Global solvability of the 1d Cosserat-Bingham fluid equations

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Constitutive laws for Micropolar Bingham fluid

- Each material point – Lagrangian coordinates $(\boldsymbol{\xi}, t)$: Position vector $\mathbf{x}(\boldsymbol{\xi}, t)$

  Orthogonal tensor $\mathrm{Q}(\boldsymbol{\xi}, t)$

- Rotation velocity tensor $\Omega(\mathbf{x}, t) = \dfrac{\partial \mathrm{Q}}{\partial t}\mathrm{Q}^*$ (skew-symmetric)

$$\Omega(\mathbf{h}) = \boldsymbol{\omega} \times \mathbf{h}, \quad \forall \mathbf{h} \in \mathbb{R}^3 :$$

$\boldsymbol{\omega}(\mathbf{x}, t)$ - micro-rotational velocity

$\mathbf{v}(\mathbf{x}, t)$ - velocity of the mass center of $(\boldsymbol{\xi}, t)$

- Micropolar viscoplastic fluid by $\mathrm{B} = \dfrac{\partial \mathbf{v}}{\partial \mathbf{x}} - \Omega, \quad A = \dfrac{\partial \boldsymbol{\omega}}{\partial \mathbf{x}}$

- Instant stress state by the Cauchy stress tensor $\mathrm{T} = -p\mathrm{I} + \mathrm{S}$ and the couple stress tensor $\mathrm{N}(\mathbf{x}, t)$

$$\frac{1}{2}\left(\frac{\partial \mathbf{v}}{\partial \mathbf{x}} + \left(\frac{\partial \mathbf{v}}{\partial \mathbf{x}}\right)^*\right) \equiv \left(\frac{\partial \mathbf{v}}{\partial \mathbf{x}}\right)_s, \qquad \frac{1}{2}\left(\frac{\partial \mathbf{v}}{\partial \mathbf{x}} - \left(\frac{\partial \mathbf{v}}{\partial \mathbf{x}}\right)^*\right) \equiv \left(\frac{\partial \mathbf{v}}{\partial \mathbf{x}}\right)_a$$

- Constitutive laws for a simple micropolar (Eringen, 1999)

$$\mathrm{S} = 2\mu_1 \mathrm{B}_s + \mu_2 \mathrm{B}_a, \quad \mathrm{N} = \varkappa_1 A_s^d + \varkappa_2 A_a + \varkappa_3 (\mathrm{tr}\, A)\, \mathrm{I}, \tag{1}$$

$$A^d = A - \frac{\mathrm{tr}\, A}{3} \cdot I$$

$\mu_i, \varkappa_j$ - viscosities

$$\mathrm{B} = \frac{\partial \mathbf{v}}{\partial \mathbf{x}} - \Omega, \quad A = \frac{\partial \omega}{\partial \mathbf{x}}$$

We denote

$$B_0 = B_s + \varkappa B_a, \quad \varkappa = \frac{\mu_2}{2\mu_1},$$
$$A_0 = A_s^d + \bar{\varkappa}_2 A_a + \bar{\varkappa}_3 (\operatorname{tr} A) I, \quad \bar{\varkappa}_2 = \frac{\varkappa_2}{\varkappa_1}, \quad \bar{\varkappa}_3 = \frac{\varkappa_3}{\varkappa_1},$$

and modify (1) by incorporating an yield stress τ_* and an yield couple-stress τ_n ,

introducing the potencials

$$V = \mu_1 |B_0|^2 + \tau_* |B_0|, \quad V_n = \frac{\varkappa_1}{2} |A_0|^2 + \tau_n |A_0|.$$

$$|S|^2 \equiv S : S \equiv S_{ij} S_{ij}$$

The constitutive laws for micropolar viscoelastic fluid

$$S \in \partial V(B_0), \quad N \in \partial V_n(A_0)$$

Constitutive laws for micropolar viscoelastic fluids

$$\mathrm{S} = \begin{cases} 2\mu_1 \mathrm{B}_0 + \tau_* \frac{\mathrm{B}_0}{|\mathrm{B}_0|}, & \mathrm{B}_0(\mathbf{x},t) \neq 0, \\ \mathrm{S}_p(\mathbf{x},t), & \mathrm{B}_0(\mathbf{x},t) = 0, \end{cases}$$

$$\mathrm{N} = \begin{cases} \varkappa_1 A_0 + \tau_n \frac{A_0}{|A_0|}, & A_0(\mathbf{x},t) \neq 0, \\ \mathrm{N}_p(\mathbf{x},t), & A_0(\mathbf{x},t) = 0. \end{cases}$$

Tensors $S_p(y,t)$, $N_p(y,t)$ obey the restrictions

$$|\mathrm{S}_p| \leq \tau_*, \quad |\mathrm{N}_p| \leq \tau_n$$

The momentum and the spin balance equations **(incompressible)**

$$\begin{aligned} \rho\dot{\mathbf{v}} &= \operatorname{div}\mathrm{T} + \rho\mathbf{f}, \\ \rho J\dot{\boldsymbol{\omega}} &= \operatorname{div}\mathrm{N} + \boldsymbol{\varepsilon} : \mathrm{T}^* + \mathbf{l}. \end{aligned}$$

+ boundary and initial conditions.

Special 1-d case. Flow with the properties

$$\begin{aligned} v_2 &= v_3 = 0,\ v_1 = v(y,t),\ \omega_1 = \omega_2 = 0, \omega_3 = \omega(y,t), \\ \partial p/\partial x_1 &\equiv p_x = const < 0. \end{aligned}$$

In the layer

$$\Omega = \{y : |y| < H\}$$

the flow equations

$$\rho v_t = -p_x + \frac{\partial}{\partial y} S_{12}, \quad \rho J \omega_t = \frac{\partial}{\partial y} N_{32} + S_{21} - S_{12}$$

Denoting $\alpha = v_y/2, \quad \beta = \varkappa(\alpha + \omega),$

then the consitutive laws in the special 2d-case

$$S_{12} = 2\mu_1(\alpha + \beta) + \tau_* \frac{\alpha + \beta}{\sqrt{2}(|\alpha| + |\beta|)} \quad \text{if} \quad |\alpha| + |\beta| \neq 0,$$
$$S_{21} = 2\mu_1(\alpha - \beta) + \tau_* \frac{\alpha - \beta}{\sqrt{2}(|\alpha| + |\beta|)} \quad \text{if} \quad |\alpha| + |\beta| \neq 0,$$

$$\left|\frac{S_{12} + S_{21}}{\sqrt{2}}\right| + \left|\frac{S_{12} - S_{21}}{\sqrt{2}}\right| \leq \tau_* \quad \text{if} \quad |\alpha| + |\beta| = 0.$$

and

$$N_{32} = \bar{\gamma}\omega_y + \bar{\tau}_n \operatorname{sign}\omega_y, \quad N_{23} = \hat{\gamma}\omega_y + \hat{\tau}_n \operatorname{sign}\omega_y \quad \text{if} \quad \omega_y \neq 0,$$
$$N_{32}^2(y,t) + N_{23}^2(y,t) \leq \tau_n^2 \quad \text{if} \quad \omega_y = 0,$$

with $\bar{\gamma} = \frac{\varkappa_1 + \varkappa_2}{2}, \quad \hat{\gamma} = \frac{\varkappa_1 - \varkappa_2}{2}, \quad \bar{\tau}_n = \tau_n \frac{1 + \bar{\varkappa}_2}{\sqrt{2(1 + \bar{\varkappa}_2^2)}}, \quad \hat{\tau}_n = \tau_n \frac{1 - \bar{\varkappa}_2}{\sqrt{2(1 + \bar{\varkappa}_2^2)}}.$

Periodic boundary and initial conditions

$$\begin{aligned} v|_{y=-H} &= v|_{y=H}, \quad \omega|_{y=-H} = \omega|_{y=H} \\ v|_{t=0} &= v_0(y), \quad \omega|_{t=0} = \omega_0(y). \end{aligned}$$

Theorem 3.1. *Assume that the functions $v_0(y)$ and $\omega_0(y)$ are periodical and*

$$v_0, \omega_0 \in W^{1,2}(\Omega). \tag{19}$$

Then there are y-periodical functions v, ω, S_{12}, S_{21}, N_{32}, N_{23} such that

$$v, \omega \in L^\infty(0,T; W^{1,2}(\Omega)), \quad S_{ij} \in L^2(Q), \quad N_{ij} \in L^2(Q),$$

$$\omega \in L^2(0,T; W^{2,2}(\Omega)) \cap H^{\nu,\nu/2}(\overline{Q})$$

for some ν such that $0 < \nu < 1$. These functions satisfy the constitutive laws (12)- (16) *and they solve equations* (11) *in the weak sense:*

$$\int_Q \rho v \varphi_t - S_{12}\varphi_y - p_x \varphi \, dydt = -\int_\Omega \rho v_0(y) \varphi(y,0)\, dy,$$

$$\int_Q \rho J \omega \psi_t - N_{32}\psi_y + (S_{21} - S_{12})\psi \, dydt = -\int_\Omega \rho J \omega_0(y) \psi(y,0)\, dy,$$

for any y-periodical functions $\varphi(y,t)$ and $\psi(y,t)$, such that

$$\varphi \in C^1(0,T; W^{1,2}(\Omega)), \quad \psi \in C^1(0,T; W^{1,2}(\Omega)), \, \varphi(y,T) = 0, \, \psi(y,T) = 0.$$

Given $\delta > 0$ we introduce

$$S_{12}^{\delta} = 2(\alpha+\beta)\mu_1^{\delta}(\alpha,\beta), \quad S_{21}^{\delta} = 2(\alpha-\beta)\mu_1^{\delta}(\alpha,\beta),$$
$$N_{32}^{\delta} = \omega_y \bar{\gamma}^{\delta}(\omega_y), \quad N_{23}^{\delta} = \omega_y \hat{\gamma}^{\delta}(\omega_y),$$

$$\mu_1^{\delta} = \mu_1 + \frac{\tau_*}{2\sqrt{2}(|\alpha|+|\beta|)+2\delta}, \quad \bar{\gamma}^{\delta} = \bar{\gamma} + \frac{\bar{\tau}_n}{\sqrt{\omega_y^2+\delta^2}},$$
$$\hat{\gamma}^{\delta} = \hat{\gamma} + \frac{\hat{\tau}_n}{\sqrt{\omega_y^2+\delta^2}}.$$

where $\alpha = \frac{v_y}{2}, \quad \beta = \varkappa(\alpha+\omega)$

The approximate equations with the periodic boundary conditions

$$\rho v_t = -p_x + \frac{\partial}{\partial y} S_{12}^{\delta}(v_y,\omega), \quad \rho J \omega_t = \frac{\partial}{\partial y} N_{32}^{\delta}(\omega_y) + S_{21}^{\delta}(v_y,\omega) - S_{12}^{\delta}(v_y,\omega)$$

$$v|_{t=0} = v_0^{\delta}, \quad \omega|_{t=0} = \omega_0^{\delta},$$

The energy equality

$$\frac{d}{dt}\int\limits_{\Omega}\frac{\rho(v^2+J\omega^2)}{2}dy+\int\limits_{\Omega}\{4\mu_1^\delta\left[\alpha^2+\varkappa(\alpha+\omega)^2\right]+\bar{\gamma}^\delta\omega_y^2\}\,dy=-\int\limits_{\Omega}p_x v\,dy.$$

implies

Lemma 4.1. *Solutions of the problem* (20)- (21) *satisfies the estimates*

$$\begin{aligned}\sup_{0<t<T}(\|v\|^2_{2,\Omega}+\|\omega\|^2_{2,\Omega})+\|v_y\|^2_{2,Q}+\|\omega_y\|^2_{2,Q} &\le c,\\ \|S^\delta_{21}-S^\delta_{12}\|^2_{2,Q} &\le c,\\ \sup_{0<t<T}\|\omega_y\|^2_{2,\Omega}+\|\omega_{yy}\|^2_{2,Q}+\|\omega_t\|^2_{2,Q}\le c,\quad \|\omega\|_{\infty,Q} &\le c\end{aligned}$$

$$\|\omega_y\|_{4,Q}\le c,$$

The limit transition

$$v^{\delta} \rightarrow v$$

$$\omega^{\delta} \rightarrow \omega$$

$$\begin{aligned} v^\delta &\to v \quad \text{in} \quad L^2(Q), \quad \omega^\delta \to \omega \quad \text{in} \quad L^2(Q), \quad \omega_y^\delta \to \omega_y \quad \text{a.e. in} \quad Q, \\ v_y^\delta &\to v_y \quad \text{a. e. in} \quad E_{v,\omega}, \quad S_{12}^\delta \to S_{12} \quad \text{weakly in} \quad L^2(Q), \\ S_{21}^\delta &\to S_{21} \quad \text{weakly in} \quad L^2(Q), \quad N_{32}^\delta \to N_{32} \quad \text{weakly in} \quad L^2(Q), \\ &N_{23}^\delta \to N_{23} \quad \text{weakly in} \quad L^2(Q), \end{aligned}$$

The limit functions satisfies

$$\rho \partial_t v = -p_x + \frac{\partial S_{12}}{\partial y}, \quad \rho J \partial_t \omega = \frac{\partial N_{32}}{\partial y} + S_{21} - S_{12}$$

Let $$\xi^\delta = \frac{S_{12}^\delta + S_{21}^\delta - 4\mu_1\alpha^\delta}{\sqrt{2}}, \quad \eta^\delta = \frac{S_{12}^\delta - S_{21}^\delta - 4\mu_1\beta^\delta}{\sqrt{2}}.$$

that satisfies

$$\int_Q \left(|\xi^\delta| + |\eta^\delta|\right)\varphi \; dydt \le \int_Q \tau_*\varphi \; dydt$$ for any positive $\varphi(y,t) \in C_0(Q)$

By weakly lower semi-continuity we get $$\int_Q \left(|\xi^\delta| + |\eta^\delta|\right)\varphi \; dydt \le \int_Q \tau_*\varphi \; dydt$$

which is equivalente to

$$\left|\frac{S_{12} + S_{21}}{\sqrt{2}}\right| + \left|\frac{S_{12} - S_{21}}{\sqrt{2}}\right| \le \tau_* \quad \text{a.e. in} \quad \{(y,t) \in Q : |\alpha(y,t)| = 0, \quad |\omega(y,t)| = 0\}.$$

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Thank YOU!