The Role of Pressure in the Theory of Weak Solutions to the Navier–Stokes Equations

Jiří Neustupa

Mathematical Institute of the Czech Academy of Sciences

Prague, Czech Republic

Fluids under Pressure

Summer school, organized by T. Bodnár, G. P. Galdi and Š. Nečasová

August 29 – September 2, 2016, Prague

Contents

1. Notation, auxiliary lemmas.

$\Omega \dots$ a domain in $\mathbb{R}^3$

- $\mathbf{C}^\infty_{0,\sigma}(\Omega) \dots$ the linear space of infinitely differentiable divergence–free vector functions in Ω, with a compact support in Ω,
- $\mathbf{L}^2_\sigma(\Omega)$ (for $1 < q < \infty$) $\dots$ the closure of $\mathbf{C}^\infty_{0,\sigma}(\Omega)$ in $\mathbf{L}^q(\Omega)$,
- $\mathbf{W}^{1,2}_{0,\sigma}(\Omega) \dots$ the closure of $\mathbf{C}^\infty_{0,\sigma}(\Omega)$ in $\mathbf{W}^{1,2}(\Omega)$,
- $\mathbf{W}^{-1,2}_{0,\sigma}(\Omega) \dots$ the dual space to $\mathbf{W}^{1,2}_{0,\sigma}(\Omega)$,
- $\mathbf{W}^{-1,2}_{0}(\Omega) \dots$ the dual space to $\mathbf{W}^{1,2}_{0}(\Omega)$.
- The duality between elements of $\mathbf{W}^{-1,2}_{0,\sigma}(\Omega)$ and $\mathbf{W}^{1,2}_{0,\sigma}(\Omega)$ is denoted by $\langle\, .\, ,\, .\,\rangle_\sigma$, while the duality between elements of $\mathbf{W}^{-1,2}_{0}(\Omega)$ and $\mathbf{W}^{1,2}_{0}(\Omega)$ is denoted by $\langle\, .\, ,\, .\,\rangle$.

Obviously, $\mathbf{W}^{1,2}_{0,\sigma}(\Omega) \subset \mathbf{W}^{1,2}_0(\Omega)$ (with the same norms). Hence, if $\mathbf{f} \in \mathbf{W}^{-1,2}_0(\Omega)$ then $\mathbf{f}$ is also a bounded linear functional on $\mathbf{W}^{1,2}_{0,\sigma}(\Omega)$ in the sense that the duality between $\mathbf{f}$ and $\boldsymbol{\varphi} \in \mathbf{W}^{1,2}_{0,\sigma}(\Omega)$ is $\langle \mathbf{f}, \boldsymbol{\varphi} \rangle$. Thus, in this sense $\mathbf{W}^{-1,2}_0(\Omega) \subset \mathbf{W}^{-1,2}_{0,\sigma}(\Omega)$ (algebraically).

However, in order to avoid confusion, it is further advantageous to distinguish between 1) $\mathbf{f}$ as an element of $\mathbf{W}^{-1,2}_0(\Omega)$ (which acts on elements of $\mathbf{W}^{1,2}_0(\Omega)$ through the duality $\langle\, .\, ,\, .\, \rangle$) and 2) $\mathbf{f}$ as an element of $\mathbf{W}^{-1,2}_{0,\sigma}(\Omega)$ (which acts on elements of $\mathbf{W}^{1,2}_{0,\sigma}(\Omega)$ through the duality $\langle\, .\, ,\, .\, \rangle_\sigma$). Therefore we prefer to write $\mathcal{P}_\sigma \mathbf{f}$ instead of $\mathbf{f}$ in 2).

More precisely, we define $\mathcal{P}_\sigma$ as a linear mapping of $\mathbf{W}^{-1,2}_0(\Omega)$ to $\mathbf{W}^{-1,2}_{0,\sigma}(\Omega)$ by the equation

$$\langle \mathcal{P}_\sigma \mathbf{f}, \boldsymbol{\varphi} \rangle_\sigma := \langle \mathbf{f}, \boldsymbol{\varphi} \rangle \qquad \text{for all } \mathbf{f} \in \mathbf{W}^{-1,2}_0(\Omega) \text{ and } \boldsymbol{\varphi} \in \mathbf{W}^{1,2}_{0,\sigma}(\Omega).$$

Lemma 1. *$\mathcal{P}_\sigma$ is bounded, $R(\mathcal{P}_\sigma) = \mathbf{W}^{-1,2}_{0,\sigma}(\Omega)$, $\mathcal{P}_\sigma$ is not 1–1.*

Proof. $\mathcal{P}_\sigma$ is bounded: Let $\mathbf{f} \in \mathbf{W}^{-1,2}_0(\Omega)$. Then

$$\|\mathcal{P}_\sigma \mathbf{f}\|_{\mathbf{W}^{-1,2}_{0,\sigma}} = \sup_{\boldsymbol{\varphi} \in \mathbf{W}^{1,2}_{0,\sigma};\ \boldsymbol{\varphi} \neq \mathbf{0}} \frac{|\langle \mathcal{P}_\sigma \mathbf{f}, \boldsymbol{\varphi} \rangle_\sigma|}{\|\boldsymbol{\varphi}\|_{\mathbf{W}^{1,2}_{0,\sigma}}} = \sup_{\boldsymbol{\varphi} \in \mathbf{W}^{1,2}_{0,\sigma};\ \boldsymbol{\varphi} \neq \mathbf{0}} \frac{|\langle \mathbf{f}, \boldsymbol{\varphi} \rangle|}{\|\boldsymbol{\varphi}\|_{\mathbf{W}^{1,2}_0}}$$

$$\leq \sup_{\boldsymbol{\varphi} \in \mathbf{W}^{1,2}_0;\ \boldsymbol{\varphi} \neq \mathbf{0}} \frac{|\langle \mathbf{f}, \boldsymbol{\varphi} \rangle|}{\|\boldsymbol{\varphi}\|_{\mathbf{W}^{1,2}_0}} = \|\mathbf{f}\|_{\mathbf{W}^{-1,2}}.$$

$\mathcal{P}_\sigma$ is onto: given $\mathbf{w} \in \mathbf{W}^{-1,2}_{0,\sigma}(\Omega)$, there exists (by the Hahn–Banach theorem) an extension $\mathbf{w}_{\mathrm{ext}} \in \mathbf{W}^{-1,2}_0(\Omega)$ such that $\langle \mathbf{w}_{\mathrm{ext}}, \boldsymbol{\varphi} \rangle := \langle \mathbf{w}, \boldsymbol{\varphi} \rangle_\sigma$ for all $\boldsymbol{\varphi} \in \mathbf{W}^{1,2}_{0,\sigma}(\Omega)$. Hence $\mathbf{w} = \mathcal{P}_\sigma(\mathbf{w}_{\mathrm{ext}})$.

$\mathcal{P}_\sigma$ is not 1–1: taking $\mathbf{f} = \nabla g$ for $g \in L^2(\Omega)$, we get $\mathcal{P}_\sigma \mathbf{f} = \mathbf{0}$. ∎

Remark 1. If $\mathbf{v} \in \mathbf{L}^2(\Omega)$ and $\boldsymbol{\varphi} \in \mathbf{W}^{1,2}_{0,\sigma}(\Omega)$ then

$$\begin{aligned} \langle \mathcal{P}_\sigma \mathbf{v}, \boldsymbol{\varphi} \rangle_\sigma \;&=\; \langle \mathbf{v}, \boldsymbol{\varphi} \rangle \;=\; \int_\Omega \mathbf{v} \cdot \boldsymbol{\varphi} \, \mathrm{d}\mathbf{x} \\ &=\; \int_\Omega \mathbf{v} \cdot P_\sigma \boldsymbol{\varphi} \, \mathrm{d}\mathbf{x} \;=\; \int_\Omega P_\sigma \mathbf{v} \cdot \boldsymbol{\varphi} \, \mathrm{d}\mathbf{x} \;=\; \langle P_\sigma \mathbf{v}, \boldsymbol{\varphi} \rangle_\sigma. \end{aligned}$$

Hence $\mathcal{P}_\sigma \mathbf{v} = P_\sigma \mathbf{v}$, where P_σ is the Helmholtz projection in $\mathbf{L}^2(\Omega)$.

Particularly, if $\mathbf{v} \in \mathbf{L}^2_\sigma(\Omega)$, we have $\mathcal{P}_\sigma \mathbf{v} = \mathbf{v}$. ■

In order to obtain more information on the space $\mathbf{W}^{-1,2}_{0,\sigma}(\Omega)$, its relation to $\mathbf{W}^{-1,2}_0(\Omega)$, and on mapping $\mathcal{P}_\sigma$,

- we denote $\mathbf{W}^{1,2}_{0,\sigma}(\Omega)^\perp := \big\{ \mathbf{f} \in \mathbf{W}^{-1,2}_0(\Omega);\, \forall \boldsymbol{\varphi} \in \mathbf{W}^{1,2}_{0,\sigma}(\Omega) : \langle \mathbf{f}, \boldsymbol{\varphi} \rangle = 0 \big\}$ (the space of annihilators of $\mathbf{W}^{1,2}_{0,\sigma}(\Omega)$),

- for $\mathbf{f} \in \mathbf{W}^{-1,2}_{0,\sigma}(\Omega)$, we denote by $\mathbf{f}_{\mathrm{ext}}$ an extension of $\mathbf{f}$ to a bounded linear functional on $\mathbf{W}^{1,2}_0(\Omega)$ – the extension exists due to the Hahn–Banach theorem and satisfies

$$\langle \mathbf{f}_{\mathrm{ext}}, \boldsymbol{\varphi} \rangle \;=\; \langle \mathbf{f}, \boldsymbol{\varphi} \rangle_\sigma \quad \text{for all } \boldsymbol{\varphi} \in \mathbf{W}^{1,2}_{0,\sigma}(\Omega),$$

- we define mapping $\sigma : \mathbf{W}^{-1,2}_{0,\sigma}(\Omega) \longrightarrow \mathbf{W}^{-1,2}_{0}(\Omega)\big|_{\mathbf{W}^{1,2}_{0,\sigma}(\Omega)^\perp}$ (the quotient space) by the equation

$$\sigma(\mathbf{f}) \;:=\; \mathbf{f}_{\mathrm{ext}} + \mathbf{W}^{1,2}_{0,\sigma}(\Omega)^\perp.$$

In fact, the definition of mapping σ is independent of the choice of a concrete extension of $\mathbf{f} \in \mathbf{W}^{1,2}_{0,\sigma}(\Omega)$ to $\mathbf{f}_{\mathrm{ext}} \in \mathbf{W}^{-1,2}_{0}(\Omega)$: let $\mathbf{f}'_{\mathrm{ext}}$ and $\mathbf{f}''_{\mathrm{ext}}$ be two such extensions. They coincide on $\mathbf{W}^{1,2}_{0,\sigma}(\Omega)$, hence $\mathbf{f}'_{\mathrm{ext}} - \mathbf{f}''_{\mathrm{ext}} \in \mathbf{W}^{1,2}_{0,\sigma}(\Omega)^\perp$. Denote by σ' the mapping defined by means of the extension $\mathbf{f}'_{\mathrm{ext}}$ and by σ'' the mapping defined by the extension $\mathbf{f}''_{\mathrm{ext}}$. Then, for $\mathbf{f} \in \mathbf{W}^{-1,2}_{0,\sigma}(\Omega)$, we have

$$\sigma'\mathbf{f} - \sigma''\mathbf{f} \;=\; (\mathbf{f}'_{\mathrm{ext}} - \mathbf{f}''_{\mathrm{ext}}) + \mathbf{W}^{1,2}_{0,\sigma}(\Omega)^{\perp} \;=\; \mathbf{W}^{1,2}_{0,\sigma}(\Omega)^{\perp},$$

which is the zero element of the quotient space $\mathbf{W}_0^{-1,2}(\Omega)\big|_{\mathbf{W}^{1,2}_{0,\sigma}(\Omega)^{\perp}}$.

Applying [24, Theorem 4.9], we obtain

Lemma 2. *σ is an isometric isomorphism of $\mathbf{W}^{-1,2}_{0,\sigma}(\Omega)$ onto $\mathbf{W}_0^{-1,2}(\Omega)\big|_{\mathbf{W}^{1,2}_{0,\sigma}(\Omega)^{\perp}}$.*

If we denote by q the so called *quotient mapping* of $\mathbf{W}_0^{-1,2}(\Omega)$ onto $\mathbf{W}_0^{-1,2}(\Omega)\big|_{\mathbf{W}^{-1,2}_{0,\sigma}(\Omega)^{\perp}}$, which is the mapping defined by the equation

$$q(\mathbf{g}) \;:=\; \mathbf{g} + \mathbf{W}^{-1,2}_{0,\sigma}(\Omega)^{\perp} \quad \text{for } \mathbf{g} \in \mathbf{W}_0^{-1,2}(\Omega),$$

then we naturally have $\mathbf{W}_0^{-1,2}(\Omega)\big|_{\mathbf{W}_{0,\sigma}^{-1,2}(\Omega)^\perp} = q\big(\mathbf{W}_0^{-1,2}(\Omega)\big)$.

Moreover, we also have $\mathbf{W}_0^{-1,2}(\Omega)\big|_{\mathbf{W}_{0,\sigma}^{-1,2}(\Omega)^\perp} = \sigma\big(\mathbf{W}_{0,\sigma}^{-1,2}(\Omega)\big)$.

Hence $\sigma^{-1}q$ maps $\mathbf{W}_0^{-1,2}(\Omega)$ onto $\mathbf{W}_{0,\sigma}^{-1,2}(\Omega)$. Concretely, if $\mathbf{g} \in \mathbf{W}_0^{-1,2}(\Omega)$ then $\sigma^{-1}q(\mathbf{g})$ is an element $\mathbf{f} \in \mathbf{W}_{0,\sigma}^{-1,2}(\Omega)$ such that $\sigma(\mathbf{f}) = q(\mathbf{g})$, which means that

$$\mathbf{f}_{\mathrm{ext}} + \mathbf{W}_{0,\sigma}^{-1,2}(\Omega)^\perp = \mathbf{g} + \mathbf{W}_{0,\sigma}^{-1,2}(\Omega)^\perp.$$

Hence $\mathbf{f}_{\mathrm{ext}} - \mathbf{g} \in \mathbf{W}_{0,\sigma}^{-1,2}(\Omega)^\perp$. Consequently,

$$\langle \mathbf{g}, \boldsymbol{\varphi}\rangle = \langle \mathbf{f}_{\mathrm{ext}}, \boldsymbol{\varphi}\rangle = \langle \mathbf{f}, \boldsymbol{\varphi}\rangle_\sigma,$$

for all $\boldsymbol{\varphi} \in \mathbf{W}_{0,\sigma}^{1,2}(\Omega)$. This yields $\mathbf{f} = \mathcal{P}_\sigma \mathbf{g}$. Since $\mathbf{f} = \sigma^{-1}q(\mathbf{g})$, we obtain:

Lemma 3. $\mathcal{P}_\sigma = \sigma^{-1}q$

The next lemma provides more information on elements of $\mathbf{W}^{1,2}_{0,\sigma}(\Omega)^{\perp}$ in the case of domain Ω that satisfies the cone condition.

Lemma 4. *Let Ω be bounded or exterior domain in $\mathbb{R}^n$ ($n \geq 2$), that satisfies the cone condition. Let $\mathcal{F} \in \mathbf{W}^{-1,2}_0(\Omega)$ be a bounded linear functional on $\mathbf{W}^{1,2}_0(\Omega)$, that vanishes on $\mathbf{W}^{1,2}_{0,\sigma}(\Omega)$ (i.e. $\mathcal{F} \in \mathbf{W}^{1,2}_{0,\sigma}(\Omega)^{\perp}$). Then there exists a function $p \in L^2(\Omega)$ such that*

$$\mathcal{F}(\mathbf{v}) \; = \int_{\Omega} p \, \mathrm{div}\, \mathbf{v} \, \mathrm{d}\mathbf{x}$$

for all $\mathbf{v} \in \mathbf{W}^{1,2}_0(\Omega)$. Function p is determined by functional $\mathcal{F}$ uniquely (up to an additive constant in the case of Ω bounded).

The lemma follows from [10, Corollary III.5.1]. The right hand side can also be written in the form $\langle \nabla p, \mathbf{v} \rangle$, where ∇p is the gradient of p in the sense of distributions.

The next lemma holds for any domain Ω. It tells us what form has a distribution that vanishes on divergence–free functions.

Lemma 5. *Let Ω be any domain in $\mathbb{R}^n$ ($n \geq 2$) and $\mathbf{f} = (f_1, \dots, f_n)$, where f_i ($i = 1, \dots, n$) are distributions in Ω. Then $\mathbf{f}$ has the form $\mathbf{f} = \nabla p$ (where p is a distribution in Ω and ∇p is the distributional gradient) if and only if*

$$\langle \mathbf{f}, \boldsymbol{\varphi} \rangle = 0 \qquad \text{for all } \boldsymbol{\varphi} \in \mathbf{C}^{\infty}_{0,\sigma}(\Omega).$$

The lemma coincides with Proposition I.1.1 in [36]. It comes from G. De Rham.

The next lemma plays an important role in studies of non–steady problems. It coincides with Lemma III.1.1 in [36].

Lemma 6. *Let X be a Banach space with the dual X' and let $\mathbf{u}, \mathbf{g} \in L^1(a,b;\, X)$. Then the following three conditions are equivalent:*

- $\mathbf{u}$ *is a.e. in (a,b) equal to a primitive function of* $\mathbf{g}$,
- $\displaystyle\int_a^b \dot{\vartheta}(t)\,\mathbf{u}(t)\;\mathrm{d}t = -\int_a^b \vartheta(t)\,\mathbf{g}(t)\;\mathrm{d}t$ *for all* $\vartheta \in C_0^\infty((a,b))$,
- $\displaystyle\frac{\mathrm{d}}{\mathrm{d}t}\,\langle \boldsymbol{\eta}, \mathbf{u}\rangle \;=\; \langle \boldsymbol{\eta}, \mathbf{g}\rangle$ *in the sense of distributions in (a,b) for each $\boldsymbol{\eta} \in X'$.*

(Here, $\langle\, .\,,\, .\,\rangle$ denotes the duality between X' and X.)

2. A pressure associated with a weak solution to the Navier–Stokes equations.

$\Omega \ldots$ a domain in $\mathbb{R}^3$, $\quad T > 0$,

$Q_T := \Omega \times (0, T)$,

$\Gamma_T := \partial\Omega \times (0, T)$

The Navier–Stokes initial–boundary value problem in Q_T:

$$\partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} + \nabla p = \nu \Delta \mathbf{u} + \mathbf{f} \qquad \text{in } Q_T, \tag{2.1}$$

$$\operatorname{div} \mathbf{u} = 0 \qquad \text{in } Q_T, \tag{2.2}$$

$$\mathbf{u} = \mathbf{0} \qquad \text{in } \Gamma_T, \tag{2.3}$$

$$\mathbf{u}\big|_{t=0} = \mathbf{u}_0 \qquad \text{in } \Omega. \tag{2.4}$$

The weak formulation of the IBVP problem (2.1)–(2.4):

Given $\mathbf{u}_0 \in \mathbf{L}^2_\sigma(\Omega)$ and $\mathbf{f} \in L^2\big(0,T;\ \mathbf{W}^{-1,2}_{0,\sigma}(\Omega)\big)$.

We look for $\mathbf{u} \in L^\infty\big(0,T;\ \mathbf{L}^2_\sigma(\Omega)\big) \cap L^2\big(0,T;\ \mathbf{W}^{1,2}_{0,\sigma}(\Omega)\big)$ (the so called *weak solution*) such that

$$\int_0^T \int_\Omega \big[-\mathbf{u}\cdot\partial_t\boldsymbol{\phi} + \nu\nabla\mathbf{u} : \nabla\boldsymbol{\phi} + \mathbf{u}\cdot\nabla\mathbf{u}\cdot\boldsymbol{\phi}\big]\,\mathrm{d}\mathbf{x}\,\mathrm{d}t = \int_\Omega \mathbf{u}_0\cdot\boldsymbol{\phi}(\mathbf{x},0)\,\mathrm{d}\mathbf{x} + \int_0^T \langle\mathbf{f},\boldsymbol{\phi}\rangle_\sigma\,\mathrm{d}t \tag{2.5}$$

for all $\boldsymbol{\phi} \in C^\infty\big([0,T];\ \mathbf{W}^{1,2}_{0,\sigma}(\Omega)\big)$ such that $\boldsymbol{\phi}(\,.\,,T) = \mathbf{0}$.

1st equivalent formulation (see e.g. [36]) :

Given $\mathbf{u}_0 \in \mathbf{L}^2_\sigma(\Omega)$ and $\mathbf{f} \in L^2\big(0,T;\ \mathbf{W}^{-1,2}_{0,\sigma}(\Omega)\big)$. We look for $\mathbf{u} \in L^\infty\big(0,T;\ \mathbf{L}^2_\sigma(\Omega)\big) \cap L^2\big(0,T;\ \mathbf{W}^{1,2}_{0,\sigma}(\Omega)\big)$ such that $\mathbf{u}$ satisfies the initial condition (2.4) and

$$\frac{\mathrm{d}}{\mathrm{d}t}\,(\mathbf{u},\boldsymbol{\varphi}) + \nu\,\langle\mathcal{A}\mathbf{u},\boldsymbol{\varphi}\rangle_\sigma + \langle\mathcal{B}(\mathbf{u},\mathbf{u}),\boldsymbol{\varphi}\rangle_\sigma \;=\; \langle\mathbf{f},\varphi\rangle_\sigma \tag{2.6}$$

for all $\boldsymbol{\varphi} \in \mathbf{W}^{1,2}_{0,\sigma}(\Omega)$ and a.e. in $(0,T)$, where the operators $\mathcal{A} : \mathbf{W}^{1,2}_{0,\sigma}(\Omega) \longrightarrow \mathbf{W}^{-1,2}_{0,\sigma}(\Omega)$ and $\mathcal{B} : \mathbf{W}^{1,2}_{0,\sigma}(\Omega) \times \mathbf{W}^{1,2}_{0,\sigma}(\Omega) \longrightarrow \mathbf{W}^{-1,2}_{0,\sigma}(\Omega)$ are defined by the equations

$$\big\langle\mathcal{A}\mathbf{v},\boldsymbol{\varphi}\big\rangle_\sigma \;:=\; \int_\Omega \nabla\mathbf{v} : \nabla\boldsymbol{\varphi}\,\mathrm{d}\mathbf{x} \qquad \text{for } \mathbf{v},\boldsymbol{\varphi} \in \mathbf{W}^{1,2}_{0,\sigma}(\Omega),$$

$$\big\langle\mathcal{B}(\mathbf{v},\mathbf{w}),\boldsymbol{\varphi}\big\rangle_\sigma \;:=\; \int_\Omega \mathbf{v}\cdot\nabla\mathbf{w}\cdot\boldsymbol{\varphi}\,\mathrm{d}\mathbf{x} \qquad \text{for } \mathbf{v},\mathbf{w},\boldsymbol{\varphi} \in \mathbf{W}^{1,2}_{0,\sigma}(\Omega).$$

It is easy to verify that if $\mathbf{u}$ is a weak solution then

$$\mathcal{A}\mathbf{u} \in L^2(0,T;\, \mathbf{W}^{-1,2}_{0,\sigma}(\Omega)),$$
$$\mathcal{B}(\mathbf{u},\mathbf{u}) \in L^{4/3}(0,T;\, \mathbf{W}^{-1,2}_{0,\sigma}(\Omega)).$$

Applying Lemma 6 (with $X = \mathbf{W}^{-1,2}_{0,\sigma}(\Omega)$), we deduce that (2.6) is equivalent to

$$\mathbf{u}' + \nu\mathcal{A}\mathbf{u} + \mathcal{B}(\mathbf{u},\mathbf{u}) \;=\; \mathbf{f}, \tag{2.7}$$

which is an equation in $\mathbf{W}^{-1,2}_{0,\sigma}(\Omega)$, satisfied for a.a. $t \in (0,T)$. Here, $\mathbf{u}'$ denotes the derivative with respect to t of $\mathbf{u}$, as a function from $(0,T)$ to $\mathbf{W}^{-1,2}_{0,\sigma}(\Omega)$. Thus, we obtain:

2nd equivalent formulation: Given $\mathbf{u}_0 \in \mathbf{L}^2_\sigma(\Omega)$ and $\mathbf{f} \in L^2\big(0,T;\, \mathbf{W}^{-1,2}_{0,\sigma}(\Omega)\big)$. We look for $\mathbf{u} \in L^\infty\big(0,T;\, \mathbf{L}^2_\sigma(\Omega)\big) \cap L^2\big(0,T;\, \mathbf{W}^{1,2}_{0,\sigma}(\Omega)\big)$ such that $\mathbf{u}$ satisfies the initial condition (2.4) and the equation (2.7) is satisfied for a.a. $t \in (0,T)$.

Since

$$\left\langle \mathcal{A}\mathbf{v}, \boldsymbol{\varphi} \right\rangle_\sigma = \langle -\Delta \mathbf{v}, \boldsymbol{\varphi} \rangle \qquad \text{for } \mathbf{v}, \boldsymbol{\varphi} \in \mathbf{W}^{1,2}_{0,\sigma}(\Omega),$$
$$\left\langle \mathcal{B}(\mathbf{v}, \mathbf{w}), \boldsymbol{\varphi} \right\rangle_\sigma = \langle \mathbf{v} \cdot \nabla \mathbf{w}, \boldsymbol{\varphi} \rangle \qquad \text{for } \mathbf{v}, \mathbf{w}, \boldsymbol{\varphi} \in \mathbf{W}^{1,2}_{0,\sigma}(\Omega).$$

we have: $\mathcal{A}\mathbf{v} = \mathcal{P}_\sigma(-\Delta \mathbf{v})$ and $\mathcal{B}(\mathbf{v}, \mathbf{w}) = \mathcal{P}_\sigma(\mathbf{v} \cdot \nabla \mathbf{w})$. Thus, equation (2.7) can also be written in the form

$$\mathbf{u}' - \nu \mathcal{P}_\sigma(\Delta \mathbf{u}) + \mathcal{P}_\sigma(\mathbf{u} \cdot \nabla \mathbf{u}) = \mathbf{f}. \tag{2.8}$$

Since all the terms $\mathcal{P}_\sigma(\Delta \mathbf{u})$, $\mathcal{P}_\sigma(\mathbf{u} \cdot \nabla \mathbf{u})$ and $\mathbf{f}$ are in $L^{4/3}(0, T;\ \mathbf{W}^{-1,2}_{0,\sigma}(\Omega))$, $\mathbf{u}'$ is in $L^{4/3}(0, T;\ \mathbf{W}^{-1,2}_{0,\sigma}(\Omega))$, too.

Associated pressure. Let $\mathbf{u}$ be a weak solution to the problem (2.1)–(2.4). If there exists a distribution p in Q_T such that the Navier–Stokes equation (2.1) is satisfied in Q_T in the sense of distributions then p is called an *associated pressure* to the weak solution $\mathbf{u}$.

Remark 2. $\mathbf{u}'$ cannot be identified with the distributional derivative of $\mathbf{u}$, as a function from $(0,T)$ to $\mathbf{W}_0^{-1,2}(\Omega)$.

In order to show it, denote the distributional derivative of $\mathbf{u}$, as a function from (a,b) to $\mathbf{W}_0^{-1,2}(\Omega)$, by $\dot{\mathbf{u}}$ and assume that $\dot{\mathbf{u}} \in L^1(a,b;\, \mathbf{W}_0^{-1,2}(\Omega))$. Then we have

$$\frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{u},\boldsymbol{\varphi}) \;=\; \frac{\mathrm{d}}{\mathrm{d}t}\langle \mathbf{u},\boldsymbol{\varphi}\rangle \;=\; \langle \dot{\mathbf{u}},\boldsymbol{\varphi}\rangle$$

for $\boldsymbol{\varphi} \in \mathbf{W}_0^{1,2}(\Omega)$.

Particularly, if $\boldsymbol{\varphi} \in \mathbf{W}_{0,\sigma}^{1,2}(\Omega)$ then $\langle \dot{\mathbf{u}},\boldsymbol{\varphi}\rangle \;=\; \langle \mathcal{P}_\sigma \dot{\mathbf{u}},\boldsymbol{\varphi}\rangle_\sigma$.

For these $\boldsymbol{\varphi}$, we also have

$$\frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{u},\boldsymbol{\varphi}) \;=\; \frac{\mathrm{d}}{\mathrm{d}t}\langle \mathbf{u},\boldsymbol{\varphi}\rangle_\sigma \;=\; \langle \mathbf{u}',\boldsymbol{\varphi}\rangle_\sigma.$$

From this, we obtain: $\mathbf{u}' = \mathcal{P}_\sigma \dot{\mathbf{u}}$.

Existence of an associated pressure. Let $\mathbf{u}$ be a weak solution. Assume that $\mathbf{f} = \mathcal{P}_\sigma \widetilde{\mathbf{f}}$, where $\widetilde{\mathbf{f}} \in L^2(0,T;\, \mathbf{W}_0^{-1,2}(\Omega))$. Then equation (2.8) takes the form

$$\mathbf{u}' - \nu \mathcal{P}_\sigma(\Delta \mathbf{u}) + \mathcal{P}_\sigma(\mathbf{u} \cdot \nabla \mathbf{u}) \;=\; \mathcal{P}_\sigma \widetilde{\mathbf{f}}.$$

Integrating with respect to time from 0 to t, we get

$$\mathbf{u}(\,.\,,t) - \mathbf{u}(\,.\,,0) + \int_0^t \mathcal{P}_\sigma\big[-\nu\Delta\mathbf{u} + \mathbf{u}\cdot\nabla\mathbf{u}\big]\,\mathrm{d}\tau \;=\; \int_0^t \mathcal{P}_\sigma \widetilde{\mathbf{f}}\,\mathrm{d}\tau.$$

Since $\mathbf{u}(\,.\,,t)$ and $\mathbf{u}(\,.\,,0)$ are in $\mathbf{L}^2_\sigma(\Omega)$, they coincide with $\mathcal{P}_\sigma\mathbf{u}(\,.\,,t)$ and $\mathcal{P}_\sigma\mathbf{u}(\,.\,,0)$, respectively. Hence

$$\mathcal{P}_\sigma\Big[\mathbf{u}(\,.\,,t) - \mathbf{u}(\,.\,,0) + \int_0^t \big[-\nu\Delta\mathbf{u} + \mathbf{u}\cdot\nabla\mathbf{u}\big]\,\mathrm{d}\tau\Big] \;=\; \mathcal{P}_\sigma \int_0^t \widetilde{\mathbf{f}}\,\mathrm{d}\tau.$$

Define for $\boldsymbol{\varphi} \in \mathbf{C}_0^\infty(\Omega)$

$$\mathcal{F}(\boldsymbol{\varphi}) \;:=\; \Big\langle \mathbf{u}(\,.\,,t) - \mathbf{u}(\,.\,,0) + \int_0^t \big[-\nu\Delta\mathbf{u} + \mathbf{u}\cdot\nabla\mathbf{u}\big]\,\mathrm{d}\tau - \int_0^t \widetilde{\mathbf{f}}\,\mathrm{d}\tau,\; \boldsymbol{\varphi}\Big\rangle$$

$\mathcal{F}$ is a distribution in Ω, that vanishes for $\boldsymbol{\varphi} \in \mathbf{C}^{\infty}_{0,\sigma}(\Omega)$. Hence, due to Lemma 5, there exists a distribution $P(t)$ in Ω such that

$$\mathbf{u}(\,.\,,t) - \mathbf{u}(\,.\,,0) + \int_0^t \left[-\nu\Delta\mathbf{u} + \mathbf{u}\cdot\nabla\mathbf{u}\right]\mathrm{d}\tau - \int_0^t \widetilde{\mathbf{f}}\,\mathrm{d}\tau \;=\; -\nabla P(t).$$

The left hand side is, in dependence on t, in $L^{\infty}([0,T);\ \mathbf{W}_0^{-1,2}(\Omega))$. Hence the right hand side is also in $L^{\infty}([0,T);\ \mathbf{W}_0^{-1,2}(\Omega))$.

Let $\boldsymbol{\phi} \in \mathbf{C}^{\infty}_0(Q_T)$. Applying the sides of the equation to $\partial_t\boldsymbol{\phi}(\,.\,,t)$, we get

$$\begin{aligned}
&\int_{\Omega}\left[\mathbf{u}(\,.\,,t) - \mathbf{u}(\,.\,,0)\right]\cdot\partial_t\boldsymbol{\phi}(\,.\,,t)\,\mathrm{d}\mathbf{x} \\
&\quad + \int_0^t\int_{\Omega}\left[\nu\nabla\mathbf{u}(\,.\,,\tau) : \nabla\partial_t\boldsymbol{\phi}(\,.\,,t) + \mathbf{u}(\,.\,,\tau)\cdot\nabla\mathbf{u}(\,.\,,\tau)\cdot\partial_t\boldsymbol{\phi}(\,.\,,t)\right]\mathrm{d}\mathbf{x}\,\mathrm{d}\tau \\
&\qquad = \int_0^t \big\langle\widetilde{\mathbf{f}}(\tau), \partial_t\boldsymbol{\phi}(\,.\,,t)\big\rangle\,\mathrm{d}\tau - \big\langle\nabla P(t), \partial_t\boldsymbol{\phi}(\,.\,,t)\big\rangle.
\end{aligned}$$

Integrating this equation with respect to t from 0 to T and applying the integration by parts, we obtain

$$\int_0^T\int_\Omega \mathbf{u}(\,.\,,t)\cdot\partial_t\boldsymbol{\phi}(\,.\,,t)\;\mathrm{d}\mathbf{x}\,\mathrm{d}t-\int_0^T\int_\Omega\big[\nu\nabla\mathbf{u}:\nabla\boldsymbol{\phi}+\mathbf{u}\cdot\nabla\mathbf{u}\cdot\boldsymbol{\phi}\big]\;\mathrm{d}\mathbf{x}\,\mathrm{d}\tau$$

$$=\;-\int_0^T\big\langle\widetilde{\mathbf{f}},\boldsymbol{\phi}(\,.\,,t)\big\rangle\;\mathrm{d}\tau-\int_0^T\big\langle\nabla P,\partial_t\boldsymbol{\phi}\big\rangle\;\mathrm{d}t$$

If we denote by $\langle\!\langle\,.\,,\,.\,\rangle\!\rangle_{Q_T}$ the duality between a distribution in Q_T and a function from $\mathbf{C}_0^\infty(Q_T)$ then the last term can be written:

$$\ldots\;=\;-\langle\!\langle\nabla P,\partial_t\boldsymbol{\phi}\rangle\!\rangle_{Q_t}\;=\;\langle\!\langle\nabla\partial_t P,\boldsymbol{\phi}\rangle\!\rangle_{Q_T}.$$

(We identify $\nabla P\in L^\infty(0,T;\,\mathbf{W}_0^{-1,2}(\Omega))$ with a distribution in Q_T.) The whole equation can be written in the form

$$\langle\!\langle\partial_t\mathbf{u}-\nu\Delta\mathbf{u}+\mathbf{u}\cdot\nabla\mathbf{u}-\widetilde{\mathbf{f}},\;\boldsymbol{\phi}\rangle\!\rangle_{Q_T}\;=\;-\langle\!\langle\nabla\partial_t P,\boldsymbol{\phi}\rangle\!\rangle_{Q_T}.$$

From this, we observe that $\mathbf{u}$ and $p\equiv\partial_t P$ satisfy the Navier–Stokes equation (with the right hand side $\widetilde{\mathbf{f}}$) in the sense of distributions in Q_T.

Remark 3. The weak solution $\mathbf{u}$ can be redefined on a set of measure zero so that it satisfies

$$\begin{aligned}\int_0^t \int_\Omega \big[\nu\nabla\mathbf{u} : \nabla\boldsymbol{\psi} + \mathbf{u}\cdot\nabla\mathbf{u}\cdot\boldsymbol{\psi}\big]\,\mathrm{d}\mathbf{x}\,\mathrm{d}\tau - \int_0^t \langle\mathbf{f},\boldsymbol{\psi}\rangle\,\mathrm{d}\tau \\ = -\int_\Omega \mathbf{u}(\,.\,,t)\cdot\boldsymbol{\psi}\,\mathrm{d}\mathbf{x} + \int_\Omega \mathbf{u}_0\cdot\boldsymbol{\psi}\,\mathrm{d}\mathbf{x}\end{aligned} \tag{2.9}$$

for all $t \in (0,T)$ and $\boldsymbol{\psi} \in \mathbf{W}^{1,2}_{0,\sigma}(\Omega)$. (See [11, Lemma 2.2].) From this, one can deduce that $\mathbf{u}$ is weakly continuous from $[0,T)$ to $\mathbf{L}^2_\sigma(\Omega)$.

One can also prove a reverse statement, i.e. that function $\mathbf{u} \in L^\infty\big(0,T;\, \mathbf{L}^2_\sigma(\Omega)\big) \cap L^2\big(0,T;\, \mathbf{W}^{1,2}_{0,\sigma}(\Omega)\big)$, satisfying (2.9) for all $\boldsymbol{\psi} \in \mathbf{W}^{1,2}_{0,\sigma}(\Omega)$ and for all $t \in (0,T)$, is a weak solution. (See [11, Lemma 2.4].) ■

The next Theorem 1 (which is taken from the paper [11]) by G. P. Galdi) brings a more detailed information on distribution P. Further information is also provided by the papers [28] (by J. Simon) and [37] (by J. Wolf).

More on the associated pressure:

Theorem 1. *Let* $\mathbf{f} \in L^2\big(0,T;\, \mathbf{W}_0^{-1,2}(\Omega)\big)$ *and let* $\mathbf{u}$ *be a weak solution to the problem (2.1)–(2.4). Then there exists a scalar function* P *in* Q_T*, unique up to an additive function of* t*, such that*

- $P(\,.\,,t) \in L^2_{loc}(\Omega)$ *for all* $t \in [0,T)$*,*
- *if* $\Omega' \subset \Omega$ *satisfies the cone condition then* $P \in L^\infty(0,T;\, L^2(\Omega'))$*,*
- *(2.10) holds with* $P(\,.\,,t)$ *instead of* $\big(\int_0^t p \, \mathrm{d}\tau\big)$ *for all* $\boldsymbol{\chi} \in \mathbf{W}_0^{1,2}(\Omega)$ *and all* $t \in [0,T)$*:*

$$\int_0^t \int_\Omega \big[\nu \nabla \mathbf{u} : \nabla \boldsymbol{\chi} + \mathbf{u} \cdot \nabla \mathbf{u} \cdot \boldsymbol{\chi}\big] \, \mathrm{d}\mathbf{x} \, \mathrm{d}\tau - \int_0^t \langle \mathbf{f}, \boldsymbol{\chi} \rangle \, \mathrm{d}\tau = \int_\Omega P(\,.\,,t) \, \mathrm{div}\, \boldsymbol{\chi} \, \mathrm{d}\mathbf{x} - \int_\Omega \mathbf{u}(\,.\,,t) \cdot \boldsymbol{\chi} \, \mathrm{d}\mathbf{x} + \int_\Omega \mathbf{u}_0 \cdot \boldsymbol{\chi} \, \mathrm{d}\mathbf{x}. \tag{2.10}$$

Remark 4. If $\partial\Omega$ is bounded and satisfies the cone condition then P can be chosen so that $P \in L^\infty(0,T;\, L^2(\Omega))$.

Remark 5. Assuming that $\mathbf{f} \in L^2(0,T;\, \mathbf{W}_0^{-1,2}(\Omega))$, we can obtain formally (2.10) (with $P(\,.\,,t) = \int_0^t p(\,.\,,\tau)\, \mathrm{d}\tau$) from the Navier–Stokes equation (2.1) if we multiply it by a function $\boldsymbol{\chi} \in \mathbf{W}_0^{1,2}(\Omega)$ and integrate in $\Omega \times (0,t)$.

On the other hand, if $\mathbf{u}$ and P satisfy (2.10) then, considering the distributional derivative of (2.10) with respect to t, one can deduce that $\mathbf{u}$, $\partial_t P$ satisfy equation (2.1) in the sense of distributions in Q_T. (The arguments are analogous to those used in order to show that function $\mathbf{u}$, satisfying (2.9) for all $\boldsymbol{\psi} \in \mathbf{W}_{0,\sigma}^{1,2}(\Omega)$ and all $t \in (0,T)$, is a weak solution – see [11, Lemma 2.4].)

Example.

Here, we give an example of a simple weak solution to the system (2.1), (2.2), that is not smooth in dependence on time and $\partial_t P(\mathbf{x},t)$ does not exist as a function. The solution, however, does not satisfy the boundary condition (2.3).

Let $a \in C([0,T))$ be such a function that $\dot{a} \notin L^1_{loc}([0,T))$. Let $\mathbf{v} \in \mathbf{W}^{1,2}_{0,\sigma}(\Omega)$ have the form $\mathbf{v} = \nabla\varphi$, where φ is a harmonic function in Ω. Put

$$\mathbf{u}(\mathbf{x},t) \ := \ a(t)\,\mathbf{v}(\mathbf{x}).$$

Then $\mathbf{u}$ is a weak solution to the system (2.1), (2.2) with the initial velocity $\mathbf{u}_0 = a(0)\,\mathbf{v}$ and $\mathbf{f} = \mathbf{0}$ in $\Omega \times (0,T)$.

(It means that $\mathbf{u}$ satisfies (2.5) with $\mathbf{u}_0 = a(0)\,\mathbf{v}$ and $\mathbf{f} = \mathbf{0}$ for all $\boldsymbol{\phi} \in C^\infty\big([0,T);\ \mathbf{W}^{1,2}_{0,\sigma}(\Omega)\big)$ such that $\boldsymbol{\phi}(\,.\,,T) = \mathbf{0}$.)

Equation (2.6) takes the form

$$\int_0^t \int_\Omega \left[\nu a\, \Delta\varphi + \tfrac{1}{2} a^2 \varphi^2\right] \operatorname{div} \boldsymbol{\chi}\, \mathrm{d}\mathbf{x}\, \mathrm{d}\tau$$
$$= \int_\Omega P(\,.\,, t) \operatorname{div} \boldsymbol{\chi}\, \mathrm{d}\mathbf{x} + \big[a(t) - a(0)\big] \int_\Omega \varphi \operatorname{div} \boldsymbol{\chi}\, \mathrm{d}\mathbf{x}.$$

It means that

$$P(\,.\,, t) \;=\; \int_0^t \left[\nu a\, \Delta\varphi + \tfrac{1}{2} a^2 \varphi^2\right] \mathrm{d}\tau - \big[a(t) - a(0)\big]\, \varphi \quad \text{in } \Omega.$$

As $\dot{a} \notin L^1_{loc}([0,T))$, $\partial_t P$ (which is a pressure associated with the weak solution $\mathbf{u}$) exists only as a distribution.

Principle of the proof of Theorem 1. (See the proof of Theorem 2.1 in [11].) Ω is expressed: $\Omega = \bigcup_{k=1}^{\infty} \Omega_k$, where $\Omega_k \subset \Omega_{k+1}$ (for $k = 1, 2, \dots$), all Ω_k are bounded and satisfy the cone condition. Define for $\boldsymbol{\chi} \in \mathbf{W}_0^{1,2}(\Omega_k)$

$$\mathcal{F}(\boldsymbol{\chi}) := \int_0^t \left(\int_{\Omega_k} \left[\nu \nabla \mathbf{u} : \nabla \boldsymbol{\chi} + \mathbf{u} \cdot \nabla \mathbf{u} \cdot \boldsymbol{\chi} \right] \mathrm{d}\mathbf{x} - \langle \mathbf{f}, \boldsymbol{\chi} \rangle \right) \mathrm{d}\tau + \int_{\Omega_k} \mathbf{u}(\,.\,, t) \cdot \boldsymbol{\chi} \, \mathrm{d}\mathbf{x} - \int_{\Omega_k} \mathbf{u}_0 \cdot \boldsymbol{\chi} \, \mathrm{d}\mathbf{x}.$$

$\mathcal{F}$ is a linear functional on $\mathbf{W}_0^{1,2}(\Omega_k)$. It satisfies the estimate

$$|\mathcal{F}(\boldsymbol{\chi})| \leq c \, \|\boldsymbol{\chi}\|_{1,2;\,\Omega_k} \left[\int_0^t \left(\|\nabla \mathbf{u}\|_{2;\,\Omega_k} + \sqrt{M} \, \|\nabla \mathbf{u}\|_{2;\,\Omega_k}^{3/2} + \|\mathbf{f}\|_{-1,2} \right) \mathrm{d}\tau + 2M \right],$$

where $M := \operatorname*{ess\,sup}_{0 \leq \tau \leq T} \|\mathbf{u}(\,.\,,\tau)\|_2$. This shows that functional $\mathcal{F}$ is bounded. Moreover,

$$\mathcal{F}(\boldsymbol{\chi}) = 0 \quad \text{for } \boldsymbol{\chi} \in \mathbf{W}_{0,\sigma}^{1,2}(\Omega_k) \quad \text{(due to (2.9))}.$$

Hence, considering $k = 1$, there exists $P_1 \in L^2(\Omega_1)$ such that

$$\forall\, \boldsymbol{\chi}_1 \in \mathbf{W}_0^{1,2}(\Omega_1) \;:\qquad \mathcal{F}(\boldsymbol{\chi}_1) \;=\; \int_{\Omega_1} P_1 \,\mathrm{div}\, \boldsymbol{\chi}_1 \;\mathrm{d}\mathbf{x}.$$

Similarly, considering $k = 2$, there exists $P_2 \in L^2(\Omega_2)$ such that

$$\forall\, \boldsymbol{\chi}_2 \in \mathbf{W}_0^{1,2}(\Omega_2) \;:\qquad \mathcal{F}(\boldsymbol{\chi}_2) \;=\; \int_{\Omega_2} P_2 \,\mathrm{div}\, \boldsymbol{\chi}_2 \;\mathrm{d}\mathbf{x}.$$

Extending $\boldsymbol{\chi}_1$ by zero to $\Omega_2 \smallsetminus \Omega_1$, we have $\boldsymbol{\chi}_1 \in \mathbf{W}_0^{1,2}(\Omega_2)$. Hence

$$\forall\, \boldsymbol{\chi}_1 \in \mathbf{W}_0^{1,2}(\Omega_1) \;:\qquad \int_{\Omega_1} P_1 \,\mathrm{div}\, \boldsymbol{\chi}_1 \;\mathrm{d}\mathbf{x} \;=\; \int_{\Omega_1} P_2 \,\mathrm{div}\, \boldsymbol{\chi}_1 \;\mathrm{d}\mathbf{x}.$$

Consequently, $P_2(\mathbf{x},t) = P_1(\mathbf{x},t) + c(t)$ for $\mathbf{x} \in \Omega_1$. Thus, modifying appropriately P_2 by an additive function of t, we get $P_2(\mathbf{x},t) = P_1(\mathbf{x},t)$ for $\mathbf{x} \in \Omega_1$.

Proceeding in the same way, we obtain function $P(\,.\,,t)$ in Ω such that $P(\,.\,,t) \in L^2(\Omega_k)$ for all $k \in \mathbb{N}$ and

$$\forall\, \boldsymbol{\chi}_k \in \mathbf{W}_0^{1,2}(\Omega_k)\ : \qquad \mathcal{F}(\boldsymbol{\chi}_k)\ =\ \int_{\Omega_k} P\, \mathrm{div}\, \boldsymbol{\chi}_k\ \mathrm{d}\mathbf{x}.$$

The norm of P is

$$\|P\|_{2;\,\Omega_k}\ =\ \sup_g \biggl| \int_{\Omega_k} P g\ \mathrm{d}\mathbf{x} \biggr|,$$

where the supremum is taken over all $g \in L^2(\Omega_k)$ such that $\|g\|_{2;\,\Omega_k} = 1$.

Since each such function g can be expressed in the form $g = \mathrm{div}\, \boldsymbol{\chi}_k$, where $\boldsymbol{\chi}_k \in \mathbf{W}_0^{1,2}(\Omega_k)$ and $\|\boldsymbol{\chi}_k\|_{1,2;\,\Omega_k} \leq c_k\, \|g\|_{2;\,\Omega_k} = c_k$, we deduce that

$$\begin{aligned} \|P\|_{2;\,\Omega_k}\ &\leq\ c_k\, \|\mathcal{F}\|_{-1,2;\,\Omega_k} \\ &\leq\ c_k \int_0^t \bigl(\|\nabla\mathbf{u}\|_{2;\,\Omega_k} + M^{1/2}\, \|\nabla\mathbf{u}\|_{2;\,\Omega_k}^{3/2} + \|\mathbf{f}\|_{-1,2} \bigr)\ \mathrm{d}\tau + 2M. \end{aligned}$$

On some other results.

- If Ω is any domain in $\mathbb{R}^3$ and $\mathbf{f} = \mathbf{f}_0 + \mathrm{Div}\,\mathbf{F}$, where $\mathbf{f}_0 \in L^1_{loc}([0,T);\, \mathbf{L}^2(\Omega))$, $\mathbf{F} \in L^{4/3}_{loc}([0,T);\, \mathbb{L}^2(\Omega)^{3\times 3})$ then P can be chosen so that $P \in L^{4/3}_{loc}([0,T);\, L^2_{loc}(\Omega))$. (See H. Sohr [31, Theorem V.1.7.1] (2001).)

- If Ω is bounded then there exists at least one weak solution $\mathbf{u}$ with an associated pressure $p \in W^{-1,\infty}(0,T;\, L^2_{loc}(\Omega))$. If Ω is locally Lipschitzian then $p \in W^{-1,\infty}(0,T;\, L^2(\Omega))$. (See J. Simon [28] (1999).)

- If Ω is a smooth bounded or exterior domain in $\mathbb{R}^3$ and $\mathbf{u}$ is a weak solution then, under some assumptions on the smoothness of $\mathbf{u}_0$ and $\mathbf{f}$, the associated pressure p is in $L^{5/3}(Q_T)$. (See H. Sohr and W. von Wahl [30] (1986).)

- If Ω is a smooth bounded or exterior domain or a half–space or the whole $\mathbb{R}^3$ and $\mathbf{f} = \mathbf{0}$ then the associated pressure p is in $L^\alpha(\epsilon, T; L^\beta(\Omega'))$ for $1 < \alpha < 2$, $\frac{3}{2} < \beta < 3$, $2/\alpha + 3/\beta = 3$. (See Taniuchi [35] (1997).)

- If Ω is a smooth domain then a series of authors have shown that the problem (2.1)–(2.4) has the so called *suitable weak solution,* which is a pair of functions $\mathbf{u}$, p such that $\mathbf{u}$ is a weak solution (in the sense of the previous definition), p is an associated pressure, and $\mathbf{u}$, p satisfy the so called "generalized energy inequality" (which is also called the "localized energy inequality" or "local energy inequality"). Particularly,

  – L. Caffarelli, R. Kohn, L. Nirenberg [5] (1982) obtained $p \in L^{5/4}(Q_T)$, provided that $\mathbf{f} \in \mathbf{L}^2(Q_T) \cap \mathbf{L}^q_{loc}(Q_T)$ for some $q > \frac{5}{2}$ and $\operatorname{div} \mathbf{f} = 0$,

  – F. Lin [17] (1998) obtained $p \in L^{5/3}(\Omega)$, with $\mathbf{f} = \mathbf{0}$.

- If Ω is any domain in $\mathbb{R}^3$ then J. Wolf [37] (2007), considering $\mathbf{f} = \mathbf{0}$ and $T = \infty$, proved the existence of a "generalized suitable weak solution" u, p such that $p = p_0 + \partial_t \widetilde{p}_h$, where $p_0 \in L^{4/3}(0,\infty;\, L^2(\Omega))$ and $\widetilde{p}_h \in C(Q_\infty)$, $\Delta \widetilde{p}_h = 0$.

More on the paper [28] (1999) by J. Simon.

Assume that $\mathbf{f} \in L^2(0,T;\, \mathbf{W}_0^{-1,2}(\Omega))$.

If $\mathbf{u}$ is a weak solution then

$$\partial_t \mathbf{u} \in W^{-1,\infty}(0,T; \mathbf{L}^2_\sigma(\Omega)),$$

$$\mathbf{u}\cdot\nabla\mathbf{u} \in L^1(0,T;\, \mathbf{L}^{4/3}(\Omega))$$

$$\Delta \mathbf{u} \in L^2(0,T;\, \mathbf{W}_0^{-1,2}(\Omega)).$$

This yields $\partial_t \mathbf{u} + \mathbf{u}\cdot\nabla\mathbf{u} - \nu\Delta\mathbf{u} - \mathbf{f} \in W^{-1,\infty}(0,T;\, \mathbf{W}_0^{-1,2}(\Omega))$.

Assume that Ω is locally Lipschitzian and denote

$$\begin{aligned}\mathbf{G}(t) &:= \int_0^t \big[\partial_t\mathbf{u} + \mathbf{u}\cdot\nabla\mathbf{u} - \nu\Delta\mathbf{u} - \mathbf{f}\big]\, \mathrm{d}\tau \\ &= \mathbf{u}(t) - \mathbf{u}_0 + \int_0^t \big[\mathbf{u}\cdot\nabla\mathbf{u} - \nu\Delta\mathbf{u} - \mathbf{f}\big]\, \mathrm{d}\tau.\end{aligned}$$

Obviously, $\mathbf{G} \in L^\infty(0,T;\, \mathbf{W}_0^{-1,2}(\Omega))$ and $\mathbf{G}(t)$ vanishes on $\mathbf{W}_{0,\sigma}^{1,2}(\Omega)$. Hence, due to Lemma 4, there exists $P \in L^\infty(0,T;\, L^2(\Omega))$ such that

$$\langle \mathbf{G}(t), \boldsymbol{\varphi}\rangle \;=\; \int_\Omega P(\,.\,,t)\,\mathrm{div}\,\boldsymbol{\varphi}\;\mathrm{d}\mathbf{x} \;=\; -\langle \nabla P(\,.\,,t), \boldsymbol{\varphi}\rangle$$

for all $\boldsymbol{\varphi} \in \mathbf{W}_0^{-1,2}(\Omega)$. Thus, $\displaystyle\int_0^t \big[\partial_t\mathbf{u} + \mathbf{u}\cdot\nabla\mathbf{u} - \nu\Delta\mathbf{u} - \mathbf{f}\big]\, \mathrm{d}\tau = -\nabla P.$

If Ω is not locally Lipschitzian then a similar consideration can be applied only locally inside Ω. ■

More on the paper [30] (1986) by H. Sohr and W. von Wahl.

Theorem 3.3 in [30] concerns the case $\Omega \subset \mathbb{R}^n$. If $n = 3$ then it says:

Theorem 2. *Let Ω be a smooth bounded domain in $\mathbb{R}^3$, $T > 0$, and $\mathbf{u}$ be a weak solution to the Navier–Stokes problem (2.1)–(2.4).*

- *Let* $\frac{3}{2} < r < 3$, $1 < s < 2$, $1 < q < \frac{3}{2}$ *with* $3 \le 2/s + 3/r$, $1/q := 1/r + \frac{1}{3}$,
- *let* $\mathbf{f} \in L^s(0,T;\, \mathbf{L}^q(\Omega))$ *and* $\mathbf{u}_0 \in D(A_q^{1-\frac{1}{s}+\epsilon}) \cap \mathbf{L}^2_\sigma(\Omega)$ *for some* $\epsilon > 0$.

Then we have

- $\partial_t \mathbf{u}$, $\Delta \mathbf{u}$, $\mathbf{u} \cdot \nabla \mathbf{u} \in L^s(0,T;\, \mathbf{L}^q(\Omega))$,
- *there exists an associated pressure* $p \in L^s(0,T;\, W^{1,q}(\Omega))$.

Particularly, choosing $s = r = \frac{5}{3}$, we have $q = \frac{15}{14}$ and

$$\partial_t \mathbf{u},\ \Delta\mathbf{u},\ \mathbf{u}\cdot\nabla\mathbf{u} \in L^{5/3}(0,T;\mathbf{L}^{15/14}(\Omega)),\ \ p \in L^{5/3}(0,T;\ W^{1,15/14}(\Omega))$$

provided that $\mathbf{f} \in L^{5/3}(0,T;\ \mathbf{L}^{15/14}(\Omega))$ and $\mathbf{u}_0 \in D(A^{\frac{2}{5}+\epsilon}) \cap \mathbf{W}^{2,2}(\Omega)$ for some $\epsilon \in (0, \frac{3}{5})$.

Applying the imbedding $W^{1,15/14}(\Omega) \hookrightarrow L^{5/3}(\Omega)$, we obtain $p \in L^{5/3}(0,T;$ $L^{5/3}(\Omega))$.

Principle of the proof. For $1 < a < \infty$, let A_a be the Stokes operator in $\mathbf{L}^a_\sigma(\Omega)$, i.e. $A_a := -P_a\Delta$ with the domain $D(A_a) := \mathbf{W}^{1,a}_{0,\sigma}(\Omega) \cap \mathbf{W}^{2,a}(\Omega)$. ($P_a$ is the Helmholtz projection in $\mathbf{L}^a(\Omega)$.) The imbedding properties yield

$$\|A_a^\alpha \mathbf{v}\|_a \ \leq\ c\,\|A_b^\beta \mathbf{v}\|_b \qquad \text{for} \quad 1 < b \leq a < \infty, \quad \frac{2\alpha}{3} - \frac{1}{a} \leq \frac{2\beta}{3} - \frac{1}{b}.$$

Since $A_b\mathbf{v} = A_a\mathbf{v}$ for $b \leq a$ and $\mathbf{v} \in D(A_a)$, we further write only A instead of A_a or A_b.

Consider the test function in (2.5) in the form $\boldsymbol{\phi}(\mathbf{x},t) = \mathbf{w}(\mathbf{x})\,h(t)$, where $\mathbf{w} \in D(A^{3/4}) \cap \mathbf{C}_0^\infty(\overline{\Omega})$ and $h \in C^1([0,T])$, $h(T) = 0$.

$\mathbf{w} = A^{-3/4}A^{3/4}\mathbf{w} = A^{-3/4}\widetilde{\mathbf{w}}$, where $\widetilde{\mathbf{w}} := A^{3/4}\mathbf{w}$
$\dots \quad \boldsymbol{\phi}(\mathbf{x},t) = A^{-3/4}\widetilde{\mathbf{w}}(\mathbf{x})\,h(t)$

The integral equation (2.5) takes the form

$$-\int_0^T (\mathbf{u}, A^{-3/4}\widetilde{\mathbf{w}})\,\dot{h}\;\mathrm{d}t + \nu\int_0^T (\nabla\mathbf{u}, \nabla A^{-3/4}\widetilde{\mathbf{w}})\,h\;\mathrm{d}t$$
$$+\int_0^T (\mathbf{u}\cdot\nabla\mathbf{u}, A^{-3/4}\widetilde{\mathbf{w}})\;\mathrm{d}t \;=\; (\mathbf{u}_0, A^{-3/4}\widetilde{\mathbf{w}})\,h(0) + \int_0^T (\mathbf{f}, A^{-3/4}\widetilde{\mathbf{w}})\,h\;\mathrm{d}t.$$

One can show that $\mathbf{u}\cdot\nabla\mathbf{u} \in \mathbf{L}^r(Q_T)$ for $1 < r \leq \frac{5}{4}$. From this, one can deduce that $A^{-3/4}P_r(\mathbf{u}\cdot\nabla\mathbf{u}) \in \mathbf{L}^2(Q_T)$. Similarly, $\Delta A^{-3/4}\mathbf{u} \in L^2(Q_T)$, which means that $P_2\Delta A^{-3/4}\mathbf{u} \in \mathbf{L}^2(Q_T)$. Moreover, $A^{-3/4}P_q\mathbf{f} \in L^s(0,T;\,\mathbf{L}^2(\Omega))$.

Thus, the integral equation can be rewritten:

$$-\int_0^T (\mathbf{u}, A^{-3/4}\widetilde{\mathbf{w}})\,\dot{h}\;\mathrm{d}t - \nu\int_0^T (A^{-3/4}P_2\Delta\mathbf{u}, \widetilde{\mathbf{w}})\,h\;\mathrm{d}t$$

$$+\int_0^T \big(A^{-3/4}P_r(\mathbf{u}\cdot\nabla\mathbf{u}), \widetilde{\mathbf{w}}\big)\;\mathrm{d}t$$

$$= \;(A^{-3/4}\mathbf{u}_0, \widetilde{\mathbf{w}})\,h(0) + \int_0^T (A^{-3/4}P_q\mathbf{f}, \widetilde{\mathbf{w}})\,h\;\mathrm{d}t.$$

Considering at first functions h such that $h(0)=0$, we observe that

$$\frac{\mathrm{d}}{\mathrm{d}t}\,(A^{-3/4}\mathbf{u}, \widetilde{\mathbf{w}}) - \nu\,(P_2\Delta A^{-3/4}\mathbf{u}, \widetilde{\mathbf{w}}) + \big(A^{-3/4}P_r(\mathbf{u}\cdot\nabla\mathbf{u}), \widetilde{\mathbf{w}}\big)$$

$$= \;(A^{-3/4}P_q\mathbf{f}, \widetilde{\mathbf{w}})$$

a.e. in $(0,T)$. Furthermore, considering all “admissible” functions h, we obtain

$$(A^{-3/4}\mathbf{u}(0), \widetilde{\mathbf{w}}) \;=\; (A^{-3/4}\mathbf{u}_0, \widetilde{\mathbf{w}}).$$

Since

$$\frac{\mathrm{d}}{\mathrm{d}t}\,(A^{-3/4}\mathbf{u}, \widetilde{\mathbf{w}}) \;=\; \Big(\partial_t(A^{-3/4}\mathbf{u}), \widetilde{\mathbf{w}}\Big),$$

we also get

$$\partial_t(A^{-3/4}\mathbf{u}) - \nu\, P_2\Delta A^{-3/4}\mathbf{u} + A^{-3/4}P_r(\mathbf{u}\cdot\nabla\mathbf{u}) \;=\; A^{-3/4}P_q\mathbf{f}$$

(equation in $\mathbf{L}_\sigma(\Omega)$) for a.a. $t \in (0,T)$. From this, we get $\partial_t(A^{-3/4}\mathbf{u}) \in L^\gamma(0,T;\, \mathbf{L}^2(\Omega))$ for $\gamma := \min\{s;2\}$.

Considering $J_k := (I + k^{-1}\, A)^{-3/4}$ instead of $A^{-3/4}$, we obtain

$$\partial_t\, J_k\mathbf{u} - \nu P_2\Delta J_k\mathbf{u} + J_kP_r(\mathbf{u}\cdot\nabla\mathbf{u}) \;=\; J_kP_q\mathbf{f}. \tag{2.11}$$

This is the “regularized” Navier–Stokes equation.

The inclusion $\mathbf{u}\cdot\nabla\mathbf{u} \in L^s(0,T;\, \mathbf{L}^q(\Omega))$ follows from the definition of the weak solution, particularly from $\mathbf{u} \in L^2(0,T;\, \mathbf{W}^{1,2}_{0,\sigma}(\Omega)) \cap L^\infty(0,T;\, \mathbf{L}^2_\sigma(\Omega))$.

Thus, (2.11) can be treated as a linear equation

$$\partial_t\, J_k\mathbf{u} - \nu P_q \Delta J_k \mathbf{u} \;=\; J_k\big(P_q\mathbf{f} - P_q(\mathbf{u}\cdot\nabla\mathbf{u})\big). \tag{2.12}$$

(Since $q < 2$ and $q < r$, we may write $P_q\Delta J_k\mathbf{u}$ instead of $P_q\Delta J_k\mathbf{u}$ and $P_q(\mathbf{u}\cdot\nabla\mathbf{u}$ instead of $P_r(\mathbf{u}\cdot\nabla\mathbf{u}$.)

Applying estimates which hold for this linear equation, we obtain

$$\int_0^T \big(\|\partial_t(J_k\mathbf{u})\|_q^s + \|J_k\mathbf{u}\|_{2,q}^s\big)\,\mathrm{d}t \\ \leq\; c\,\|A^{1-\frac{1}{s}+\epsilon}J_k\mathbf{u}_0\|_q^s + c\int_0^T \big\|J_kP_q(\mathbf{f}-\mathbf{u}\cdot\nabla\mathbf{u})\big\|_q^s\,\mathrm{d}t.$$

The right hand side is $\;\leq\; c\,\|A^{1-\frac{1}{s}+\epsilon}\mathbf{u}_0\|_q^s + c\displaystyle\int_0^T \big\|P_q(\mathbf{f}-\mathbf{u}\cdot\nabla\mathbf{u})\big\|_q^s\,\mathrm{d}t$.

Thus, considering the limit for $k\longrightarrow\infty$ in the left hand side, we get $\partial_t\mathbf{u}$, $\Delta\mathbf{u}\in L^s(0,T;\,\mathbf{L}^q(\Omega))$.

The same procedure in (2.12) yields

$$\partial_t \mathbf{u} \;=\; \nu P_q \Delta \mathbf{u} + P_q(\mathbf{f} - \mathbf{u} \cdot \nabla \mathbf{u}).$$

Since

$$\nu \Delta \mathbf{u} + P_q(\mathbf{f} - \mathbf{u} \cdot \nabla \mathbf{u}) \;=\; P_q\big(\nu \Delta \mathbf{u} + \mathbf{f} - \mathbf{u} \cdot \nabla \mathbf{u}\big) + \nabla p,$$

where ∇p is uniquely determined, we obtain the equation

$$\partial_t \mathbf{u} \;=\; -\nabla p + \nu \Delta \mathbf{u} + \mathbf{f} - \mathbf{u} \cdot \nabla \mathbf{u} \tag{2.13}$$

in $L^s(0, T;\, \mathbf{L}^q(\Omega))$. ■

More on the paper [37] (2008) by J. Wolf.

Function $\mathbf{u}$ satisfies formally the N-S equation (2.1) (with $\mathbf{f} = \mathbf{0}$ and $p = p_0 + \partial_t \widetilde{p}_h$, where $\widetilde{p}_h$ is harmonic) if and only if $\mathbf{v} := \mathbf{u} + \nabla \widetilde{p}_h$ satisfies

$$\partial_t \mathbf{v} + \mathbf{u} \cdot \nabla \mathbf{u} + \nabla p_0 \;=\; \nu \Delta \mathbf{v}. \tag{2.14}$$

Multiplying this equation by $\boldsymbol{\varphi} \in \mathbf{C}_0^\infty(Q_\infty)$ and integrating by parts, we get

$$\int_{Q_\infty} \left[-\mathbf{v} \cdot \partial_t \boldsymbol{\varphi} + \mathbf{u} \cdot \nabla \mathbf{u} + \nu \nabla \mathbf{v} : \nabla \boldsymbol{\varphi} \right] \mathrm{d}\mathbf{x} \, \mathrm{d}t \; = \; \int_{Q_\infty} p_0 \, \mathrm{div} \, \boldsymbol{\varphi} \, \mathrm{d}\mathbf{x} \, \mathrm{d}t. \tag{2.15}$$

On the other hand, if $\mathbf{v}$ satisfies (2.15) for all $\boldsymbol{\varphi} \in \mathbf{C}_0^\infty(Q_\infty)$ then function $\mathbf{u}$ in $L^\infty(0,T;\, \mathbf{L}^2_\sigma(\Omega)) \cap L^2(0,\infty;\, \mathbf{W}_{0,\sigma}(\Omega))$ that differs from $\mathbf{v}$ at most by a gradient of a harmonic function, is a weak solution to the Navier–Stokes problem (2.1)–(2.4). If the difference $\mathbf{v} - \mathbf{u}$ is denoted by $\nabla \widetilde{p}_h$ then $p_0 + \partial_t \widetilde{p}_h$ is a pressure associated with the weak solution $\mathbf{u}$.

Using the representation $\widetilde{p} := p_0 + \frac{1}{2}|\mathbf{u}|^2$ and $\mathbf{u} \cdot \nabla \mathbf{u} = \frac{1}{2}|\mathbf{u}|^2 + \mathbf{curl}\; \mathbf{u} \times \mathbf{u}$, we can rewrite equation (2.14):

$$\partial_t \mathbf{v} + \mathbf{curl}\; \mathbf{u} \times \mathbf{u} + \nabla \widetilde{p} \; = \; \nu \Delta \mathbf{v}. \tag{2.16}$$

Let $\mathbf{u} \in L^\infty(0,T;\, \mathbf{L}^2_\sigma(\Omega)) \cap L^2(0,\infty;\, \mathbf{W}^{1,2}_{0,\sigma}(\Omega))$ be a weak solution to the Navier–Stokes problem (2.1)–(2.4). Then $\mathbf{u}$ is said to be a *suitable weak solution* if

- there exists $p_0 \in L^{4/3}(0,\infty;\, L^2(\Omega))$ and $\widetilde{p}_h \in C(Q_\infty)$ such that $\Delta\widetilde{p}_h = 0$,
- function $\mathbf{v}$ satisfies (2.15) for all $\boldsymbol{\varphi} \in \mathbf{C}_0^\infty(Q_\infty)$,
- and the *local energy inequality*

$$\begin{aligned} &\int_\Omega |\mathbf{v}(\,.\,,t)|^2\, \phi(\,.\,,t)\; \mathrm{d}\mathbf{x} + 2\nu \int_0^t \int_\Omega |\nabla\mathbf{v}|^2\, \phi\; \mathrm{d}\mathbf{x}\, \mathrm{d}t \\ &\leq \int_0^t \int_\Omega |\mathbf{v}|^2\, (\partial_t\phi + \nu\Delta\phi)\; \mathrm{d}\mathbf{x}\, \mathrm{d}t - \int_0^t \int_\Omega (\nabla\widetilde{p}_h \times \mathbf{curl}\ \mathbf{v}) \cdot \phi\mathbf{v}\; \mathrm{d}\mathbf{x}\, \mathrm{d}t \\ &\quad + \int_0^t \int_\Omega \widetilde{p}\,\mathbf{v} \cdot \nabla\phi\; \mathrm{d}\mathbf{x}\, \mathrm{d}t \end{aligned} \tag{2.17}$$

for each nonnegative function $\phi \in C_0^\infty(Q_\infty)$.

Inequality (2.17) can be formally obtained from (2.16) if one uses the formula $\mathbf{u} = \mathbf{v} + \nabla\widetilde{p}_h$, multiples (2.16) by $\phi\mathbf{v}$ and integrates in Q_∞.

Theorem 1.3 in Wolf's paper [37] says:

Theorem 3. *For all $\mathbf{u}_0 \in \mathbf{L}^2_\sigma(\Omega)$, a suitable weak solution of the Navier–Stokes problem (2.1)–(2.4) (in the sense of the previous definition) exists.*

Here, we restrict ourselves only to the question of existence of an appropriate pressure, associated to a weak solution $\mathbf{u}$ of the problem (2.1)–(2.4).

Let G be a domain in $\mathbb{R}^3$, $G \subset \Omega$. Let $W^{2,2}_0(G)$ be the closure of $C^\infty_0(G)$ in the norm $\|\Delta\,.\,\|_2$. Define

$$A^2(G) \;:=\; \{\Delta u \in L^2(G);\ u \in W^{2,2}_0(G)\},$$
$$B^2(G) \;:=\; \{p \in L^2(G);\ \Delta p = 0 \text{ in } G\}.$$

Then $L^2(G) = A^2(G) \oplus B^2(G)$ (by Weyl's lemma).

Lemma 7. *If* $\mathbf{v}^* \in \mathbf{W}_0^{-1,2}(\Omega)$ *and* $p_0 \in A^2(G)$ *are such that*

$$\langle \mathbf{v}^*, \nabla\phi \rangle \;=\; \int_G p_0\, \Delta\phi \, \mathrm{d}\mathbf{x} \qquad \textit{for all } \phi \in C_0^\infty(G) \tag{2.18}$$

then $\|p_0\|_{2;\,G} \leq \|\mathbf{v}^*\|_{-1,2;\,\Omega}$.

Proof. (2.18) means that $\Delta p_0 = \operatorname{div} \mathbf{v}^*$ in the sense of distributions in G.

$$\begin{aligned}
\|\mathbf{v}^*\|_{-1,2;\,\Omega} &= \sup_{\boldsymbol{\varphi}\in\mathbf{W}_0^{1,2}(\Omega);\ \boldsymbol{\varphi}\neq\mathbf{0}} \frac{|\langle \mathbf{v}^*, \boldsymbol{\varphi}\rangle|}{\|\boldsymbol{\varphi}\|_{1,2;\,\Omega}} \geq \sup_{\boldsymbol{\varphi}\in\mathbf{W}_0^{1,2}(G);\ \boldsymbol{\varphi}\neq\mathbf{0}} \frac{|\langle \mathbf{v}^*, \boldsymbol{\varphi}\rangle|}{\|\boldsymbol{\varphi}\|_{1,2;\,G}} \\
&\geq \sup_{\boldsymbol{\varphi}=\nabla\phi;\ \phi\not\equiv 0;\ \phi\in C_0^\infty(G)} \frac{|\langle \mathbf{v}^*, \nabla\phi\rangle|}{\|\nabla\phi\|_{1,2;\,\Omega}} \\
&\geq c \sup_{\boldsymbol{\varphi}=\nabla\phi;\ \phi\not\equiv 0;\ \phi\in C_0^\infty(G)} \frac{|\langle \mathbf{v}^*, \nabla\phi\rangle|}{\|\Delta\phi\|_{2;\,G}}
\end{aligned}$$

$$\geq c \sup_{\boldsymbol{\varphi}=\nabla\phi;\ \phi\not\equiv 0;\ \phi\in C_0^\infty(G);\ g\in B^2(G)} \frac{\left|\int_\Omega p_0\,(\Delta\phi+g)\ \mathrm{d}\mathbf{x}\right|}{\|\Delta\phi+g\|_{1,2;\,G}}$$

$$= c \sup_{f\in L^2(G)} \frac{\left|\int_\Omega p_0\, f\ \mathrm{d}\mathbf{x}\right|}{\|f\|_{2;\,G}} = c\,\|p_0\|_{2;\,G}.$$

If we consider $\mathbf{W}_0^{1,2}(\Omega)$ and $W_0^{2,2}(\Omega)$ with the equivalent norms $\|\nabla\boldsymbol{\varphi}\|_2$ and $\|\Delta\phi\|_2$, respectively, then $c = 1$. ∎

Lemma 8. (= Theorem 2.2 in [37]) *Let* $\mathbf{u} \in C_w([0,\infty);\, \mathbf{L}^2_\sigma(\Omega))$ *and* $\mathbf{h} \in L^1_{loc}([0,\infty);\, \mathbb{L}^2(\Omega)^{3\times 3})$ *satisfy*

$$\int_{Q_\infty} \left[-\mathbf{u}\cdot\partial_t\boldsymbol{\varphi} + \mathbf{h} : \nabla\boldsymbol{\varphi}\right]\, \mathrm{d}\mathbf{x}\, \mathrm{d}t = 0 \tag{2.19}$$

for all $\boldsymbol{\varphi} \in \mathbf{C}_0^\infty(Q_\infty)$ *such that* $\operatorname{div}\boldsymbol{\varphi} = 0$. *Let* $B \subset\subset \Omega$.

Then there exist functions $p_0 \in L^1_{loc}([0,\infty);\, A^2(\Omega))$ *and* $\widetilde{p}_h \in C(\Omega \times [0,\infty))$ *such that* $\Delta\widetilde{p}_h = 0$, $(\widetilde{p}_h)_B = 0$ *(i.e. the mean value of* $\widetilde{p}_h$ *in* B *is zero) and*

$$\int_{Q_\infty} \big[-\mathbf{u}\cdot\partial_t\boldsymbol{\varphi} + \mathbf{h} : \nabla\boldsymbol{\varphi}\big]\, \mathrm{d}\mathbf{x}\, \mathrm{d}t = \int_{Q_\infty} \big[p_0\, \mathrm{div}\, \boldsymbol{\varphi} + \nabla\widetilde{p}_h \cdot \partial_t\boldsymbol{\varphi}\big]\, \mathrm{d}\mathbf{x}\, \mathrm{d}t + \int_\Omega \mathbf{u}(\,.\,,0)\cdot\boldsymbol{\varphi}(\,.\,,0)\, \mathrm{d}\mathbf{x} \qquad \textbf{(2.20)}$$

for all $\boldsymbol{\varphi} \in \mathbf{C}^\infty(Q_\infty)$ *with* $\mathrm{supp}\,\boldsymbol{\varphi} \subset\subset \Omega \times [0,\infty)$.
Moreover, $\|p_0(\,.\,,t)\|_{2;\,\Omega} \leq \|\mathbf{h}(\,.\,,t)\|_{2;\,\Omega}$ *for a.a.* $t \in [0,\infty)$*, and for each Lipschitzian domain* $G \subset\subset \Omega$ *there exists a constant* c_G *such that*

$$\|\widetilde{p}_h(\,.\,,t)\|_{2;\,G} \;\leq\; c_G \left(\|\mathbf{u}(\,.\,,t) - \mathbf{u}(\,.\,,0)\|_{2;\,G} + \left\| \int_0^t \mathbf{h}(\,.\,,s)\, \mathrm{d}s \right\|_{2;\,\Omega} \right)$$

for a.a. $t \in [0,\infty)$.

Roughly speaking, Lemma 8 says that if $\mathbf{u} \in C_w([0,\infty);\, \mathbf{L}^2_\sigma(\Omega))$ and $\mathbf{h} \in L^1_{loc}([0,\infty);\, \mathbb{L}^2(\Omega)^{3\times 3})$ satisfy the equation

$$\mathbf{u}' \;=\; \mathcal{P}_\sigma(\operatorname{div}\mathbf{h}) \tag{2.21}$$

as an equation in $\mathbf{W}^{-1,2}_{0,\sigma}(\Omega)$, satisfied a.e. in $(0,\infty)$, then there exist appropriate functions p_0 and $\widetilde{p}_h$ ($\widetilde{p}_h$ harmonic) such that the equation

$$\partial_t\mathbf{u} \;=\; -\nabla p + \operatorname{div}\mathbf{h} \tag{2.22}$$

holds in the sense of distributions in Q_∞ with $p = p_0 + \partial_t\widetilde{p}_h$.

The proof essentially uses the next Lemma 9, which, again roughly speaking, says that if $\mathbf{U} \in \mathbf{L}^2_\sigma(\Omega)$ and $\mathbf{v}^* \in \mathbf{W}_0^{-1,2}(\Omega)$ are such that

$$\mathbf{U} + \mathcal{P}_\sigma \mathbf{v}^* \;=\; \mathbf{0}$$

(an equation in $\mathbf{W}_{0,\sigma}^{-1,2}(\Omega)$) then there exist unique appropriate functions $p_0 \in A^2(\Omega)$ and $p_h \in C^\infty(\Omega)$ with $\Delta p_h = 0$, $(p_h)_B = 0$ (where B is any domain $\subset\subset \Omega$) such

$$\mathbf{U} + \mathbf{v}^* \;=\; -\nabla p$$

(an equation in $\mathbf{W}_0^{-1,2}(\Omega)$) with $p = p_0 + p_h$.

Lemma 9. *Let* $B \subset\subset \Omega$ *be an arbitrary domain and let* $\mathbf{U} \in \mathbf{L}^2_\sigma(\Omega)$ *and* $\mathbf{v}^* \in \mathbf{W}_0^{-1,2}(\Omega)$ *satisfy*

$$\int_\Omega \mathbf{U} \cdot \boldsymbol{\varphi} \, \mathrm{d}\mathbf{x} + \langle \mathbf{v}^*, \boldsymbol{\varphi} \rangle \; = \; 0 \tag{2.23}$$

for all $\boldsymbol{\varphi} \in \mathbf{C}^\infty_{0,\sigma}(\Omega)$. *Then there exist unique functions* $p_0 \in A^2(\Omega)$ *and* $p_h \in C^\infty(\Omega)$ *with* $\Delta p_h = 0$ *and* $(p_h)_B = 0$, *such that*

$$\int_\Omega \mathbf{U} \cdot \boldsymbol{\varphi} \, \mathrm{d}\mathbf{x} + \langle \mathbf{v}^*, \boldsymbol{\varphi} \rangle \; = \; \int_\Omega (p_0 + p_h) \, \mathrm{div}\, \boldsymbol{\varphi} \, \mathrm{d}\mathbf{x} \tag{2.24}$$

for all $\boldsymbol{\varphi} \in \mathbf{C}^\infty_0(\Omega)$. *Moreover,* $\|p_0\|_2 \leq \|\mathbf{v}^*\|_{-1,2}$ *and for each Lipschitzian domain* $G \subset\subset \Omega$ *there exists a constant* c_G *such that* $\|p_0\|_2 \leq c_G \left(\|\mathbf{U}\|_2 + \|\mathbf{v}^*\|_{-1,2} \right)$.

Principle of the proof of Lemma 9. Let $\Omega_1 \subset \Omega_2 \subset \dots$ be a sequence of bounded Lipschitzian domains such that $\Omega = \bigcup_{k=1}^{\infty} \Omega_k$. For $k \in \mathbb{N}$, define

$$\langle \mathbf{w}_k^*, \boldsymbol{\varphi} \rangle \;:=\; \int_{\Omega_k} \mathbf{U} \cdot \boldsymbol{\varphi} \, \mathrm{d}\mathbf{x} + \langle \mathbf{v}^*, \boldsymbol{\varphi} \rangle \qquad \text{for } \boldsymbol{\varphi} \in \mathbf{W}_0^{1,2}(\Omega_k).$$

Clearly, $\mathbf{w}_k^*$ vanishes on $\mathbf{W}_{0,\sigma}^{1,2}(\Omega_k)$. Hence there exists $p_k \in L^2(\Omega_k)$ such that

$$\langle \mathbf{w}_k^*, \boldsymbol{\varphi} \rangle \;=\; \int_{\Omega_k} p_k \, \mathrm{div}\, \boldsymbol{\varphi} \, \mathrm{d}\mathbf{x} \qquad \text{for all } \boldsymbol{\varphi} \in \mathbf{W}_0^{1,2}(\Omega_k).$$

Moreover, there exist unique $p_{0k} \in A^2(\Omega_k)$ and $p_{hk} \in B^2(\Omega_k)$ such that $p_k = p_{0k} + p_{hk}$.

Finally, extending p_{0k} and p_{hk} by zero to $\Omega \smallsetminus \Omega_k$ and considering the limits for $k \to \infty$, one obtains functions p_0 and p_h. ∎

Principle of the proof of Lemma 8. Applying (2.21) to $\boldsymbol{\psi} \in \mathbf{C}^{\infty}_{0,\sigma}(\Omega)$ and integrating with respect to time from 0 to t, we obtain

$$\begin{aligned}\int_{\Omega} \big[\mathbf{u}(\,.\,,t) - \mathbf{u}(\,.\,,0)\big] \cdot \boldsymbol{\psi}\; \mathrm{d}\mathbf{x} \;&=\; -\int_0^t \big\langle \mathcal{P}_{\sigma}(\operatorname{div}\mathbf{h}), \boldsymbol{\psi}\big\rangle_{\sigma}\; \mathrm{d}t \\ &=\; \int_0^t \langle \operatorname{div}\mathbf{h}, \boldsymbol{\psi}\rangle\; \mathrm{d}t \;=\; -\int_0^t \int_{\Omega} \mathbf{h} : \nabla\boldsymbol{\psi}\; \mathrm{d}\mathbf{x}\, \mathrm{d}t \\ &=\; -\int_{\Omega} \widetilde{\mathbf{h}}(\,.\,,t) : \nabla\boldsymbol{\psi}\; \mathrm{d}\mathbf{x},\end{aligned}$$

where $\widetilde{\mathbf{h}}(\,.\,,t) := \int_0^t \mathbf{h}(\,.\,,\tau)\; \mathrm{d}\tau$. Due to Lemma 9, there exist unique functions $\widetilde{p}_0(\,.\,,t) \in A^2(\Omega)$ and $\widetilde{p}_h(\,.\,,t) \in B^2(\Omega)$ such that $\forall\, \boldsymbol{\psi} \in \mathbf{C}^{\infty}_0(\Omega)$:

$$\begin{aligned}&\int_{\Omega} \big[\mathbf{u}(\,.\,,t) - \mathbf{u}(\,.\,,0)\big] \cdot \boldsymbol{\psi}\; \mathrm{d}\mathbf{x} + \int_{\Omega} \widetilde{\mathbf{h}}(\,.\,,t) : \nabla\boldsymbol{\psi}\; \mathrm{d}\mathbf{x} \\ &\quad=\; \int_{\Omega} \big[\widetilde{p}_0(t) + \widetilde{p}_h(t)\big]\; \operatorname{div}\boldsymbol{\psi}\; \mathrm{d}\mathbf{x} \;=\; -\big\langle \nabla\widetilde{p}_0 + \nabla\widetilde{p}_h,\; \boldsymbol{\psi}\big\rangle. \qquad (2.25)\end{aligned}$$

Particularly, considering $\boldsymbol{\psi}$ in the form $\boldsymbol{\psi} = \nabla\phi$ for some $\phi \in C_0^\infty(\Omega)$, we get

$$\int_\Omega \widetilde{\mathbf{h}}(\,.\,,t) : \nabla^2\phi \, \mathrm{d}\mathbf{x} \; = \; \int_\Omega \widetilde{p}_0(\,.\,,t)\,\Delta\phi\,\mathrm{d}\mathbf{x}.$$

From this, one can deduce that

$$\|\widetilde{p}_0(\,.\,,s) - \widetilde{p}_0(\,.\,,t)\|_2 \; \leq \; \|\widetilde{\mathbf{h}}(\,.\,,s) - \widetilde{\mathbf{h}}(\,.\,,t)\|_2 \; \leq \; \int_s^t \|\mathbf{h}(\tau)\|_2\,\mathrm{d}\tau$$

for $0 \leq t \leq s < \infty$. Furthermore, $\widetilde{p}_0 \in C([0,\infty);\, L^2(\Omega))$ and $p_0 := \partial_t\widetilde{p}_0 \in L^1_{loc}([0,\infty);\, L^2(\Omega))$.

Considering $\boldsymbol{\psi}$ in (2.25) in the form $\boldsymbol{\psi} = \partial_t\boldsymbol{\varphi}(\,.\,,t)$, where $\boldsymbol{\varphi} \in \mathbf{C}_0^\infty(Q_\infty)$, and integrating over $[0,\infty)$, we get (2.20). For more details (including the information that $\widetilde{p}_h \in C(\Omega \times [0,\infty))$), see [37]. ■

Existence of a pressure with the properties stated in Wolf's definition of a suitable weak solution:

Consider the Navier–Stokes equation in the form (2.8) with $\mathbf{f} = \mathbf{0}$, i.e.

$$\mathbf{u}' - \nu \mathcal{P}_\sigma(\Delta \mathbf{u}) + \mathcal{P}_\sigma(\mathbf{u} \cdot \nabla \mathbf{u}) \;=\; \mathbf{0}.$$

It corresponds to equation (2.21). Due to Lemma 8, there exists p such that $p = p_0 + \partial_t \widetilde{p}_h$, where p_0 and $\widetilde{p}_h$ have the properties stated in Lemma 8, and $\mathbf{u}$ and p satisfy the Navier–Stokes equation in the sense of distributions in Q_T.

3. An interior regularity of pressure under Serrin's conditions and relation to the boundary conditions.

In this section, we assume for simplicity that $\mathbf{f} = \mathbf{0}$. Thus, the considered Navier–Stokes system is

$$\partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} + \nabla p \;=\; \nu \Delta \mathbf{u}, \tag{3.1}$$

$$\operatorname{div} \mathbf{u} \;=\; 0. \tag{3.2}$$

If Ω' is a domain in $\mathbb{R}^3$ and $-\infty < t_1 < t_2 < \infty$ then $\mathbf{u} \in$ is said to be a *weak solution to the system (2.1), (2.2) in $\Omega' \times (t_1, t_2)$* (with $\mathbf{f} = \mathbf{0}$) if it satisfies

$$\int_0^T \int_\Omega \big[-\mathbf{u} \cdot \partial_t \boldsymbol{\phi} + \nu \nabla \mathbf{u} : \nabla \boldsymbol{\phi} + \mathbf{u} \cdot \nabla \mathbf{u} \cdot \boldsymbol{\phi} \big] \, \mathrm{d}\mathbf{x} \, \mathrm{d}t \;=\; 0$$

for all $\boldsymbol{\phi} \in C_0^\infty\big(\Omega' \times (t_1, t_2)\big)$ such that $\boldsymbol{\phi}(\,.\,, T) = \mathbf{0}$ such that $\operatorname{div} \boldsymbol{\phi} = 0$.

The next lemma on the interior regularity concerns the weak solution to the Navier–Stokes system (2.1), (2.2) in $\Omega' \times (t_1, t_2) \subset \Omega \times (0, T)$. (See J. Serrin [26].)

Lemma 10. *Let Ω' be a domain in $\mathbb{R}^3$ and let $\mathbf{u}$ be a weak solution to the Navier–Stokes system (2.1), (2.2) in $\Omega' \times (t_1, t_2)$. Suppose that $\mathbf{u}$ satisfies Serrin's condition*

- $\mathbf{u} \in L^r(t_1, t_2; \mathbf{L}^s(\Omega'))$ *for some* $r, s \in \mathbb{R}$ *such that* $2/r + 3/s = 1$, $s > 3$.

Then $\mathbf{u}$ with all its space derivatives is bounded on every compact subset of $\Omega' \times (t_1, t_2)$.

Moreover, if $\partial_t \mathbf{u} \in L^2(t_1, t_2; \mathbf{L}^q(\Omega'))$ for some $q \geq 1$ then $\mathbf{u}$ and each its space derivative are absolutely continuous functions of time.

A similar statement can also be proven if Ω' is bounded and $\mathbf{u} \in L^\infty(t_1, t_2;\, \mathbf{L}^3(\Omega'))$ with the norm of $\mathbf{u}$ in this space "sufficiently small".

Typically, the lemma provides information on the regularity of $\mathbf{u}$ in $\Omega'' \times (t_1 + \xi, t_2 - \xi)$, where $\Omega'' \subset\subset \Omega'$ and $t_1 < t_1 + \xi < t_2 - \xi < t_2$. It implies that $\mathbf{u}$ is in $\mathbf{C}^\infty(\Omega'')$ at each time $t \in (t_1 + \xi, t_2 - \xi)$. However, it does not say anything about the regularity of $\mathbf{u}$ in the direction of t and it says nothing about the associated pressure.

On interior regularity of $\partial_t \mathbf{v}$ and p.

Theorem 4. *Let Ω be a bounded or exterior domain with a smooth boundary, or a half–space. Let $\mathbf{u}$ be a weak solution to the Navier–Stokes problem (2.1)–(2.4), satisfying the assumptions of Lemma 10. Then $\partial_t \mathbf{v}$ and p have all spatial derivatives in $L^\alpha(t_1 + \xi, t_2 - \xi;\, L^\infty(\Omega''))$ for each $\alpha \in [1, 2)$, where $\Omega'' \subset\subset \Omega'$ is a bounded domain.*

The theorem is essentially due to J.N. and P. Penel [19] (2001) and Z. Skalák, P. Kučera [29] (2003).

Principle of the proof. Due to [35], the associated pressure p is in $L^\alpha(t_1+\xi, t_2-\xi;\ L^\beta(\Omega))$ for all $1<\alpha<2$, $\frac{3}{2}<\beta<3$, $2/\alpha+3/\beta=3$. Applying the operator div to the Navier–Stokes equation, we obtain the equation

$$\Delta p \;=\; -\partial_i u_j\, \partial_j u_i, \tag{3.3}$$

satisfied in Ω in the sense of distributions at all times $t\in(t_1+\xi, t_2-\xi)$. Since the right hand side is "smooth" in Ω' (due to Lemma 10), p is also "smooth" in Ω'.

Let $\delta>0$ be so small that $U_{4\delta}(\Omega'')\subset\Omega'$. Denote by ψ an infinitely differentiable cut–off function defined in $\mathbb{R}^3$ such that $0\le\psi\le 1$ and

$$\psi \;=\; \begin{cases} 1 & \text{on } U_\delta(\Omega''), \\ 0 & \text{on } \mathbb{R}^3 \smallsetminus U_{3\delta}(\Omega''). \end{cases}$$

The product $\psi\, p$ satisfies

$$\psi(\mathbf{x})\, p(\mathbf{x},t) \;=\; -\frac{1}{4\pi}\int_{\mathbb{R}^3} \frac{1}{|\mathbf{x}-\mathbf{y}|}\,\big[\,\Delta(\psi p)\,\big](\mathbf{y},t)\,\mathrm{d}\mathbf{y}. \tag{3.4}$$

Using equation (3.3), we get

$$\Delta(\psi p) \;=\; (\Delta\psi)p + 2\nabla\psi\cdot\nabla p - \psi\,\partial_i\partial_j(u_iu_j).$$

Substituting this to (3.4) and applying integration by parts, we get (for $\mathbf{x}\in\Omega''$):

$$\psi(\mathbf{x})\, p(\mathbf{x},t) \;=\; p^I(\mathbf{x},t) + p^{II}(\mathbf{x},t)$$

where

$$p^I(\mathbf{x},t) \;=\; \frac{1}{4\pi}\int_{\mathbb{R}^3} \frac{1}{|\mathbf{x}-\mathbf{y}|}\,\big[\psi\,\partial_iu_j\,\partial_ju_i\big](\mathbf{y},t)\,\mathrm{d}\mathbf{y},$$

$$
\begin{aligned}
p^{II}(\mathbf{x},t) \; &= \; \frac{1}{4\pi}\int_{\mathbb{R}^3}\frac{1}{|\mathbf{x}-\mathbf{y}|}\,\big[p\,\Delta\psi\big](\mathbf{y},t) \\
&\quad + \frac{1}{2\pi}\int_{\mathbb{R}^3}\frac{\mathbf{x}-\mathbf{y}}{|\mathbf{x}-\mathbf{y}|^3}\cdot\big[\nabla\psi\,p\big](\mathbf{y},t)\;\mathrm{d}\mathbf{y} \\
&= \; \frac{1}{4\pi}\int_{\mathbb{R}^3}\bigg(\frac{\Delta\psi(\mathbf{y})}{|\mathbf{x}-\mathbf{y}|}+2\,\frac{\mathbf{x}-\mathbf{y}}{|\mathbf{x}-\mathbf{y}|^3}\cdot\nabla\psi(\mathbf{y})\bigg)\,p(\mathbf{y},t)\;\mathrm{d}\mathbf{y}.
\end{aligned}
$$

$\mathbf{u}$ and its spatial derivatives are bounded on $\operatorname{supp}\psi\times(t_1+\xi,t_2-\xi)$, hence

$$
\begin{aligned}
\big|\nabla^k p^I(\mathbf{x},t)\big| \equiv \big|\nabla_{\mathbf{x}}^k p^I(\mathbf{x},t)\big| = \bigg|\frac{1}{4\pi}\int_{\mathbb{R}^3}\frac{1}{|\mathbf{x}-\mathbf{y}|}\;\Big[\nabla_{\mathbf{y}}^k\big(\psi\,\partial_i u_j\,\partial_j u_i\big)\Big](\mathbf{y},t)\;\mathrm{d}\mathbf{y}\bigg| \\
\le c(k).
\end{aligned}
$$

The first two integrals in the expression of p^{II} can be estimated in the same way.

The integrand in the last integral the representation of p^{II} is supported only for $\mathbf{y} \in U_{3\delta}(\Omega'') \smallsetminus U_\delta(\Omega'')$, where $|\mathbf{x} - \mathbf{y}| > \delta$. Hence the modulus of any spatial derivative of this integral (i.e. the derivative with respect to components of $\mathbf{x}$) is

$$\leq \int_{t_1}^{t_2} \left| \int_{\mathbb{R}^3} \Big(\dots \Big) \, p(\mathbf{y},t) \, \mathrm{d}\mathbf{y} \right|^\alpha \mathrm{d}t \;\leq\; c(k) \int_{t_1}^{t_2} \bigg(\int_{\mathrm{supp}\, \nabla\psi} |p(\mathbf{y},t)| \, \mathrm{d}\mathbf{y} \bigg)^\alpha \mathrm{d}t$$

$$\leq\; c(k) \int_{t_1}^{t_2} \bigg(\int_{U_\delta(\Omega'')} |p(\mathbf{y},t)|^\beta \, \mathrm{d}\mathbf{y} \bigg)^{\alpha/\beta} \mathrm{d}t,$$

where β is chosen so that $2/\alpha + 3/\beta = 3$. Due to [35], $p \in L^\alpha(t_1,t_2; L^\beta(U_\delta(\Omega')))$ for $1 < \alpha < 2$, $\frac{3}{2} < \beta < 3$ such that $2/\alpha + 3/\beta = 3$. Hence the last integral is finite. ■

Remark 6. The assumptions on the smoothness of $\partial\Omega$ are needed when we apply [35].

Remark 7. The information on the rate of integrability of p and $\partial_t \mathbf{u}$ in time can be improved if $\partial\Omega = \emptyset$, i.e. $\Omega = \mathbb{R}^3$. In this case, $\partial_t \mathbf{v}$ and p have all spatial derivatives in $L^\infty(t_1 + \delta, t_2 - \delta;\, L^\infty(\Omega''))$. (See Z. Skalák and P. Kučera [29] (2003) and J.N. [20] (2003).)

A similar result can also be obtained if $\mathbf{u}$ is assumed to satisfy the so called Navier–type boundary conditions, i.e.

$$\mathbf{u} \cdot \mathbf{n} = 0$$

$$\mathbf{curl}\ \mathbf{u} \times \mathbf{n} = \mathbf{0}$$

on Γ_T ($= \partial\Omega \times (0,T)$).

4. An influence of pressure on the regularity of a weak solution. Regularity criteria in terms of pressure.

Most of the authors consider $\mathbf{f} = \mathbf{0}$. Many results are formulated for $\Omega \subset \mathbb{R}^n$. However, here we consider only $n = 3$.

- S. Kaniel [14] (1969) assumed that $\mathbf{u}$ is a Leray–Hopf weak solution to the Navier–Stokes problem (with $\mathbf{f} = \mathbf{0}$) in $\Omega \times (0, T)$, where Ω is a bounded domain in $\mathbb{R}^3$, and p is an associated pressure. He proved that the condition $p \in L^\infty(0, T;\, L^q(\Omega))$, $q > \frac{12}{5}$ guarantees that solution $\mathbf{u}$ is regular.
- The result was later improved by L. Berselli [2] (1999), who proved that if $p \in L^\alpha(0, T;\, L^{3\alpha/(\alpha+1)}(\Omega))$ for some $\alpha > 3$ then $\mathbf{u}$ is regular.

- For other improvements (using the weak L^γ–space over Q_T) see H. Beirão da Veiga [1] (1998).

- D. Chae and J. Lee [7] (2001): the Navier–Stokes problem with $\mathbf{f} = \mathbf{0}$ in $\mathbb{R}^3 \times (0,T)$. They showed that if $\mathbf{u}_0 \in \mathbf{L}^2_\sigma(\mathbb{R}^3) \cap \mathbf{L}^q(\mathbb{R}^3)$ for some $q > 3$, $\mathbf{u}$ is the Leray–Hopf weak solution in $(0,T)$ and p is in $L^r(0,T;\ L^s(\mathbb{R}^3))$ for some $1 < r \leq \infty$, $\frac{3}{2} < s < \infty$, satisfying $2/r + 3/s < 2$, or $p \in L^1(0,T;\, L^\infty(\mathbb{R}^3))$, or if $p \in L^\infty(0,T;\, L^{3/2}(\mathbb{R}^3))$ (with the corresponding norm sufficiently small) then $\mathbf{u}$ is a regular solution.

- L. Berselli and G. P. Galdi [3] (2002) extended the previous result to the case $2/r + 3/s = 2$ (with $1 \leq r < \infty$). They also considered the analogous condition on ∇p with $2/r + 3/s = 3$, $1 \leq r \leq 3$.

- G. Seregin and V. Šverák [25] (2002) considered a suitable weak solution $\mathbf{u}$, p in $\mathbb{R}^3 \times (0,T)$.

  The authors say that a scalar function $g : \mathbb{R}^3 \times (0,\infty) \longrightarrow [0,\infty)$ satisfies condition (C) if to any $t_0 > 0$ there exists $R_0 > 0$ such that

$$A(t_0) := \sup_{\mathbf{x}_0 \in \mathbb{R}^3} \sup_{t_0 - R_0^2 \leq t \leq t_0} \int_{|\mathbf{x}-\mathbf{x}_0|<R_0} \frac{g(\mathbf{x},t)}{|\mathbf{x}-\mathbf{x}_0|}\, \mathrm{d}\mathbf{x} < \infty$$

  and for each fixed $\mathbf{x}_0 \in \mathbb{R}^3$ and each fixed $R \in (0, R_0]$ the function

$$t \mapsto \int_{|\mathbf{x}-\mathbf{x}_0|<R} \frac{g(\mathbf{x},t)}{|\mathbf{x}-\mathbf{x}_0|}\, \mathrm{d}\mathbf{x}$$

  is continuous at t_0 from the left.

The main result of [25] says that if there exists function g satisfying condition (C) so that the normalized pressure

$$p(\mathbf{x},t) \;:=\; \frac{1}{4\pi}\int_{\mathbb{R}^3} \frac{1}{|\mathbf{x}-\mathbf{y}|}\,[\partial_i u_j\,\partial_j u_i](\mathbf{y},t)\;\mathrm{d}\mathbf{y}$$

satisfies

$$|\mathbf{u}(\mathbf{x},t)|^2 + 2p(\mathbf{x},t) \;\leq g(\mathbf{x},t) \qquad \text{for all } \mathbf{x} \in \mathbb{R}^3,\; 0<t<\infty \tag{4.1}$$

or

$$p(\mathbf{x},t) \;\geq\; -g(\mathbf{x},t) \qquad \text{for all } \mathbf{x} \in \mathbb{R}^3,\; 0<t<\infty \tag{4.2}$$

then $\mathbf{u}$ is regular in $\mathbb{R}^3 \times (0,\infty)$.

- J. Nečas and J. Neustupa [18] (2002) proved the regularity of a suitable weak solution $\mathbf{u}$, p at the space–time point $(\mathbf{x}_0, t_0)$ under the conditions that there exist $r > 0$ (arbitrarily small) and $\rho > 0$ (arbitrarily large) such that
  - the negative part p_- of the pressure is integrable with exponents $\alpha \in [\frac{3}{2}, \infty)$ (in time) and $\beta \in (\frac{3}{2}, \infty)$ (in space), such that $2/\alpha + 3/\beta = 2$, over

    $$V_r^\rho \;:=\; \big\{(\mathbf{x}, t) \in Q_T;\; t_0 - r^2/\rho^2 < t < t_0,\; |\mathbf{x} - \mathbf{x}_0| < \epsilon(t)\rho\big\}$$

    where $\epsilon(t) := \sqrt{t_0 - t}$,
  - the velocity is integrable with the exponents a (in time) and b (in space) such that $a \in [3, \infty)$, $\beta \in (3, \infty)$, such that $2/a + 3/b = 1$, over

    $$U_r^\rho \;:=\; \big\{(\mathbf{x}, t) \in Q_T;\; t_0 - r^2/\rho^2 < t < t_0,\; \epsilon(t)\rho < |\mathbf{x} - \mathbf{x}_0| < r\big\}.$$

- K. Kang, J. Lee [12] (2006) and [13] (2010): considered a smooth bounded domain $\Omega \subset \mathbb{R}^3$ and the initial velocity $\mathbf{u}_0$ in $\mathbf{L}^2_\sigma(\Omega) \cap \mathbf{L}^3(\Omega)$. They assumed that $\mathbf{u}$ is a weak solution with an associated pressure p, and they proved regularity of $\mathbf{u}$ provided that

$$p \in L^r(0,T;\, L^s(\Omega)) \text{ with } 2/r + 3/s \le 2,\ \tfrac{3}{2} < s \le \infty,$$

$$\text{or } \nabla p \in L^r(0,T;\, \mathbf{L}^s(\Omega)) \text{ with } 2/r + 3/s \le 3,\ 1 < s \le \infty.$$

- Y. Zhou [38, 39] (2006): technical improvements of the results from [3] . . . ∇p is supposed to be in $L^r(0,T;\, \mathbf{L}^s(\mathbb{R}^3))$ with $2/r + 3/s = 3$, $1 < s < \infty$, $\frac{2}{3} < r < \infty$.

- J. Fan, S. Jiang and G. Ni [9] (2008): considered $\Omega = \mathbb{R}^3$, obtained results in Morrey and Besov spaces.

- Z. Cai, J. Fan and J. Zhai [6] (2010) proved the regularity of a Leray–Hopf weak solution $\mathbf{u}$ (with an associated pressure p) if one of the conditions holds

  - $p \in L^r(0,T;\ L^{s,\infty}(\mathbb{R}^3))$ with $2/r + 3/s = 2$, $1 \leq r < \infty$,
  - $p \in L^\infty(0,T;\ L^{3/2,\infty}(\mathbb{R}^3))$ with the corresponding norm "sufficiently small",
  - $\nabla p \in L^r(0,T;\ \mathbf{L}^{s,\infty}(\mathbb{R}^3))$ with $2/r + 3/s = 3$, $\frac{2}{3} < s < \infty$, $1 < r < \infty$,
  - $\nabla p \in L^{2/3}(0,T;\ \mathbf{L}^{\infty,\infty}(\mathbb{R}^3))$ with the corresponding norm "sufficiently small".

- S. Suzuki [34] (2012) considered a bounded smooth domain Ω and the initial velocity in $\mathbf{W}^{1,2}_{0,\sigma}(\Omega) \cap \mathbf{L}^{\infty}(\Omega)$. She proved the regularity of an existing weak solution $\mathbf{u}$ if an associated pressure p is in the Lorentz space $L^{r,\infty}(0,T;\, L^{q,\infty}(\Omega))$ with the corresponding norm "sufficiently small", where

$$\frac{2}{r} + \frac{3}{q} = 2, \quad \frac{5}{2} \leq q \leq 3, \qquad 2 \leq r \leq \frac{5}{2},$$

or, alternatively, if ∇p is in $L^{r,\infty}(0,T;\, \mathbf{L}^{q,\infty}(\Omega))$ with the corresponding norm "sufficiently small", where

$$\frac{2}{r} + \frac{3}{q} = 3, \quad \frac{5}{2} \leq q < 3, \qquad 1 \leq r \leq \frac{5}{3}.$$

- S. Bosia, M. Conti and V. Pata [4] (2014) considered either Ω bounded or $\Omega = \mathbb{R}^3$, and a Leray–Hopf weak solution $\mathbf{u}$. They proved the regularity criterion, saying that the identity

$$\liminf_{\epsilon \longrightarrow 0} \; \epsilon^{3/2} \int_0^{T_\epsilon} \|\nabla p(\,.\,,t)\|_s^{r(1-\epsilon)} \, \mathrm{d}t \; = \; 0,$$

where $T_\epsilon := T - \mathrm{e}^{-1/\epsilon}$, $\epsilon > 0$, and $2/r + 3/s = 3$, $1 < s \leq 3$, implies that solution $\mathbf{u}$ is regular in Q_T.

More on the paper [3] (2002) by L. Berselli and G. P. Galdi.

The results and proofs are formulated for $\Omega = \mathbb{R}^n$. However, here we consider only the case $n = 3$.

The advantage of the whole space $\Omega = \mathbb{R}^3$: one can use the inequality

$$\|p\|_q \;\leq\; c\,\|\mathbf{u}\|_{2q}^2 \qquad \text{for } 1 < q < \infty, \tag{4.3}$$

obtained from the equations $\Delta p = -\partial_i\partial_j(u_i u_j)$ by means of the Calderon–Zygmund inequality.

The initial velocity is assumed to be in $\mathbf{L}^2_\sigma(\mathbb{R}^3) \cap \mathbf{L}^3(\Omega)$. This guarantees the existence of a "strong" solution on some time interval $(0, T_0)$. The authors show that, under some conditions on p, one has $T_0 = T$.

Multiplying the Navier–Stokes equation (2.1) (with $\mathbf{f} = \mathbf{0}$) by $|\mathbf{u}|\mathbf{u}$ and integrating in $\mathbb{R}^3$, one gets

$$\frac{1}{3}\,\frac{\mathrm{d}}{\mathrm{d}t}\,\|\mathbf{u}\|_3^3 + \nu \int_{\mathbb{R}^3} |\mathbf{u}||\nabla\mathbf{u}|^2\,\mathrm{d}\mathbf{x} + \frac{4\nu}{9}\,\big\|\nabla|\mathbf{u}|^{3/2}\big\|_2^2$$

$$\leq \frac{2}{3}\int_{\mathbb{R}^3} |p|\,|\mathbf{u}|^{1/2}\,\big|\nabla|\mathbf{u}|^{3/2}\big|\,\mathrm{d}\mathbf{x}. \tag{4.4}$$

Applying (4.3), interpolation inequalities and the Young inequality, the authors estimate the right hand side from above by something that can be absorbed by the left hand side + something that is integrable with respect to t on $(0,T)$ times $\|\mathbf{u}\|_3^3$. Then, from Gronwall's inequality, one gets $\mathbf{u} \in L^\infty(0,T;\, \mathbf{L}^3(\mathbb{R}^3))$, and especially

$$\nabla|\mathbf{u}|^{3/2} \in L^2(0,T;\, \mathbf{L}^2(\mathbb{R}^3)) \implies |\mathbf{u}|^{3/2} \in L^2(0,T;\, \mathbf{L}^6(\mathbb{R}^3))$$
$$\implies \mathbf{u} \in L^3(0,T;\, \mathbf{L}^9(\mathbb{R}^3)),$$

which is the class of regularity.

Recall that p is supposed to be in $L^r(0,T;\ L^s(\mathbb{R}^3))$ for r, s, satisfying certain conditions. Since (4.3) is not needed if $\frac{9}{4} \leq s < 3$, the results hold, for these s, also in a half–space or in a smooth bounded domain Ω in $\mathbb{R}^3$ or in a smooth exterior domain Ω in $\mathbb{R}^3$.

Finally, the authors show that if $\Omega = \mathbb{R}^3$ or Ω is any of the domains named above and if $\nabla p \in L^r(0,T;\ \mathbf{L}^s(\Omega))$ for some r, s such that $2/r + 3/s = 3$, $\frac{9}{7} \leq s \leq 3$, then $\mathbf{u}$ is regular. This generalizes previous results from [22]. The trick is that the right hand side of (4.4) is treated in the form $\int_\Omega \nabla p \cdot \mathbf{u}|\mathbf{u}|\, \mathrm{d}\mathbf{x}$ and is not modified by the integration by parts. ■

More on the papers [12, 13] (2006, 2010) by K. Kang and J. Lee.

The authors, in addition to inequality (4.4), also multiplied the Navier–Stokes equation by $\mathbf{u}\,|\mathbf{u}|^2$. By analogy with (4.4), they obtained

$$\frac{1}{4}\,\frac{\mathrm{d}}{\mathrm{d}t}\,\|\mathbf{u}\|_4^4+\nu\int_{\mathbb{R}^3}|\mathbf{u}|^2|\nabla\mathbf{u}|^2\,\mathrm{d}\mathbf{u}\;\leq\;c\int_\Omega|p|\,|\nabla\mathbf{u}|\,|\mathbf{u}|^2\,\mathrm{d}\mathbf{x}.$$

In the case $s>3$, the right hand side is

$$\leq\;c\,\|\nabla\mathbf{u}\|_2\,\|p\|_s\,\big\||\mathbf{u}|^2\big\|_{2s/(s-2)}\;\leq\;c\,\|\nabla\mathbf{u}\|_2\,\|p\|_s\,\|\mathbf{u}\|_{4s/(s-2)}^2\;\leq\;\dots.$$

Finally, applying interpolation inequalities, Young's inequality, imbedding inequalities and Gronwall's inequality, the authors estimated $\mathbf{u}$ in $L^\infty(\epsilon,T;\mathbf{L}^4(\Omega))$ (for any $\epsilon>0$), which is the class of regularity.

This procedure enabled the authors to avoid inequality (4.3), obtained by means of the Calderon–Zygmund theorem.

More on the paper [25] (2002) by G. Seregin and V. Šverák.

The authors fix a representative of $\mathbf{u}$ such that

$$\liminf_{t\to t_0} \|\mathbf{u}(\,.\,,t)\|_2 \;\geq\; \|\mathbf{u}(\,.\,,t_0)\|_2 \qquad \text{for all } 0<t_0<T$$

and $\mathbf{u}$ is weakly continuous from $(0,T)$ to $\mathbf{L}^2_\sigma(\mathbb{R}^3)$. They prove the two lemmas (= Lemma 3.2 and Lemma 3.3 in [25]):

Lemma 11. *Given* $\Omega_0 \subset\subset \Omega$, $0<t_0\leq T$ *and* $0<\delta_0<\sqrt{t_0}$. *If*

$$a(\Omega_0,t_0,\delta_0) \;:=\; \sup\bigg\{\frac{1}{R}\,\|\mathbf{u}(\,.\,,t)\|_{2;\,B_R(\mathbf{x}_0)};\; x_0\in\Omega_0,\; t\in[t_0-\delta_0^2,t_0],\; 0<R\leq d_0:=\frac{1}{2}\,\mathrm{dist}(\partial\Omega,\Omega_0)\bigg\} \;<\; \infty$$

then $$\lim_{t\to t_0-} \|\mathbf{u}(\,.\,,t)-\mathbf{u}(\,.\,,t_0)\|_{2;\,\Omega_0} \;=\; 0.$$

Lemma 12. *Let* $\mathbf{u}$, p *be a suitable weak solution to the Navier–Stokes problem in* $Q_T \equiv \mathbb{R}^3 \times (0,T)$ *and* $\mathbf{f} \in \mathbf{M}_{2,\gamma}(Q_T)$ *(the Morrey space) for some* $\gamma > 0$. *There exists* $\epsilon_* > 0$ *(depending only on* γ*) such that if for some* $R_* > 0$, $Q(\mathbf{z}_0, R_*) \subset Q_T$ *and*

$$\sup_{0<R<R_*} \sup_{t_0-R^2 \leq t \leq t_0} \frac{1}{R} \, \|\mathbf{u}(\,.\,,t)\|^2_{B_R(\mathbf{x}_0)} \; < \; \epsilon_* \tag{4.5}$$

then $\mathbf{z}_0$ *is a regular point of solution* $\mathbf{u}$.

Here, $\mathbf{z}_0 \equiv (\mathbf{x}_0, t_0)$ and $Q(\mathbf{z}_0, R) \equiv B_R(\mathbf{x}_0) \times (t_0 - R^2, t_0)$.

The principle idea of the proof is to show that (4.5) implies that

$$\frac{1}{\rho^2} \int_{Q(\mathbf{z}_0,\rho} \big(|\mathbf{u}|^3 + |p|^{3/2}\big) \, \mathrm{d}\mathbf{x} \, \mathrm{d}t \; \leq \; c \, \epsilon_*$$

(where c is independent of ρ) for all ρ sufficiently small.

The authors prove the identities

$$\begin{aligned} &\int_{B_R(\mathbf{x}_0)} \frac{1}{|\mathbf{y}-\mathbf{x}_0|} \left(2p(\mathbf{y},t) + |\widehat{\mathbf{u}}^{\mathbf{x}_0}(\mathbf{y},t)|^2\right) \, \mathrm{d}\mathbf{y} \\ &\quad = \int_{B_R(\mathbf{x}_0)} \frac{1}{R} \left(3p(\mathbf{y},t) + |\mathbf{u}(\mathbf{y},t)|^2\right) \, \mathrm{d}\mathbf{y} \\ &\quad = R^2 \int_{\mathbb{R}^3 \smallsetminus B_R(\mathbf{x}_0)} \nabla_{\mathbf{y}}^2 \Big(\frac{1}{|\mathbf{y}-\mathbf{x}_0|}\Big) : \big[\mathbf{u}(\mathbf{y},t) \otimes \mathbf{u}(\mathbf{y},t)\big] \, \mathrm{d}\mathbf{y}, \qquad (4.6) \end{aligned}$$

where

$$\widehat{\mathbf{u}}^{\mathbf{x}_0}(\mathbf{y},t) \; := \; \mathbf{u}(\mathbf{y},t) - \frac{[\mathbf{u}(\mathbf{y},t)\cdot(\mathbf{y}-\mathbf{x}_0)]\,(\mathbf{y}-\mathbf{x}_0)}{|\mathbf{y}-\mathbf{x}_0|^2}.$$

Assume that t_0 is the first instant of time when a singularity appears.

For $0 < t < t_0$, applying energy estimates, Hölder's inequality and estimates of singular integrals following from the Calderon–Zygmund theory, the authors derive

$$\lim_{R\to\infty} \sup_{0\leq t<t_0} \int_{\mathbb{R}^3 \smallsetminus B_R(\mathbf{0})} |\mathbf{u}(\mathbf{x},t)|^2 \, \mathrm{d}\mathbf{x} \; = \; 0.$$

Since $\displaystyle \liminf_{t\to t_0-} \int_{\mathbb{R}^3 \smallsetminus B_R(\mathbf{0})} |\mathbf{u}(\mathbf{y},t)|^2 \, \mathrm{d}\mathbf{x} \;\geq\; \int_{\mathbb{R}^3 \smallsetminus B_R(\mathbf{0})} |\mathbf{u}(\mathbf{x},t_0)|^2 \, \mathrm{d}\mathbf{x}$,
we get

$$\lim_{R\to\infty} \sup_{0\leq t\leq t_0} \int_{\mathbb{R}^3 \smallsetminus B_R(\mathbf{0})} |\mathbf{u}(\mathbf{x},t)|^2 \, \mathrm{d}\mathbf{x} \; = \; 0. \tag{4.7}$$

Further, the authors use identities (4.6), Lemma 11 and either condition (4.3) or condition (4.4) and prove that for any $\rho > 0$

$$\lim_{t\to t_0-} \|\mathbf{u}(\,.\,,t) - \mathbf{u}(\,.\,,t_0)\|_{2;\,B_\rho(\mathbf{x}_0)} \; = \; 0.$$

From this and (4.7), they deduce that

$$\lim_{t \to t_0 -} \|\mathbf{u}(\,.\,, t) - \mathbf{u}(\,.\,, t_0)\|_2 \; = \; 0. \tag{4.8}$$

(Recall that $\|\,.\,\|_2 = \|\,.\,\|_{2;\,\Omega}$, which is now $\|\,.\,\|_{2;\,\mathbb{R}^3}$.)

As an intermediate result, using identities (4.6) and condition (4.3), the authors derive the inequality

$$\frac{1}{2R}\,\|\mathbf{u}(\,.\,, t)\|_{2;\,B_R(\mathbf{x}_0)} \; \leq \; \frac{1}{2}\int_{B_R(\mathbf{x}_0)} \frac{g(\mathbf{x}, t)}{|\mathbf{x}-\mathbf{x}_0|}\,\mathrm{d}\mathbf{x} + \int_{B_R(\mathbf{x}_0)} \frac{|\widetilde{\mathbf{u}}^{\mathbf{x}_0}(\mathbf{x}, t_0)|^2}{|\mathbf{x}-\mathbf{x}_0|}\,\mathrm{d}\mathbf{x}$$

$$\leq \; \int_{B_R(\mathbf{x}_0)} \frac{1}{|\mathbf{x}-\mathbf{x}_0|}\,\big[g(\mathbf{x}, t) - \big(|\mathbf{u}(\mathbf{x}, t)|^2 + 2p(\mathbf{x}, t)\big)\big]\,\mathrm{d}\mathbf{x}, \tag{4.9}$$

where

$$\widetilde{\mathbf{u}}^{\mathbf{x}_0}(\mathbf{x}, t) \; := \; \frac{[\mathbf{u}(\mathbf{y}, t)\cdot(\mathbf{y}-\mathbf{x}_0)]\,(\mathbf{y}-\mathbf{x}_0)}{|\mathbf{y}-\mathbf{x}_0|^2}.$$

Let ϵ_* be the number from Lemma 12. There exists $R_* > 0$ such that

$$\frac{1}{2}\int_{B_{R_*}(\mathbf{x}_0)} \frac{g(\mathbf{x},t_0)}{|\mathbf{x}-\mathbf{x}_0|}\,\mathrm{d}\mathbf{x} + \int_{B_{R_*}(\mathbf{x}_0)} \frac{|\widetilde{\mathbf{u}}^{\mathbf{x}_0}(\mathbf{x},t_0)|^2}{|\mathbf{x}-\mathbf{x}_0|}\,\mathrm{d}\mathbf{x}$$

$$+ \int_{B_{R_*}(\mathbf{x}_0)} \frac{1}{|\mathbf{x}-\mathbf{x}_0|}\,\big[g(\mathbf{x},t_0) - \big(|\mathbf{u}(\mathbf{x},t_0)|^2 + 2p(\mathbf{x},t_0)\big)\big]\,\mathrm{d}\mathbf{x}$$

$$= \frac{3}{2}\int_{B_{R_*}(\mathbf{x}_0)} \frac{g(\mathbf{x},t_0)}{|\mathbf{x}-\mathbf{x}_0|}\,\mathrm{d}\mathbf{x}$$

$$- R_*^2 \int_{\mathbb{R}^3\setminus B_{R_*}(\mathbf{x}_0)} K(\mathbf{x},\mathbf{x}_0) : \big(\mathbf{u}(\mathbf{x},t_0)\otimes\mathbf{u}(\mathbf{x},t_0)\big)\,\mathrm{d}\mathbf{x} \;<\; \frac{\epsilon_*}{2},$$

where

$$K(\mathbf{x},\mathbf{x}_0) \;:=\; \nabla_{\mathbf{x}}^2\Big(\frac{1}{|\mathbf{x}-\mathbf{x}_0|}\Big).$$

Due to (4.8) and the weak L^2–continuity of $\mathbf{u}$, the function

$$t \quad \longmapsto \quad \frac{3}{2}\int_{B_{R_*}(\mathbf{x}_0)} \frac{g(\mathbf{x},t)}{|\mathbf{x}-\mathbf{x}_0|}\,\mathrm{d}\mathbf{x} - R_*^2 \int_{\mathbb{R}^3 \setminus B_{R_*}(\mathbf{x}_0)} K(\mathbf{x},\mathbf{x}_0) : \big(\mathbf{u}(\mathbf{x},t) \otimes \mathbf{u}(\mathbf{x},t)\big)\,\mathrm{d}\mathbf{x}$$

is left–continuous at point t_0. Hence there exists $\delta_* > 0$ (sufficiently small) such that

$$\frac{1}{2}\int_{B_{R_*}(\mathbf{x}_0)} \frac{g(\mathbf{x},t)}{|\mathbf{x}-\mathbf{x}_0|}\,\mathrm{d}\mathbf{x} + \int_{B_{R_*}(\mathbf{x}_0)} \frac{|\widetilde{\mathbf{u}}^{\mathbf{x}_0}(\mathbf{x},t)|^2}{|\mathbf{x}-\mathbf{x}_0|}\,\mathrm{d}\mathbf{x} + \int_{B_{R_*}(\mathbf{x}_0)} \frac{1}{|\mathbf{x}-\mathbf{x}_0|}\,\Big[g(\mathbf{x},t) - \big(|\mathbf{u}(\mathbf{x},t)|^2 + 2p(\mathbf{x},t)\big)\Big]\,\mathrm{d}\mathbf{x} \; < \; \frac{\epsilon^*}{2}$$

for all $t_0 - \delta_*^2 \leq t \leq t_0$.

However, then (4.9) yields

$$\frac{1}{2R}\,\|\mathbf{u}(\,.\,,t)\|_{2;\,B_R(\mathbf{x}_0)}\;<\;\frac{\epsilon_*}{2}$$

for $t_0-\delta_*^2\leq t\leq t_0$.

Due to Lemma 12, $\mathbf{z}_0\equiv(\mathbf{x}_0,t_0)$ is a regular point of solution $\mathbf{u}$.

Recall that we have used inequality (4.9), which follows from (4.6) and condition (4.3). Moreover, condition (4.3) has also been used in order to obtain (4.8).

If one uses condition (4.4) instead of (4.3) then the procedure is analogous. ■

References

[1] H. Beirao da Veiga: Concerning the regularity of the solutions to the Navier–Stokes equations via the truncation method II. *Équationsaux dérivées partielles et applications.* Gauthier–Villars, Éd. Sci.Méd. Elsevier, Paris 1998, pp. 127–138.

[2] L. Berselli: Sufficient conditions for the regularity of the solutions of the Navier–Stokes equations. *Math. Meth. Appl. Sci.* **22**, 1999, 1079–1085.

[3] L. Berselli, G. P. Galdi: Regularity criterions involving the pressure for the weak solutions to the Navier–Stokes equations. *Proc. Am. Math. Soc.* 2002

[4] S. Bosia, M. Conti, V. Pata: A regularity criterion for the NavierStokes equations in terms of the pressure gradient. *Central European J. Math.* **12**, 2014, 1015–1025.

[5] L. Caffarelli, R. Kohn, L. Nirenberg: Partial regularity of suitable weak solutions of the Navier–Stokes equations. *Comm. on Pure and Appl. Math.* **35**, 1982, 771–831.

[6] Z. Cai, J. Fan, J. Zhai: Regularity criteria in weak spaces for Navier-Stokes equations in R^3. *Diff. Int. Equations* **23**, 2010, 1023–1033.

[7] D. Chae, J. Lee: Regularity criterion in terms of pressure for the Navier–Stokes equations. *Nonlinear Analysis: Theory, Methods and Appl.* **46**, 2001, 727–735.

[8] J. Fan, T. Ozawa: Regularity criterion for weak solutions to the Navier–Stokes equations in terms of pressure. *J. of Inequalities and Applications*, 2008, article ID 412678, 6 pages, DOI: 10.1155/2008/412678.

[9] J. Fan, S. Jiang, G. Ni: On regularity criteria for the n-dimensional NavierStokes equations in terms of the pressure. *J. Diff. Equations* **244**, 2008, 2963-2979.

[10] G. P. Galdi: *An Introduction to the Mathematical Theory of the Navier–Stokes Equations*. 2nd edition, Springer 2011.

[11] G. P. Galdi: An Introduction to the Navier–Stokes initial–boundary value problem. In *Fundamental Directions in Mathematical Fluid Mechanics*, ed. G. P. Galdi, J. Heywood, R. Rannacher, series "Advances in Mathematical Fluid Mechanics". Birkhauser, Basel 2000, 1–98.

[12] K. Kang, J. Lee: On regularity criteria in conjunction with the pressure of the NavierStokes equations. *Int. Math. Res. Not.* Art. ID 80762, 2006, p. 25

[13] K. Kang, J. Lee: Erratum: On regularity criteria in conjunction with the pressure of the NavierStokes equations. *Int. Math. Res. Not. IMRN* **9**, 2010, 1772–1774.

[14] S. Kaniel: A sufficient condition for smoothness of solutions of the Navier–Stokes equations. *Israel J. Math.* **6**, 1969, 354–358.

[15] O. A. Ladyzhenskaya, G. Seregin: On partial regularity of suitable weak solutions to the three–dimensional Navier–Stokes equations. *J. Math. Fluid Mech.* **1**, 1999, 356–387.

[16] J. L. Lions: *Quelques méthodes de résolution des problémes âux limites non linéaire.* Dunod, Gauthier–Villars, Paris 1969.

[17] F. Lin: A new proof of the Caffarelli–Kohn–Nirenberg theorem. *Comm. on Pure and Appl. Math.* **51**, 1998, 241–257.

[18] J. Nečas, J. Neustupa: New conditions for local regularity of a suitable weak solution to the Navier–Stokes equations. *J. Math. Fluid Mech.* **4**, 2002, 237–256.

[19] J. Neustupa, P. Penel: Anisotropic and geometric criteria for interior regularity of weak solutions to the 3D Navier–Stokes equations. In *Mathematical Fluid Mechanics, Recent Results and Open Questions*, ed. J. Neustupa and P. Penel, Birkhauser, Basel 2001, 237–268.

[20] J. Neustupa: The boundary regularity of a weak solution of the Navier–Stokes equation and connection with the interior regularity of pressure. *Appl. of Math.* **6**, 48, 2003, 547–558.

[21] T. Ohyama: Interior regularity of weak solutions of the time dependent Navier–Stokes equations. *Proc. Japan. Acad.* **36**, 273

[22] M. O'Leary: Pressure conditions for the local regularity of solutions of the Navier–Stokes equations. *Electronic J. Diff. Equations*, 1998, paper No. 12, 9 pp.

[23] Chen Qionglei, Zhang Zhifei: Regularity criterion via the pressure on weak solutions to the 3D Navier–Stokes equations. *Proc. of the AMS* **136**, 6, 2007, 1829–1837.

[24] W. Rudin: *Functional Analysis*. Mc Graw–Hill Inc., New York 1973.

[25] G. Seregin, V. Šverák: Navier–Stokes equations with lower bounds on the pressure. *Arch. Rat. Mech. Anal.* **163**, 2002, 65–86.

[26] J. Serrin, *On the interior regularity of weak solutions of the Navier–Stokes equations*, Arch. Rat. Mech. Anal. **9** (1962), 187–195.

[27] J. Serrin: The initial value problem for the Navier–Stokes equations. *Nonlinear Problems*, ed. R. E. Langer, Madison: University of Wisconsin Press 9, 1963, 69–98.

[28] J. Simon: On the existence of pressure for solutions of the variational Navier–Stokes equations. *J. Math. Fluid Mech.* 1999

[29] Z. Skalák, P. Kučera: Regularity of pressure in the neighbourhood of regular points of weak solutions of the Navier–Stokes equations. *Appl. of Math.* **48**, 6, 2003, 573–586.

[30] H. Sohr, W. von Wahl: On the regularity of the pressure of weak solutions of Navier–Stokes equations. *Arch. Math.* **46**, 1986, 428–439.

[31] H. Sohr: *The Navier–Stokes Equations. An Elementary Functional Analytic Approach.* Birkhäuser Advanced Texts, Basel–Boston–Berlin 2001.

[32] M. Struwe: On partial regularity results for the Navier–Stokes equations. *Comm. Pure. Appl. Math.* **41**, 437–

[33] M. Struwe: On a Serrin–type regularity criterion for the Navier–Stokes equations in terms of the pressure. *J. Math. Fluid Mech.* **9**, 2007, 235–242.

[34] T. Suzuki: Regularity criteria of weak solutions in terms of pressure in Lorentz spaces to the Navier–Stokes equations. J. Math. Fluid Mech. **14**, 2012,

[35] Y. Taniuchi: On generalized energy inequality of Navier–Stokes equations. *Manuscripta Math*. **94**, 1997, 365–384.

[36] R. Temam: *Navier–Stokes Equations*. North–Holland, Amsterdam–New York–Oxford 1977.

[37] J. Wolf: The Caffarelli-Kohn-Nirenberg theorem – a direct proof by Campanato's method. *Parabolic and Navier–Stokes equations, Banach Center Publ*. **81**, Part 2, Polish Acad. of Sci., Inst. Math., Warsaw 2008, 533-552.

[38] Y. Zhou: On regularity criteria in terms of pressure for the Navier-Stokes equations in R^3. *Proc. Amer. Math. Soc*. **134**, 2006, 149-156.

[39] Y. Zhou: On a regularity criterion in terms of the gradient of pressure for the Navier-Stokes equations in R^N. *Z. Angew. Math. Phys*. **57**, 2006, 384-392.