Well-posedness and optimal control for stochastic second grade fluids

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Stochastic second grade fluids equations

We consider the stochastic second grade fluids equations in a bounded and simply connected domain $\mathcal{O}$ of $\mathbb{R}^2$ with a sufficiently regular boundary Γ

$$\begin{cases} d(\sigma(Y)) = (\nu\Delta Y - \mathrm{curl}(\sigma(Y)) \times Y - \nabla\pi + U)\, dt + G(t,Y)\, dW_t, \\ \mathrm{div}\, Y = 0 \\ Y \cdot \mathrm{n} = 0, \qquad (\mathrm{n} \cdot DY) \cdot \tau = 0 \\ Y(0) = Y_0 \end{cases} \tag{1}$$

$Y = (Y_1, Y_2)$ is the velocity

$$\sigma(Y) = Y - \alpha\Delta Y$$

Functional spaces

Let us introduce the following Hilbert spaces

$$\begin{aligned} H &= \left\{ v \in L^2(\mathcal{O}) \mid \operatorname{div} v = 0 \ \text{ in } \mathcal{O} \ \text{ and } \ v \cdot \mathrm{n} = 0 \ \text{ on } \Gamma \right\}, \\ V &= \left\{ v \in H^1(\mathcal{O}) \mid \operatorname{div} v = 0 \ \text{ in } \ \mathcal{O} \text{ and } \ v \cdot \mathrm{n} = 0 \ \text{ on } \ \Gamma \right\}, \\ W &= \left\{ v \in V \cap H^2(\mathcal{O}) \mid (\mathrm{n} \cdot Dv) \cdot \tau = 0 \ \text{ on } \Gamma \right\}, \\ \widetilde{W} &= W \cap H^3(\mathcal{O}). \end{aligned}$$

$$\|v\|_V^2 = (\sigma(v), v) = \|v\|_2^2 + 2\alpha \|Dv\|_2^2 \approx \|v\|_{H^1}^2$$

$$\|v\|_W^2 = \|\mathbb{P}\sigma(v)\|_2^2 + \|v\|_V^2 \approx \|v\|_{H^2}^2$$

$$\|v\|_{\widetilde{W}} = \|\operatorname{curl} \sigma(v)\|_2^2 + \|v\|_V^2 \approx \|v\|_{H^3}^2$$

Leray projection $\mathbb{P}$, rate-of-strain tensor $DY = \frac{1}{2}\left[\nabla Y + \nabla Y^\top\right]$

Well-posedness for the SDE

Theorem

Assume $U \in L^p(\Omega, L^p(0,T;H^1(\mathcal{O}))), \quad Y_0 \in L^p(\Omega, V) \cap L^2(\Omega, \widetilde{W})$
for some $4 \leq p < \infty$.
Then there exists a unique solution
$Y \in L^2(\Omega, L^\infty(0,T;\widetilde{W})) \cap L^p(\Omega, L^\infty(0,T;V))$ *such that*

$$\mathbb{E}\sup_{s\in[0,t]}\|Y(s)\|_V^2 + 8\nu\mathbb{E}\int_0^t \|DY\|_2^2\ ds \leq C\left(\mathbb{E}\,\|Y_0\|_V^2 + \mathbb{E}\|U\|_{L^2(0,t;L^2)}^2 + 1\right.$$

$$\mathbb{E}\sup_{s\in[0,t]}\|\mathrm{curl}\,\sigma\,(Y(s))\|_2^2 \leq C\left(\mathbb{E}\,\|\mathrm{curl}\,\sigma(Y_0)\|_2^2 + \mathbb{E}\,\|U\|_{L^2(0,t;H^1)}^2 + 1\right),$$

$$\mathbb{E}\sup_{s\in[0,t]}\|Y(s)\|_V^p \leq C\left(\mathbb{E}\,\|Y_0\|_V^p + \mathbb{E}\|U\|_{L^p(0,t;L^2)}^p + 1\right).$$

Stability of the solutions

Theorem

Assume that $U_1, U_2 \in L^2(\Omega, L^2(0,T; H^1(\mathcal{O})))$ *and*

$$Y_1, Y_2 \in L^2(\Omega, L^\infty(0,T; \widetilde{W}))$$

are corresponding solutions of (1). Then there exist constants C, C_3, *such that*

$$\mathbb{E} \sup_{s\in[0,t]} \xi_3(s) \, \| Y_1 - Y_2 \|_W^2 \leq C \, \mathbb{E} \int_0^t \xi_3(s) \, \| U_1 - U_2 \|_2^2 \, ds$$

where the function ξ *is defined as*

$$\xi_3(t) = e^{-C_3 \int_0^t \left(\| Y_1 \|_{H^3} + \| Y_2 \|_{H^3} + 1 \right) ds}.$$

Control problem

Cost functional

$$J(U, Y_U) = \mathbb{E} \int_0^T L\left(t, Y_U, U\right)\, dt + \mathbb{E}\, h(Y_U(T)) \qquad (2)$$

$$(\mathcal{P}) \begin{cases} \text{minimize} & J(U, Y_U) \\ \text{subject to} & \begin{cases} U \in \mathcal{U}_{ad}^b \\ Y_U \in L^2(\Omega, L^\infty\left(0, T; H^3(\mathcal{O})\right)) \text{ solution of } (1) \end{cases} \end{cases}$$

$\mathcal{U}_{ad}^b \subset L^p(\Omega, L^2(0, T; H^1(\mathcal{O}))),\ 4 \leq p < \infty,$ closed convex and bouded. A standard example is the quadratic cost, where the Lagrangian is defined by

$$L\left(t, U, Y_U\right) = \|\, Y_U - Y_d\|_2^2 + \tfrac{\lambda}{2}\, \|U\|_2^2$$

with a desired target field $Y_d \in L^2(\Omega \times \mathcal{O} \times (0, T))$ and some given $\lambda \geq 0$

Main result for the control problem

$\exists$ (U^*, Y^*) a solution for $(\mathcal{P})$. In addition, exists a unique solution $(p^*, q^*) \in L^2\left(0, T; H^2(\mathcal{O})\right) \cap C\left([0, T]; H^1(\mathcal{O})\right)$ to (BSDE)

$$\begin{cases} d\sigma(p^*) = (-\nu\Delta p^* - \operatorname{curl}\sigma(Y) \times p^* + \operatorname{curl}\left(\sigma\left(Y^* \times p^*\right)\right) + \nabla\pi - f) \\ \qquad\qquad - \left(\nabla_y L\left(t, U^*, Y^*\right) + \nabla_y G(t, Y^*)^T q^*\right)\, dt + \sigma\left(q^*\right)\, dW_t, \\ \operatorname{div} p^* = 0 \\ p^* \cdot \mathrm{n} = 0, \qquad (\mathrm{n} \cdot Dp^*) \cdot \tau = 0 \\ p^*(T) = h\left(Y(T)\right) \end{cases} \tag{3}$$

such that the following optimality condition holds

$$\mathbb{E}\int_0^T \left(p^* + \nabla_x L(t, U^*, Y^*), \Psi - U^*\right) dt \geq 0 \tag{4}$$

for all $\Psi \in \mathcal{U}_{ad}^b$.

Main idea to deduce the optimality condition

Take (U, Y) and suppose it is optimal pair. Then $\forall \Psi \in \mathcal{U}_{ad}^b$.

$$\left.\frac{dJ(U+\rho(\Psi-U))}{d\rho}\right|_{\rho=0} = \mathbb{E}\int_0^T (\nabla_x L(t,U,Y), \Psi - U) + (\nabla_y L(t,U,Y), Z)$$
$$+\mathbb{E}\,(\nabla h(Y(T)), Z(T))_V \geq 0$$

where $Z = D_{\Psi-U}Y(U)$ is the Gâteaux derivative of $U \to Y$ at point U, in the direction $\Psi - U$

- What is $Z = D_{\Psi-U}Y(U)$?
- Knowing Z, how to define the so-called adjoint state p? Verifying

$$\mathbb{E}\int_0^T (\nabla_y L(t,U,Y), Z) + \ \mathbb{E}\,(\nabla h(Y(T)), Z(T))_V = \mathbb{E}\int_0^T (p, \Psi - U)\,.$$

Derivative of $U \to Y = Y_U$

Consider U and $U_\rho = U + \rho\Phi, \qquad \rho \in (0,1).$
If $Y = Y_U$ and $Y_\rho = Y_{U_\rho}$ are the solutions of (1) corresponding to U and U_ρ then

$$Y_\rho = Y + \rho Z + \rho\,\delta_\rho \quad \text{with} \quad \lim_{\rho\to 0}\sup_{t\in[0,T]} \|\delta_\rho\|_V^2 = 0 \quad P\text{-a.e.} \quad \omega \in \Omega$$

where $Z \in L^\infty(0,T;W)$ for P-a.e. $\omega \in \Omega$ is the unique solution to

$$\begin{cases} d\sigma(Z) = (\nu\Delta Z - \operatorname{curl}\sigma(Z)\times Y - \operatorname{curl}\sigma(Y)\times Z - \nabla\pi + \Phi)\, dt \\ \qquad\qquad +\nabla_y G(t,Y) Z\, dW_t \\ \operatorname{div} Z = 0 \\ Z\cdot \mathrm{n} = 0, \qquad (\mathrm{n}\cdot DZ)\cdot\tau = 0 \\ Z(0) = 0 \end{cases} \tag{5}$$

Adjoint equation

Let us consider the following backward stochastic equation

$$
\begin{cases}
d\left(\sigma\left(p\right),\phi\right) = \left(-\nu\Delta p - \operatorname{curl}\sigma(Y)\times p + \operatorname{curl}\left(\sigma\left(Y\times p\right)\right),\phi\right)\, dt \\
\qquad\qquad - \left(\nabla_y L\left(t,U,Y\right) + \nabla_y G(t,Y)^T q,\phi\right)\, dt + \left(\sigma\left(q\right),\phi\right)\, dW_t \\
\operatorname{div} p = 0 \\
p\cdot \mathrm{n} = 0, \qquad (\mathrm{n}\cdot Dp)\cdot\tau = 0 \\
p(T) = \nabla h\left(Y(T)\right)
\end{cases}
\tag{6}
$$

where $Y = Y_U$

$\exists (p,q) \in L^\infty(0,T;W)\times L^2(0,T;W) \qquad P -$ a.e. in Ω

Duality property

We have

$$\mathbb{E}\int_0^T (\Phi, p_n)\, dt = \mathbb{E}\int_0^T \left(\nabla_y L(t, U, Y), Z_n^{\Phi}\right) dt + \mathbb{E}\left(\sigma(Z_n^{\Phi}(T)), \nabla h(Y_T)\right)$$

for any $\Phi \in L^2\left(\Omega, L^2(0, T; V)\right)$.

Lack of integrability for Z and p

Given Φ, there exists a unique Z_n^{Φ} such that

$$\mathbb{E}\sup_{s\in[0,t]}\xi_1(s)\left\|Z_n^{\Phi}(s)\right\|_V^2 + \mathbb{E}\int_0^t \xi_1(s)\left\|Z_n^{\Phi}\right\|_V^2\, ds \leq C\mathbb{E}\int_0^t \xi_1(s)\left\|\Phi\right\|_2^2\, ds$$

$$\mathbb{E}\sup_{s\in[0,t]}\xi_2(s)\left\|Z_n^{\Phi}(s)\right\|_W^2 \leq C\mathbb{E}\int_0^t \xi_2(s)\left\|\Phi\right\|_2^2\, ds$$

where the functions ξ_1, ξ_2 are defined as

$$\xi_1(t) = e^{-2C_1\int_0^t\|Y\|_{\widetilde{W}}ds}, \qquad \xi_2(t) = e^{-4C_2\int_0^t\|Y\|_{\widetilde{W}}ds}.$$

and C_1, C_2 are two fixed constants just depending of the domain.

Exponencial integrability of the state Y

Assume:

- G is bounded $||G(t, Y)||_V^2 \leq L$
- $A = \frac{1}{2\theta_1} \geq C_1$

  or $B = \frac{\gamma_2^2}{2\theta_2} \geq C_1$ and $\mathcal{O}$ non axisymmetric,

  where $\theta_1, \theta_2, \gamma_1, \bar{\gamma}_1, \gamma_2$ and τ are defined by

  $$\theta_1 = 4L\tau^2(1+\gamma_1^2), \qquad \theta_2 = 2Le^{2\varepsilon T}(1+2\bar{\gamma}_1^2),$$

  $$\gamma_1 = \frac{C_*\nu}{\alpha}\tau, \qquad \bar{\gamma}_1 = \frac{K_*}{4\alpha}\tau, \qquad \gamma_2 = \frac{C_{**}\nu}{\alpha} \quad \text{and} \quad \tau = Te^{\varepsilon T}.$$

with

$$C_{**}\,\|u\|_{\widetilde{W}}^2 \leq 2\alpha\,\|Du\|_2^2 + \|\mathrm{curl}\,\sigma(u)\|_2^2, \qquad \|u\|_{H^1}^2 \leq K_*\,\|Du\|_2^2$$

Then

$$\mathbb{E}\exp\left\{C_1\int_0^t \|Y(t)\|_{\widetilde{W}}^2\,ds\right\} < \infty$$

Existence of an optimal control

Let $\lambda = \inf_{u \in \mathcal{U}_{ad}^b} J(U, Y(U))$. Taking a minimizing sequence (U_n, Y_n) $(U_n) \to U^*$ weakly $L^p(\Omega, L^2(0, T; H^1(\mathcal{O})))$. Then (Y_n) is uniformly bounded in

$$L^2(\Omega, L^\infty(0, T; \widetilde{W})) \cap L^p(\Omega, L^\infty(0, T; V)).$$

There exists $Y^* \in L^2(\Omega, L^\infty(0, T; \widetilde{W})) \cap L^p(\Omega, L^\infty(0, T; V))$ such that $Y_n \rightharpoonup Y^*$ weakly*, we may verify that Y^* is solution of the stochastic differential equation (1). On the other hand, since the functional J is left continuous with respect to the weak toplogy, we have

$$J(U^*, Y^*) \leq \lim J(U_n, Y_n) = \lambda$$

which implies that (U^*, Y^*) is an optimal pair.

References

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