Error estimates for two types of Lagrange-Galerkin scheme for the Peterlin viscoelastic model

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Mathematical model

The Peterlin viscoelastic model (P)

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}\cdot\nabla)\mathbf{u} = \nu\Delta\mathbf{u} + \operatorname{div}(\psi(\operatorname{tr}\mathbf{C})\mathbf{C}) - \nabla p$$

$$\operatorname{div}\mathbf{u} = 0$$

$$\frac{\partial \mathbf{C}}{\partial t} + (\mathbf{u}\cdot\nabla)\mathbf{C} - (\nabla\mathbf{u})\mathbf{C} - \mathbf{C}(\nabla\mathbf{u})^T = \chi(\operatorname{tr}\mathbf{C})\mathbf{I} - \phi(\operatorname{tr}\mathbf{C})\mathbf{C} + \varepsilon\Delta\mathbf{C} \quad \text{on } \Omega\times(0,T)$$

$$\left(\mathbf{u}, \frac{\partial \mathbf{C}}{\partial \mathbf{n}}\right) = (\mathbf{0},\mathbf{0}) \quad \text{on } \partial\Omega \qquad (\mathbf{u}(0),\mathbf{C}(0)) = (\mathbf{u}_0,\mathbf{C}_0) \quad \text{on } \Omega.$$

$\Omega\subset\mathbb{R}^d$ bounded smooth domain, ν fluid viscosity, ε elastic stress diffusivity
$\mathbf{u}\in\mathbb{R}^d$ velocity, $p\in\mathbb{R}$ pressure, $\mathbf{C}\in\mathbb{R}^{d\times d}$ conformation tensor

$\psi\in C^1([0,\infty))$, $\psi'\geq 0$ and $\chi,\ \phi\in C([0,\infty))$ positive functions satisfying polynomial growth conditions

The Oseen-type Peterlin viscoelastic model

The convective terms in (P) are linearised by a given velocity $\mathbf{w}:\Omega\times(0,T)\to\mathbb{R}^d$.

Existence of global weak solutions

Let $d=2,3$ and let $(\mathbf{u}_0,\mathbf{C}_0)\in L^2_{div}(\Omega)\times L^2(\Omega)$.
There exists a weak solution to (P) such that

$$\mathbf{u}\in L^\infty(0,T;L^2(\Omega))\cap L^2(0,T;H^1_{0,div}(\Omega)), \qquad \mathbf{u}'\in L^q(0,T;W^{-3,2}(\Omega))$$
$$\mathbf{C}\in L^p((0,T)\times\Omega)\cap L^{1+\delta}(0,T;W^{1,1+\delta}), \qquad \mathbf{C}'\in L^{1+\delta}(0,T;W^{-1,1+\delta}(\Omega))$$

for some $q>1$, $p>2$ and $0<\delta<<1$, see [2].

Let $d=2$, $\psi(s)=\chi(s)=s$ and $\phi(s)=s^2$. Then, for more regular data, there exists a unique regular global weak solution to (P), cf. [1].

Numerical approximation of the Oseen-type Peterlin viscoelastic model by the stabilized Lagrange-Galerkin method

- $\psi(s)=\chi(s)=s$ and $\phi(s)=s^2$
- conforming finite element approximation of $\mathbf{u}$, p and $\mathbf{C}$ by continuous piecewise linear functions + pressure stabilization
- material derivative discretized by the method of characteristics

Semi-implicit linear scheme (L) [3]

$$\left(\frac{\mathbf{u}_h^n-\mathbf{u}_h^{n-1}\circ X_1^n}{\Delta t},\mathbf{v}_h\right) + 2\nu\left(\mathrm{D}(\mathbf{u}_h^n),\mathrm{D}(\mathbf{v}_h)\right) - \left(\operatorname{div}\mathbf{u}_h^n, q_h\right) - \left(\operatorname{div}\mathbf{v}_h, p_h^n\right) - \delta_0\sum_{K\in\tau_h} h_K^2\left(\nabla p_h^n,\nabla q_h\right)_K +$$
$$+\left(\operatorname{tr}\mathbf{C}_h^n\,\mathbf{C}_h^{n-1},\nabla\mathbf{v}_h\right)=0$$
$$\left(\frac{\mathbf{C}_h^n-\mathbf{C}_h^{n-1}\circ X_1^n}{\Delta t},\mathbf{D}_h\right) + \varepsilon\left(\nabla\mathbf{C}_h^n,\nabla\mathbf{D}_h\right) - 2\left((\nabla\mathbf{u}_h^n)\mathbf{C}_h^{n-1},\mathbf{D}_h\right) +$$
$$+\left((\operatorname{tr}\mathbf{C}_h^{n-1})^2\mathbf{C}_h^n,\mathbf{D}_h\right) - \left(\operatorname{tr}\mathbf{C}_h^{n-1}\,\mathbf{I},\mathbf{D}_h\right)=0$$

Fully implicit nonlinear scheme (N) [4]

$$\left(\frac{\mathbf{u}_h^n-\mathbf{u}_h^{n-1}\circ X_1^n}{\Delta t},\mathbf{v}_h\right) + 2\nu\left(\mathrm{D}(\mathbf{u}_h^n),\mathrm{D}(\mathbf{v}_h)\right) - \left(\operatorname{div}\mathbf{u}_h^n, q_h\right) - \left(\operatorname{div}\mathbf{v}_h, p_h^n\right) - \delta_0\sum_{K\in\tau_h} h_K^2\left(\nabla p_h^n,\nabla q_h\right)_K +$$
$$+\left(\operatorname{tr}\mathbf{C}_h^n\,\mathbf{C}_h^{n},\nabla\mathbf{v}_h\right)=0$$
$$\left(\frac{\mathbf{C}_h^n-\mathbf{C}_h^{n-1}\circ X_1^n}{\Delta t},\mathbf{D}_h\right) + \varepsilon\left(\nabla\mathbf{C}_h^n,\nabla\mathbf{D}_h\right) - 2\left((\nabla\mathbf{u}_h^n)\mathbf{C}_h^{n},\mathbf{D}_h\right) - \left(\operatorname{div}\mathbf{u}_h^n(\mathbf{C}_h^n)^\#,\mathbf{D}_h\right) +$$
$$+\left((\operatorname{tr}\mathbf{C}_h^{n})^2\mathbf{C}_h^n,\mathbf{D}_h\right) - \left(\operatorname{tr}\mathbf{C}_h^{n}\,\mathbf{I},\mathbf{D}_h\right)=0$$

Error estimates

There exist positive constants h_0, c_0 and c such that, for any pair $(h,\Delta t)$ satisfying

$$h\in(0,h_0], \quad \Delta t\leq c_0/(1+|\log h|)^{1/2}$$

and any solution $(\mathbf{u}_h,p_h,\mathbf{C}_h)$ of scheme (L) it holds that

$$\|\mathbf{C}_h\|_{\ell^\infty(L^\infty)}\leq\|\mathbf{C}\|_{C(L^\infty)}+1,$$

$$\|\mathbf{u}_h-\mathbf{u}\|_{\ell^\infty(L^2)},\quad \|\mathbf{u}_h-\mathbf{u}\|_{\ell^2(H^1)},\quad |p_h-p|_{\ell^2(|\cdot|_h)},$$
$$\|\mathbf{C}_h-\mathbf{C}\|_{\ell^\infty(H^1)},\quad \left\|\bar{D}_{\Delta t}\mathbf{C}_h-\frac{\partial\mathbf{C}}{\partial t}\right\|_{\ell^2(L^2)} \qquad \leq c(h+\Delta t).$$

There exist positive constants h_0, Δt_0 and c independent of ε such that, for any pair $(h,\Delta t)$ satisfying

$$h\in(0,h_0], \quad \Delta t\in(0,\Delta t_0]$$

and any solution $(\mathbf{u}_h,p_h,\mathbf{C}_h)$ of scheme (N) it holds that

$$\|\mathbf{u}_h-\mathbf{u}\|_{\ell^\infty(L^2)},\quad \sqrt{\nu}\|\mathbf{u}_h-\mathbf{u}\|_{\ell^2(H^1)},\quad |p_h-p|_{\ell^2(|.|_h)},$$
$$\|\mathbf{C}_h-\mathbf{C}\|_{\ell^\infty(L^2)},\quad \sqrt{\varepsilon}|\mathbf{C}_h-\mathbf{C}|_{\ell^2(H^1)},$$
$$\left\|\operatorname{tr}(\mathbf{C}_h-\mathbf{C})(\mathbf{C}_h-\mathbf{C})\right\|_{\ell^2(L^2)} \qquad \leq c(h+\Delta t).$$

Numerical experiments

Experimental order of convergence

Computational domain: $(x,y,t)\in(0,1)^2\times(0,0.5)$ $\qquad$ $\nu=0.1$, $\varepsilon=0.001$, $\delta_0=1$

h	$e_u\ l^\infty(L^2)$	EOC	$e_u\ l^2(H^1)$	EOC
1/32	$1.95e-02$	1.61	$2.88e-02$	1.31
1/64	$7.65e-03$	1.35	$1.20e-02$	1.26
1/128	$3.35e-03$	1.19	$5.90e-03$	1.03
1/256	$1.58e-03$	1.08	$2.66e-03$	1.15
h	$e_p\ l^2(L^2)$	EOC	$e_p\ l^2(\lvert.\rvert_h)$	EOC
1/32	$9.19e-02$	1.45	$6.08e-02$	1.76
1/64	$3.33e-02$	1.47	$2.11e-02$	1.53
1/128	$1.29e-02$	1.37	$8.78e-03$	1.26
1/256	$5.49e-03$	1.23	$3.74e-03$	1.23
h	$e_c\ l^\infty(L^2)$	EOC	$e_c\ l^2(H^1)$	EOC
1/32	$1.94e-02$	1.42	$2.55e-01$	1.09
1/64	$7.55e-03$	1.36	$1.05e-01$	1.28
1/128	$3.28e-03$	1.20	$3.88e-02$	1.44
1/256	$1.53e-03$	1.10	$1.35e-02$	1.52

h	$e_u\ l^\infty(L^2)$	EOC	$e_u\ l^2(H^1)$	EOC
1/32	$1.75e-02$	–	$2.71e-02$	–
1/64	$6.74e-03$	1.37	$1.12e-02$	1.28
1/128	$2.91e-03$	1.21	$5.49e-03$	1.03
1/256	$1.37e-03$	1.09	$2.44e-03$	1.17
h	$e_p\ l^2(L^2)$	EOC	$e_p\ l^2(\lvert.\rvert_h)$	EOC
1/32	$9.77e-02$	–	$6.56e-02$	–
1/64	$3.17e-02$	1.62	$2.22e-02$	1.56
1/128	$1.02e-02$	1.63	$9.01e-03$	1.30
1/256	$3.62e-03$	1.50	$3.78e-03$	1.25
h	$e_c\ l^\infty(L^2)$	EOC	$e_c\ l^2(H^1)$	EOC
1/32	$2.06e-02$	–	$2.76e-01$	–
1/64	$7.36e-03$	1.49	$1.16e-01$	1.25
1/128	$2.93e-03$	1.33	$4.40e-02$	1.40
1/256	$1.31e-03$	1.17	$1.51e-02$	1.54

Acknowledgement

The present work has been supported by the DFG IRTG 1529 "Mathematical Fluid Dynamics" and partially by the DFG IRTG within TRR 146 "Multiscale Simulation Methods for Soft Matter Systems". It has been realized in collaboration with Šárka Nečasová and Michael Renardy.

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