ is odd in x_j and is even in x_k, for $j, k = 1, , n-1$ with $j \neq k$ (tangential parity conditions, Y. Fujigaki and T. Miyakawa(2002))

$$1) + 2) + 3) \Rightarrow (1+|x|)^\alpha R'_j h_j \in L^\infty(\mathbb{R}^n_+)$$

$$1) + 4) \Rightarrow (1+|x|)^\alpha R'_j h_j \in L^\infty(\mathbb{R}^n_+)$$

Lemma

$$[e^{-tA}h]_i =$$

$$e^{-tB}h_i - 4\delta_{in}\sum_{j=1}^{n-1} e^{-tC}R_j' h_j + 2\sum_{j=1}^{n-1} e^{-tD_{ij}}h_j - 4\sum_{j=1}^{n-1} e^{-tD_{in}}R_j' h_j,$$

$$e^{-tB}f(x) = \int_{\mathbb{R}^n_+} \Big(\Gamma(x-y,t) - \Gamma(x-y^*,t)\Big) f(y)dy,$$

$$e^{-tC}f(x) = \int_{\mathbb{R}^n_+} \Gamma(x-y^*,t)h(y)dy,$$

$$e^{-tD_{ik}}f(x) = \int_{\mathbb{R}^n_+} K_{ik}(x,y,t)f(y)dy,$$

$$K_{ik}(x,z,t) = D_{x_i}\int_{\mathbb{R}^n_+} \Big(E(x-y) - E(x-y^*)\Big) D_{y_k}\Gamma(y-z^*,t)dy.$$

$$n < \alpha,\ (1+|x|)^\alpha h \in L^\infty(\mathbb{R}^n_+)(h \in L^1(\mathbb{R}^n_+)),\ div\, h = 0,\ h_n|_{x_n=0} = 0$$

$$\Rightarrow \int_{\mathbb{R}^n_+} h(y)dy = 0$$

$$\Rightarrow e^{-tA}h(x,t) = \int_{\mathbb{R}^n_+} \big(G(x,y,t) - G^*(x,0,t)\big) h(y)dy$$

$$
\begin{aligned}
|e^{-tA}h(x,t)| &\leq \int_{\mathbb{R}^n_+} |G(x,y,t) - G^*(x,0,t)||h(y)|dy \\
&\leq \|(1+|x|)^\alpha h_j\|_{L^\infty(\mathbb{R}^n_+)} \int_{\mathbb{R}^n_+} \Big|\Gamma(x-y,t) - \Gamma(x-y^*,t)\Big|(1+|y|)^{-\alpha}dy \\
&+\|(1+|x|)^\alpha R_j' h_j\|_{L^\infty(\mathbb{R}^n_+)} \int_{\mathbb{R}^n_+} \Big|\Gamma(x-y^*,t) - \Gamma(x,t)\Big|(1+|y|)^{-\alpha}dy \\
&+\Big(\|(1+|x|)^\alpha h\|_{L^\infty(\mathbb{R}^n_+)} + \|(1+|x|)^\alpha R_j' h_j\|_{L^\infty(\mathbb{R}^n_+)}\Big) \\
&\times \int_{\mathbb{R}^n_+} |K_{ik}(x,z,t) - K_{ik}(x,0,t)|(1+|z|)^{-\alpha}dz
\end{aligned}
$$

Lemma

Let $\alpha \geq 0$.

$$\int_{\mathbb{R}^n_+} \Big|\Gamma(x-y,t) - \Gamma(x-y^*,t)\Big|(1+|y|)^{-\alpha} dy$$

$$\leq \begin{cases} c(1+|x|+\sqrt{t})^{-\alpha} \text{ if } \alpha < n+1, \\ c(1+|x|+\sqrt{t})^{-n-1} \ln(2+\sqrt{t}) \text{ if } \alpha = n+1, \\ c(1+|x|+\sqrt{t})^{-n-1} \text{ if } \alpha > n+1. \end{cases}$$

Lemma

Let $\alpha > n$. *Then*

$$\int_{\mathbb{R}^n_+} \Big|\Gamma(x-y^*,t)-\Gamma(x,t)\Big|(1+|y|)^{-\alpha}dy$$

$$\leq \begin{cases} c(1+|x|+\sqrt{t})^{-\alpha}e^{-\frac{x_n^2}{8t}} \text{ if } n<\alpha<n+1, \\ c(1+|x|+\sqrt{t})^{-n-1}e^{-\frac{x_n^2}{8t}}\ln(2+\sqrt{t}) \text{ if } \alpha=n+1, \\ c(1+|x|+\sqrt{t})^{-n-1}e^{-\frac{x_n^2}{8t}} \text{ if } \alpha>n+1. \end{cases}$$

Lemma

Let $m \geq 0$ *and* $l =, 1 \cdots, n$. *Then*

$$\left| D_{z_l}^m \int_{\mathbb{R}^n_+} D_{x_i} \Big(E(x-y) - E(x-y^*) \Big) D_{y_k} \Gamma(y - z^*, t) dy \right|$$
$$\leq c t^{-\frac{m}{2} + \frac{1}{2}} (\sqrt{t} + |x - z^*|)^{-n-1} e^{-\frac{c z_n^2}{t}}.$$

$$\int_{\mathbb{R}^n_+} |K_{ik}(x,z,t) - K_{ik}(x,0,t)|(1+|z|)^{-\alpha} dz$$

$$\leq \begin{cases} c(|x| + \sqrt{t})^{-\alpha}, & \text{if } n < \alpha < n+1, \\ c(|x| + \sqrt{t})^{-n-1} \ln(2+\sqrt{t}), & \text{if } \alpha = n+1, \\ c(|x| + \sqrt{t})^{-n-1}, & \text{if } \alpha > n+1. \end{cases}$$

$$
\begin{aligned}
|e^{-tA}h(x,t)| &= \Big|\int_{\mathbb{R}^n_+}\big(G(x,y,t)-G(x,0,t)\big)h(y)dy\Big| \\
\le c\|(1+|x|)^{\alpha}\|_{L^\infty}\int_{\mathbb{R}^n_+}&\Big|\Gamma(x-y,t)-\Gamma(x-y^*,t)\Big|(1+|y|)^{-\alpha}dy \\
+c\|(1+|x|)^{\alpha}\|_{L^\infty}\int_{\mathbb{R}^n_+}&\Big|\Gamma(x-y^*,t)-\Gamma(x,t)\Big|(1+|y|)^{-\alpha}dy \\
+c\|(1+|x|)^{\alpha}\|_{L^\infty}\int_{\mathbb{R}^n_+}&|K_{ik}(x,z,t)-K_{ik}(x,0,t)|(1+|z|)^{-\alpha}dz.
\end{aligned}
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Thank you for your attention!