A bounded H^∞-calculus for the hydrostatic Stokes operator on L^p-spaces and applications

Based on joint work with Yoshikazu Giga, Matthias Hieber, Amru Hussein and Takahito Kashiwabara

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Fluids under Pressure, Summer School 2016, Prague
August 31

Recapping the operator

Primitive equations on $\Omega = G \times (0,h)$, $G = (0,1)^2$

$$\begin{cases} \partial_t v - \Delta v + (v \cdot \nabla_H + w\partial_z)v + \nabla_H \pi &= f, \quad \Omega \times (0,T) \\ \partial_z \pi &= 0, \quad \Omega \times (0,T) \\ \mathrm{div}_H v + \partial_z w &= 0, \quad \Omega \times (0,T) \\ v(0) &= v_0, \quad \Omega \end{cases}$$

with boundary conditions

$$\begin{cases} \partial_z v, w = 0, & z = h \\ v, w = 0, & z = 0 \\ v, w, \pi \quad \text{periodic}, & x, y \in \{0,1\} \end{cases}$$

Recapping the operator

Hydrostatic Stokes equation on $\Omega = G \times (0,h)$, $G = (0,1)^2$

$$\begin{cases} \partial_t v - \Delta v + \nabla_H \pi &= f, \quad \Omega \times (0,T) \\ \partial_z \pi &= 0, \quad \Omega \times (0,T) \\ \operatorname{div}_H \int_0^h v(\cdot, z)\, dz &= 0, \quad G \times (0,T) \\ v(0) &= v_0, \quad \Omega \end{cases}$$

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Recapping the operator

Hydrostatic stokes operator is given by

$$D(A_p) = \{v \in W^{2,p}_{\text{per}}(\Omega)^2 : \partial_z v\big|_{z=h} = 0, v\big|_{z=0} = 0\} \cap L^p_{\sigma}(\Omega),$$
$$A_p v = \Delta v + \frac{1}{h}(1 - Q_p)\left(\partial_z v\big|_{z=0}\right),$$

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where

$$L^p_{\overline{\sigma}}(\Omega) := \{v \in C^\infty_{\mathrm{per}}(\Omega)^2 : \mathrm{div}_H \int_0^h v(\cdot, z)\, dz = 0\}^{\|\cdot\|_{L^p}},$$
$$Q_p : L^p(G)^2 \to L^p_\sigma(G)$$

2D Helmholtz projection with periodic boundary conditions

What we know

Initial results: Hieber and Kashiwabara (2015)

On $L^p_\sigma(\Omega)$ for $p \in (1, \infty)$

- $-A_p$ is sectorial of angle 0
- A_p generates bounded analytic semigroup
- $(A_p)^{-1}$ exists and is bounded

Proof by solving the resolvent problem

What's new

New results: Giga, Gries, Hieber, Hussein, Kashiwabara (2016)

Main result:

- ($\mathcal{R}$-)bounded H^∞-calculus of angle 0

Corollaries

- Maximal $L^q - L^p$ regularity
- Bounded imaginary powers, descriptions for $D((-A_p)^\theta)$

Decay estimates

- $L^p - L^q -$smoothing

Proof by perturbation argument

The core observation

Study the perturbation term:

$$B_p : D(B_p) \to L^p(\Omega)^2, \quad v \mapsto \frac{1}{h}(1 - Q_p)\left(\partial_z v\big|_{z=0}\right),$$

$$D(B_p) := H^{1+\frac{1}{p}+\delta,p}(\Omega)^2, \quad \delta \in (0, 1 - 1/p)$$

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Bounded by trace theorem with estimate

$$\|B_p v\|_{L^p(\Omega)^2} \leq C\|(-\Delta_p)^{1-\delta} v\|_{L^p(\Omega)^2}, \quad v \in D(\Delta_p)$$

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By perturbation theory

Since A_p has bounded inverse, the H^∞-calculus of angle 0 is inherited from the Laplace operator

Corollary 1: Maximal $L^q - L^p$ regularity

Given $T \in (0, \infty]$, $J := (0, T)$ and $p, q \in (1, \infty)$, the Cauchy problem

$$\begin{cases} \partial_t v - A_p v &= f \\ v(0) &= 0 \end{cases}$$

has a unique solution

$$v \in W^{1,q}(J; L^p_\sigma(\Omega)) \cap L^q(J; D(A_p))$$

if and only if $f \in L^q(J; L^p_\sigma(\Omega))$.

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Corollary 1: Maximal $L^q - L^p$ regularity

Initial values by real interpolation

Admissible initial values for maximal $L^q - L^p$ regularity

$$v_0 \in L^p_{\overline{\sigma}}(\Omega) \cap B^{2-\frac{2}{q}}_{p,q,\mathrm{per}}(\Omega)^2$$

are characterised by compatibility conditions

$$\begin{aligned}
\text{no condition} \quad & \text{if} \quad 2 - \frac{2}{q} < \frac{1}{p}, \\
v_0\big|_{z=0} = 0 \quad & \text{if} \quad 2 - \frac{2}{q} > \frac{1}{p}, \\
\partial_z v_0\big|_{z=h} = 0 \quad & \text{if} \quad 2 - \frac{2}{q} > 1 + \frac{1}{p}, \\
? \quad & \text{if} \quad 2 - \frac{2}{q} \in \left\{ \frac{1}{p}, 1 + \frac{1}{p} \right\}.
\end{aligned}$$

Corollary 2: Bounded imaginary powers

Domains by complex interpolation

Functions $v \in D((-A_p)^\theta) = [L^p_\sigma(\Omega), D(A_p)]_\theta$ are characterised by regularity

$$v \in L^p_\sigma(\Omega) \cap H^{2\theta,p}_{\mathrm{per}}(\Omega)^2, \quad \theta \in [0,1]$$

and boundary conditions

$$\begin{aligned} \text{no condition} \quad & \text{if} \quad 2\theta < \frac{1}{p}, \\ v\big|_{z=0} = 0 \quad & \text{if} \quad 2\theta > \frac{1}{p}, \\ \partial_z v\big|_{z=h} = 0 \quad & \text{if} \quad 2\theta > 1 + \frac{1}{p}, \\ ? \quad & \text{if} \quad 2\theta \in \left\{\frac{1}{p}, 1 + \frac{1}{p}\right\}. \end{aligned}$$

Decay Estimates: $L^p - L^q-$smoothing

For $p \leq q$ there exists $C, \beta > 0$ such that for $t > 0$

$$\|e^{tA_p} P_p f\|_{L^q(\Omega)^2} \leq C t^{-\frac{3}{2}\left(\frac{1}{p} - \frac{1}{q}\right)} e^{-\beta t} \|f\|_{L^p(\Omega)^2}$$

$$\|\nabla e^{tA_p} P_p f\|_{L^q(\Omega)^2} \leq C t^{-\frac{3}{2}\left(\frac{1}{p} - \frac{1}{q}\right) - \frac{1}{2}} e^{-\beta t} \|f\|_{L^p(\Omega)^2}$$

$$\|e^{tA_p} P_p \mathrm{div} f\|_{L^q(\Omega)^2} \leq C t^{-\frac{3}{2}\left(\frac{1}{p} - \frac{1}{q}\right) - \frac{1}{2}} e^{-\beta t} \|f\|_{L^p(\Omega)^2}$$

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Proof analogous to Stokes-case:

$$H^{2\theta,p}(\Omega)^2 \hookrightarrow L^q(\Omega)^2, \quad \theta = \frac{3}{2}\left(\frac{1}{p} - \frac{1}{q}\right)$$

+ interpolation, analyticity of semigroup

Open questions

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- differently shaped domains?
- a different model for the primitive equations?
- a different approach to their well-posedness?

Thank you for your attention!