Local and Global Solvability of Interface Problems for Fluids of Different Types near the Equilibrium

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• We study the motion of two viscous immiscible fluids in a container Σ, one of fluids is compressible while another one is incompressible.They are separated by a closed interface Γ_t, $\Gamma_t \cap \Sigma = \emptyset$. The compressible fluid is assumed to be barotropic.

• Both cases: with and without surface tension on Γ_t, are under consideration. Global solvability in the Sobolev – Slobodetskiĭ spaces is obtained for the pr-m without surface tension [1].

• The proof is based on a local existence theorem proved in [2] and on the exponential energy inequality for a linearized prob-m. A similar estimate proved for the non-linear prob-m in [3].

• We eliminate the restrictions for liquid viscosities arisen in [2].

• The same result for two incompressible fluids was obtain in [4].

1. I. V. Den., V.A. Solonnikov, (2016) Handb. Math. Anal. in Mech-s of Viscous Fluids II, Springer (to appear).

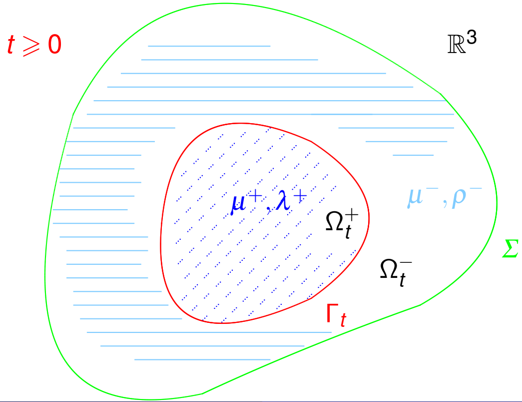
2. I. V. Denisova, *Interfaces Free Bound.*, **2**(3) (2000), 283–312.

3. I. V. Denisova, (2015) J. Math. Fluid Mech. **17**(1): 183-198.

4. I. V. Denisova, (2014) *Portugal. Math.* **71**(1), 1-24.

Statement of the problem

$t = 0$ **:** Let, for definiteness, a compressible fluid be in an interior domain $\Omega_0^+ \subset \mathbb{R}^3$;$\mu^+ > 0,\ \lambda^+ > 0$ be its dynamic viscosities, $\mu^+ > \lambda^+$. Let a incompressible fluid Ω_0^- surround Ω_0^+. It has the dynamic viscosity $\mu^- > 0$ and the density $\rho^- > 0$; $\partial\Omega_0^+ = \Gamma_0 \equiv \Gamma$; $\Sigma \equiv \partial(\Omega_0^+ \cup \Gamma \cup \Omega_0^-)$ is a given closed surface, $\Sigma \cap \Gamma = \emptyset$.



Notation

Here $\mathscr{D}_t = \partial/\partial t$, $\nabla = (\partial/\partial x_1, \partial/\partial x_2, \partial/\partial x_3)$, $\mathbf{f}$ is the given vector of mass forces, $\mathbf{v}_0$ is the initial velocity, the stress tensor $\mathbb{T}$ is

$$\mathbb{T}(\mathbf{v}, p) = \begin{cases} (-p^+(\rho^+) + \lambda^+ \nabla \cdot \mathbf{v})\mathbb{I} + \mu^+ \mathbb{S}(\mathbf{v}) & \text{in } \Omega_t^+, \\ -p^- \mathbb{I} + \mu^- \mathbb{S}(\mathbf{v}) & \text{in } \Omega_t^-, \end{cases}$$

$(\mathbb{S}(\mathbf{v}))_{ik} = \partial v_i/\partial x_k + \partial v_k/\partial x_i$, $i, k = 1, 2, 3$; $\mathbb{I}$ is the unit matrix;
ρ_0^+ is the initial density distribution of the compressible fluid;
$\mathbf{n}$ is the outward normal vector to Ω_t^+;
$\nabla \cdot \mathbb{T}$ means the vector with the components

$$(\nabla \cdot \mathbb{T})_j = \partial T_{ij}/\partial x_i, \quad T_{ij} = (\mathbb{T})_{ij}, \quad j = 1, 2, 3.$$

We imply the summation from 1 to 3 with respect to repeated indexes. A Cartesian coordinate system $\{x\}$ is introduced in $\mathbb{R}^3$. $(\cdot)$ denotes the scalar product. We mark the vectors and the vector spaces by boldface letters. $\Omega \equiv \Omega_t^- \cup \overline{\Omega_t^+}$.

Mathematical formulation

The pressure $p^{\pm}$ is the deviation from the hydrostatic pressure for both fluids. The pressure of the compressible fluid $p^{+}(\rho^{+})$ is given by a smooth increasing function of the density ρ^{+}.
For $t > 0$, the interface Γ_t-? between the domains Ω_t^+ and Ω_t^-, the velocity vector $\mathbf{v}(x,t) = (v_1, v_2, v_3)$-?, the density $\rho^+(x,t) > 0$-? of the compressible fluid, the pressure $p^-(x,t)$-? of the incompressible fluid that satisfy the initial-boundary value problem for the Navier–Stokes system:

$$\rho^+\big(\mathscr{D}_t\mathbf{v} + (\mathbf{v}\cdot\nabla)\mathbf{v}\big) - \nabla\cdot\mathbb{T} = \rho^+\mathbf{f}, \quad \mathscr{D}_t\rho^+ + \nabla\cdot(\rho^+\mathbf{v}) = 0 \quad \text{in } \Omega_t^+,$$

$$\rho^+|_{t=0} = \rho_0^+, \quad \mathbf{v}|_{t=0} = \mathbf{v}_0 \quad \text{in} \quad \Omega_0^+, \qquad (1)$$

$$\rho^-\big(\mathscr{D}_t\mathbf{v} + (\mathbf{v}\cdot\nabla)\mathbf{v}\big) - \nabla\cdot\mathbb{T} = \rho^-\mathbf{f}, \quad \nabla\cdot\mathbf{v} = 0 \quad \text{in} \quad \Omega_t^-,$$

$$\mathbf{v}|_{t=0} = \mathbf{v}_0 \quad \text{in} \quad \Omega_0^-; \quad \mathbf{v}\big|_{\Sigma} = 0;$$

$$[\mathbf{v}]\big|_{\Gamma_t} \equiv \lim_{\substack{x\to x_0\in\Gamma_t,\\ x\in\Omega_t^+}} \mathbf{v}(x) - \lim_{\substack{x\to x_0\in\Gamma_t,\\ x\in\Omega_t^-}} \mathbf{v}(x) = 0, \quad [\mathbb{T}\mathbf{n}]\big|_{\Gamma_t} = \sigma H\mathbf{n} \quad \text{on } \Gamma_t.$$

Interface equation

To exclude mass transportation through Γ_t, we assume that the liquid particles do not leave Γ_t. It means that Γ_t consists of the points $x(\xi, t)$ such that the vector $\mathbf{x}(\xi, t)$ solves the Cauchy problem

$$\mathscr{D}_t \mathbf{x} = \mathbf{v}(x(t), t), \ \ \mathbf{x}|_{t=0} = \xi, \quad \xi \in \Gamma_0, \ t > 0. \tag{2}$$

Hence, $\Gamma_t = \{x(\xi, t) | \ \xi \in \Gamma_0\}$, $\Omega_t^{\pm} = \{x(\xi, t) | \ \xi \in \Omega_0^{\pm}\}$.
We denote the mean density in Ω_t^+ by $\rho_m^+ = \frac{1}{|\Omega_t^+|} \int_{\Omega_t^+} \rho^+ \, dx$.
Observe that ρ_m^+ is independent of t because of the conservation of the mass of compressible fluid $\int_{\Omega_t^+} \rho^+ \, dx$ and that of its volume in our case: $|\Omega_t^+| = |\Omega| - |\Omega_t^-| = constant$.
Moreover, we introduce a new pressure function
$p_1 = p - p^+(\rho_m^+)$: $p_1^+ = p^+(\rho^+) - p^+(\rho_m^+)$ in Ω_t^+ and
$p_1^- = p^- - p^+(\rho_m^+)$ in Ω_t^-. Then nothing changes in system (1).

Lagrangian coordinates

We pass from the Eulerian coordinates to the Lagrangian ones:

$$\mathbf{x} = \xi + \int_0^t \mathbf{u}(\xi,\tau)d\tau \equiv \mathbf{X}_{\mathbf{u}}(\xi,t) \tag{3}$$

($\mathbf{u}(\xi,t)$ is the velocity vector field in the Lagrangian coordinates). For the Jacobian of (3)
$\mathscr{J}_{\mathbf{u}}(\xi,t) = \det\{a_{ij}\}_{i,j=1}^3$, $\quad a_{ij}(\xi,t) = \delta_j^i + \int_0^t \frac{\partial u_i}{\partial \xi_j} d\tau$, $\{\delta_j^i\}_{i,j=1}^3$ are the Kronecker symbols, we have

$$\mathscr{J}_{\mathbf{u}}(\xi,t) = 1 + \int_0^t \mathbb{A}\nabla\cdot\mathbf{u}\, d\tau, \tag{4}$$

$$\mathscr{J}_{\mathbf{u}}(\xi,t) = \exp\Big(\int_0^t \nabla\cdot\mathbf{v}|_{x=X_{\mathbf{u}}} d\tau\Big) \equiv \exp\Big(\int_0^t \nabla_{\mathbf{u}}\cdot\mathbf{u}\, d\tau\Big). \tag{5}$$

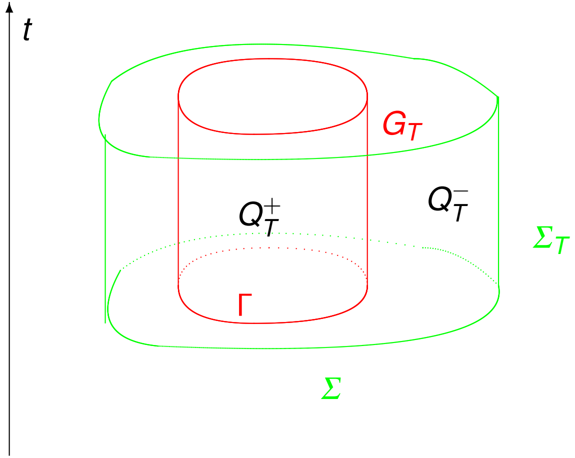
Here $\nabla_{\mathbf{u}} \equiv \mathscr{J}_{\mathbf{u}}^{-1}\mathbb{A}\nabla$; $\mathbb{A} \equiv \{A_{ij}\}_{i,j=1}^3$ is the cofactor matrix to the Jacobi m-x $\{a_{ij}\}_{i,j=1}^3$. We remark that $\mathscr{J}_{\mathbf{u}}(\xi,t) \equiv 1$ in Ω_t^-.

After (3), the equation for the density gives us the formula

$$\widehat{\rho^{+}}(\xi,t) = \rho_0^{+}(\xi)\exp\left(-\int_0^t \nabla_{\mathbf{u}}\cdot\mathbf{u}\,d\tau\right) = \rho_0^{+}(\xi)\,\mathscr{J}_{\mathbf{u}}^{-1}(\xi,t). \qquad (6)$$

We substitute (6) in the 1st eqn of (1) to exclude the density from the system. We separate the last boundary condition into the tangential and normal components.
Transformation (3) leads us to a problem for $\mathbf{u}$ and $q = p_1(X_{\mathbf{u}}, t)$ with the given interface $\Gamma \equiv \Gamma_0$.

We denote: $Q_T^{\pm} \equiv \Omega_0^{\pm} \times (0, T)$, $D_T \equiv Q_T^+ \cup Q_T^-$, $Q_T \equiv \overline{Q_T^+} \cup Q_T^-$, $G_T \equiv \Gamma \times (0, T)$, $\Sigma_T \equiv \Sigma \times (0, T)$.



Let $\mathbf{n}_0$ be the outward normal to Γ. It is connected with

$\mathbf{n} = \frac{\mathscr{J}_{\mathbf{u}}^{-1}\mathbb{A}\mathbf{n}_0}{|\mathscr{J}_{\mathbf{u}}^{-1}\mathbb{A}\mathbf{n}_0|} = \frac{\mathbb{A}\mathbf{n}_0}{|\mathbb{A}\mathbf{n}_0|}$. $\Rightarrow$ If $\mathbf{n}\cdot\mathbf{n}_0 > 0$, pr-m (1), (2) is changed into

$$\mathscr{D}_t\mathbf{u} - \frac{1}{\rho_0^+(\xi)}\mathbb{A}\nabla\cdot\widetilde{\mathbb{T}}_{\mathbf{u}}(\mathbf{u}) = \mathbf{f}(X_{\mathbf{u}},t) - \frac{1}{\rho_0^+(\xi)}\mathbb{A}\nabla p_1^+(\rho_0^+\mathscr{J}_{\mathbf{u}}^{-1}) \text{ in } Q_T^+,$$

$$\mathscr{D}_t\mathbf{u} - \nu^-\nabla_{\mathbf{u}}^2\mathbf{u} + (\rho^-)^{-1}\nabla_{\mathbf{u}}q = \mathbf{f}(X_{\mathbf{u}},t), \quad \nabla_{\mathbf{u}}\cdot\mathbf{u} = 0 \quad \text{in} \quad Q_T^-,$$

$$\mathbf{u}|_{t=0} = \mathbf{v}_0 \quad \text{in } D \equiv \Omega_0^-\cup\Omega_0^+, \qquad \mathbf{u}|_{\Sigma_T} = 0, \tag{7}$$

$$[\mathbf{u}]|_{G_T} = 0, \quad [\mu^\pm\Pi_0\Pi\mathbb{S}_{\mathbf{u}}(\mathbf{u})\mathbf{n}]|_{G_T} = 0,$$

$$[\mathbf{n}_0\cdot\widetilde{\mathbb{T}}_{\mathbf{u}}(\mathbf{u},q)\mathbf{n}]|_{G_T} = \sigma H(X_{\mathbf{u}})\mathbf{n}_0\cdot\mathbf{n} + (\mathbf{n}_0\cdot\mathbf{n})p_1^+(\rho_0^+\mathscr{J}_{\mathbf{u}}^{-1})|_{G_T}.$$

In (7) $q(\xi,t) = p_1(X_{\mathbf{u}},t)$ and $(\mathbb{S}_{\mathbf{u}}(\mathbf{w}))_{ij} = \mathscr{J}_{\mathbf{u}}^{-1}\left(A_{ik}\frac{\partial w_j}{\partial\xi_k} + A_{jk}\frac{\partial w_i}{\partial\xi_k}\right)$;

$$\left(\widetilde{\mathbb{T}}_{\mathbf{u}}(\mathbf{w},q)\right)_{i,j} = \begin{cases} (\lambda^+\nabla_{\mathbf{u}}\cdot\mathbf{w})\delta_j^i + \mu^+(\mathbb{S}_{\mathbf{u}}(\mathbf{w}))_{ij} & \text{in } Q_T^+, \\ -\delta_j^i q + \mu^-(\mathbb{S}_{\mathbf{u}}(\mathbf{w}))_{ij} & \text{in } Q_T^-; \end{cases}$$

$$\Pi_0\omega = \omega - (\mathbf{n}_0\cdot\omega)\mathbf{n}_0, \quad \Pi\omega = \omega - (\mathbf{n}\cdot\omega)\mathbf{n}.$$

The anisotropic Sobolev–Slobodetskiĭ space $W_2^{m,m/2}(Q_T)$ in $Q_T = \Omega \times (0,T)$, $0 < T \leqslant \infty$:

$$\|u\|_{W_2^{m,m/2}(Q_T)} = \left(\int_0^T \|u\|^2_{W_2^m(\Omega)} dt + \int_\Omega \|u\|^2_{W_2^{m/2}(0,T)} dx \right)^{1/2}.$$

Let $D_T \equiv Q_T^- \cup Q_T^+$, $\|u\|_{W_2^{m,m/2}(D_T)} = \|u\|_{W_2^{m,m/2}(Q_T^-)} + \|u\|_{W_2^{m,m/2}(Q_T^+)}$.

Vector norm is equal to the sum of the norms of its components.

Let $u \in W_2^{l,l/2}(Q_T)$, $l \in (0,1)$. We define

$$\|u\|_{Q_T}^{(l,l/2)} = \left(\|u\|^2_{W_2^{l,l/2}(Q_T)} + T^{-l} \|u\|^2_{Q_T} \right)^{1/2},$$

The equivalent normalization of $W_2^{2+l,1+l/2}(Q_T)$ for $T < \infty$ is

$$\left(\|u\|_{Q_T}^{(2+l,1+l/2)} \right)^2 = \|u\|^2_{W_2^{2+l,1+l/2}(Q_T)} + T^{-l} \left\{ \|\mathscr{D}_t u\|^2_{Q_T} + \sum_{|\alpha|=2} \|\mathscr{D}_x^\alpha u\|^2_{Q_T} \right\} + \sup_{t \leqslant T} \|u(\cdot,t)\|^2_{W_2^{1+l}(\Omega)}.$$

Local Existence Theorem [I. Den. (2000)]

Assume that for $l \in (1/2, 1)$ *we have* $\Gamma \in W_2^{5/2+l}$,
$\rho_0^+ \in W_2^{1+l}(\Omega_0^+)$, $0 < \rho_0^* \leqslant \rho_0^+(\xi) \leqslant \rho_\infty^* < \infty$, $\xi \in \Omega_0^+$,
$p^+ \in C^3(\mathbb{R}_+)$, $\mathbf{f}, \nabla\mathbf{f} \in \mathbf{W}_2^{l,\frac{l}{2}}(Q_T)$, $\nabla\nabla\mathbf{f} \in \mathbf{L}_2(Q_T)$, $0 < T < \infty$.
Let $\mathbf{v}_0 \in \mathbf{W}_2^{1+l}(\Omega_0^- \cup \Omega_0^+)$ *satisfy the compatibility conditions*

$$\nabla \cdot \mathbf{v}_0 = 0 \quad \text{in} \quad \Omega_0^-, \qquad \mathbf{v}_0\big|_\Sigma = 0, \qquad (8)$$
$$[\mathbf{v}_0]\big|_\Gamma = 0, \quad [\Pi_0 \mathbb{S}(\mathbf{v}_0)\mathbf{n}_0]\big|_\Gamma = 0.$$

and let the viscosities of the liquids satisfy the inequalities

$$\mu^- > \mu^+, \quad \nu^- < \mu^+/\rho_\infty^*. \qquad (9)$$

Local Existence Theorem [I. Den. (2000)]

$\Rightarrow \exists\ T_0 \in (0, T]$ *such that prbm* (7) *has a unique solution* $(\mathbf{u}, q)$ *on* $(0, T_0)$ *such that* $\mathbf{u} \in \mathbf{W}_2^{2+l,1+l/2}(D_{T_0})$, $q \in W_{2,loc}^{l,l/2}(Q_{T_0}^-)$, $\nabla q \in \mathbf{W}_2^{l,l/2}(Q_{T_0}^-)$, $q|_{G_{T_0}} \in W_2^{l+1/2,l/2+1/4}(G_{T_0})$ *and*

$$\|\mathbf{u}\|_{D_{T_0}}^{(2+l,1+l/2)} + \|\nabla q\|_{Q_{T_0}^-}^{(l,l/2)} + \|q\|_{Q_{T_0}^-}^{(l,l/2)} + \|q\|_{W_2^{l+1/2,l/2+1/4}(G_{T_0})} \leqslant$$

$$\leqslant c_1 \Big\{ \|\mathbf{f}\|_{Q_{T_0}}^{(l,\frac{l}{2})} + \|\mathbf{f}\|_{\mathbf{W}_2^{1+l,0}(Q_{T_0})} + \|\mathbf{v}_0\|_{\mathbf{W}_2^{1+l}(\Omega_0^- \cup \Omega_0^+)} \tag{10}$$

$$+ \sigma \|H_0\|_{W_2^{1/2+l}(\Gamma)} + \|p^+(\rho_0^+) - p^+(\rho_m^+)\|_{W_2^{1+l}(\Omega_0^+)} \Big\},$$

$$\|\rho^+ - \rho_m^+\|_{W_2^{1+l}(\Omega_t^+)} \leqslant c_4 \|\rho_0^+ - \rho_m^+\|_{W_2^{1+l}(\Omega_0^+)}, \quad \forall t \in (0, T_0]. \tag{11}$$

T_0 *and constants* c_1, c_4 *depend on the norms of* $\mathbf{f}$, $\mathbf{v}_0$, ρ_0^+, p^+, H_0.

Remark 1

The LE Theorem was formulated with $\Gamma \in W_2^{5/2+l}$*. In the case of* $\sigma = 0$*, the Theorem holds for any initial interface* $\Gamma \in W_2^{3/2+l}$ *because one don't need to calculate the norm of the mean curvature at initial moment which requires the higher regularity from the boundary.*

Remark 2

We show that LE Theorem is valid without restrictions (9) *which arose at the step of analysis of a linear problem with flat interface.*

Linear problem with plane interface

$$\mathscr{D}_t\mathbf{v} - \nu^+\nabla^2\mathbf{v} + (\rho_0^+)^{-1}\nabla p = 0, \quad \nabla\cdot\mathbf{v} = 0 \quad \text{in} \quad D_\infty^+ = \mathbb{R}_+^3 \times (0,\infty),$$
$$\mathscr{D}_t\mathbf{v} - \nu^-\nabla^2\mathbf{v} - (\nu^- + \varkappa^-)\nabla(\nabla\cdot\mathbf{v}) = 0 \quad \text{in} \quad D_\infty^- = \mathbb{R}_-^3 \times (0,\infty),$$
$$\mathbf{v}\big|_{t=0} = 0 \quad \text{in}\, \mathbb{R}_-^3 \cup \mathbb{R}_+^3, \quad \mathbf{v} \xrightarrow[|x|\to\infty]{} 0, \quad p \xrightarrow[|x|\to\infty]{} 0, \qquad (12)$$
$$[\mathbf{v}]\big|_{x_3=0} = 0, \quad -\left[\mu^\pm\left(\frac{\partial v_\alpha}{\partial x_3} + \frac{\partial v_3}{\partial x_\alpha}\right)\right]\bigg|_{x_3=0} = b_\alpha(x',t), \quad \alpha = 1,2;$$
$$-(p + \lambda^-\nabla\cdot\mathbf{v}) + \left[2\mu^\pm\frac{\partial v_3}{\partial x_3}\right]\bigg|_{x_3=0} + \sigma\Delta'\int_0^t v_3\big|_{x_3=0}\,d\tau = \sigma\int_0^t B\,dt$$
$$+ b_3(x',t) \quad \text{on}\, \mathbb{R}_\infty^2 = \mathbb{R}^2 \times (0,\infty).$$

$\mathbb{R}_\pm^3 = \{\pm x_3 > 0\}$; $\varkappa^- = \lambda^-/\rho_0^-$, $\rho_0^- = const > 0$, $\nu^- = \mu^-/\rho_0^- > \varkappa^-$; $x' = (x_1,x_2)$, $\Delta' = \partial^2/\partial x_1^2 + \partial^2/\partial x_2^2$ on $\mathbb{R}^2$. We take the Fourier–Laplace transform and find an explicite sol-n $(\widetilde{\mathbf{v}},\widetilde{p})$.

Non-linear problem with $\sigma = 0$

Compressible fluid density is $\rho^+(x,t) = \rho_m^+ + \theta^+(x,t)$,
incompressible fluid pressure is $p^-(x,t) = p_m^- + \theta^-(x,t)$,
where $\rho_m^+ = \frac{1}{|\Omega_t^+|}\int_{\Omega_t^+} \rho^+\,dx$ is the mean value of $\rho^+(x,t)$,
$\int_{\Omega_t^+} \theta^+(x,t)\,dx = 0$ and $p_m^- = p^+(\rho_m^+)$, $p^+(\rho^+) \equiv p(\rho^+)$.

Proposition 1 (Energy estimate)

Assume a solution of problem (1), (2) is defined on $[0, T_0]$ and $\mathbf{v}_0$ satisfies compatibility conditions (8). Let $\mathbf{f}(\cdot,\tau) \in L_2(\Omega)$, $t \in (0, T_0]$, and $\int_0^{T_0} e^{b\tau}\|\mathbf{f}(\cdot,\tau)\|^2_\Omega\,d\tau < \infty$. $\Rightarrow$ For $\forall t \in (0, T_0]$,

$$\|\mathbf{v}(\cdot,t)\|^2_\Omega + \|\theta^+(\cdot,t)\|^2_{\Omega_t^+} \leqslant ce^{-bt}\Big\{\|\mathbf{v}_0\|^2_\Omega + \|\rho_0^+ - \rho_m^+\|^2_{\Omega_0^+} + \int_0^t e^{b\tau}\|\mathbf{f}(\cdot,\tau)\|^2_\Omega\,d\tau\Big\}, \quad (13)$$

where $b > 0$ is a constant.

It is clear that $|\Omega_t^{\pm}| = mes\Omega_t^{\pm}$ are independent of t. Hence,

$$\frac{d|\Omega_t^+|}{dt} = \int_{\Omega_t^+} \nabla\cdot\mathbf{v}(x,t)\,dx = 0. \qquad (14)$$

Then the non-linear problem in the Lagrangian coordinates can be written in the form with a linear system on the left-hand side

$$\begin{aligned}
&\rho_m^+\mathscr{D}_t\mathbf{u} - \nabla\cdot\mathbb{T}_1^+(\mathbf{u}) + p'(\rho_m^+)\nabla\vartheta^+ = \mathbf{l}_1^+(\mathbf{u},\vartheta^+),\\
&\mathscr{D}_t\vartheta^+ + \rho_m^+\Big(\nabla\cdot\mathbf{u} - \frac{1}{|\Omega_0^+|}\int_{\Omega_0^+}\nabla\cdot\mathbf{u}(z,t)\,dz\Big) = l_2^+(\mathbf{u},\vartheta^+) \quad \text{in } Q_\infty^+,\\
&\rho^-\mathscr{D}_t\mathbf{u} - \nabla\cdot\mathbb{T}_1^-(\mathbf{u}) + \nabla\vartheta^- = \mathbf{l}_1^-(\mathbf{u},\vartheta^-), \qquad \nabla\cdot\mathbf{u} = l_2^-(\mathbf{u}) \quad \text{in } Q_\infty^-,\\
&[\mathbf{u}]|_{\Gamma_0} = 0, \quad \Pi_0[\mathbb{T}_1^{\pm}(\mathbf{u})\mathbf{n}_0]|_{\Gamma_0} = \mathbf{l}_3(\mathbf{u}), \qquad (15)\\
&-p'(\rho_m^+)\vartheta^+ + \vartheta^- + [\mathbf{n}_0\cdot\mathbb{T}_1^{\pm}(\mathbf{u})\mathbf{n}_0]|_{\Gamma_0} = l_4(\mathbf{u},\vartheta^+) \quad \text{on } G_\infty,\\
&\mathbf{u}|_\Sigma = 0, \qquad \mathbf{u}|_{t=0} = \mathbf{v}_0 \quad \text{in } \Omega_0^-\cup\Omega_0^+, \quad \vartheta^+|_{t=0} = \vartheta_0^+ \quad \text{in } \Omega_0^+,
\end{aligned}$$

where $\vartheta^{\pm}(y,t) = \theta^{\pm}(X(y,t),t)$, $\mathbb{T}_1^{\pm}(\mathbf{w})$ is the viscous part of str. ten.: $\mathbb{T}_1^+(\mathbf{w}) = \mu^+\mathbb{S}(\mathbf{w}) + \mu_1^+\mathbb{I}\nabla\cdot\mathbf{w}$, $\quad \mathbb{T}_1^-(\mathbf{w}) = \mu^-\mathbb{S}(\mathbf{w})$.

Global Existence Theorem for $\sigma = 0$

Assume $\Sigma, \Gamma_0 \in W_2^{\frac{3}{2}+l}$, $l \in (\frac{1}{2}, 1)$, $0 < \rho_0^* \leqslant \rho_0^+(\xi) \leqslant \rho_\infty^* < \infty$, $\xi \in \Omega_0^+$ *and let* $\mathbf{v}_0 \in W_2^{1+l}(\cup\Omega_0^\pm)$, $\vartheta_0^+ \in W_2^{1+l}(\Omega_0^+)$ *satisfying the compatibility and smallness conditions*

$$\nabla\cdot\mathbf{v}_0 = 0 \quad in \quad \Omega_0^-, \quad [\mathbf{v}_0]\big|_{\Gamma_0} = 0, \quad [\Pi_0\mathbb{T}'(\mathbf{v}_0)\mathbf{n}_0]\big|_{\Gamma_0} = 0, \qquad (16)$$

$$\int_{\Omega_0^+}\vartheta_0^+(z)\,dz = 0, \qquad \|\mathbf{v}_0\|_{W_2^{l+1}(\cup\Omega_0^\pm)} + \|\vartheta_0^+\|_{W_2^{1+l}(\Omega_0^+)} \leqslant \varepsilon \ll 1.$$

Then pr-m (15) *has a unique solution* $\forall t > 0$ *and for* $\beta > 0$

$$\|e^{\beta t}\mathbf{u}\|_{W_2^{2+l,1+l/2}(D_\infty)} + \|e^{\beta t}\vartheta^+\|_{W_2^{1+l,0}(D_\infty)} + \|e^{\beta t}\mathscr{D}_t\vartheta^+\|_{W_2^{1+l,0}(D_\infty)}$$
$$+\|e^{\beta t}\nabla\vartheta^-\|_{W_2^{l,l/2}(Q_\infty^-)} + \|e^{\beta t}\vartheta^-\|_{W_2^{l+1/2,l/2+1/4}(G_\infty)}$$
$$\leqslant c\Big\{\|\mathbf{v}_0\|_{W_2^{1+l}(\cup\Omega_0^\pm)} + \|\vartheta_0^+\|_{W_2^{1+l}(\Omega_0^+)}\Big\}.$$

The proof is based on global solvability and an exponential estimate for a generalized energy of the linear problem.

Finally, the expansion of the interface Γ_t can be estimated by evaluating the speed of interface displacement:

$$\int_0^\infty \max_{\Omega_0^+} |\mathbf{u}(\cdot, t)| dt \leqslant c \int_0^\infty \|\mathbf{u}\|_{W_2^{1+l}(\Omega_0^+)} dt \leqslant c_4 \varepsilon.$$

Consequently, if one takes ε so small that $c_4\varepsilon$ is less than the distance between the initial interface Γ_0 and the solid boundary Σ, these surfaces will never intersect. Hence, by choosing small data, one can guarantee that the bubble is always contained strictly inside the incompressible liquid. Thus, the stability of a solution to the interface problem takes place in the sense that the solution is small and decays exponentially in time under a small deviation of initial data from zero.

Conclusions

- The problem on the evolution of a bubble in an incompressible continuum has been analyzed in the spaces $W_2^{l,l/2}$. The case of a drop surrounded by a gas is studied in the same way.
- A local existence theorem for the problem has been proved in the case of non-negative surface tension without restrictions on the viscosities and the densities imposed earlier.
- Global unique solvability has been obtained for the problem without surface tension forces on the interface and with small data, the liquids being located in a container of finite volume.
- We have shown the L_2–norm of the velocity and that of the deviation of compressible fluid density from the mean value decay exponentially with respect to time.
- The maximal displacement has been estimated of the interface during the time.