Strong Periodic Solutions to non-autonomous Stokes-type Equations in Unbounded Domains

Anton Seyfert

August 31, 2016

Setting and General Problem

Consider the two differential equations:
Navier-Stokes equations: Flow of an incompressible fluid in an exterior domain $\Omega \subseteq \mathbb{R}^3$ with time dependent viscosity $\nu(t)$:

$$(NS)\begin{cases} u_t - \nu(t)\Delta u + (u\cdot\nabla)u + \nabla p &= \operatorname{div} F \quad \text{in } \mathbb{R}\times\Omega, \\ \operatorname{div} u &= 0 \quad \text{in } \mathbb{R}\times\Omega \end{cases}$$

Ornstein-Uhlenbeck-type equations: Model equation for the Navier-Stokes flow with time dependent rotating effects $\omega(t)$:

$$(OU)\begin{cases} u_t - \Delta u - (\omega(t)e_3 \times x)\cdot\nabla u + \omega(t)\times u & \\ \quad +\nabla p = \operatorname{div} F - (u\cdot\nabla)u & \text{in } \mathbb{R}\times\mathbb{R}^3, \\ \operatorname{div} u = 0 & \text{in } \mathbb{R}\times\mathbb{R}^3 \end{cases}$$

Aim: Existence and uniqueness of bounded and time-periodic strong-type solutions to these equations for suitable data. Existence of weak periodic solutions of (OU) in exterior domains were treated in [1].
Main problem: The evolution families $U(t,s)$ generated by the linearized equations are bounded but not exponentially stable due to the unbounded domains.

Solvability of the Linearized Systems

Idea:

- Adjust Duhamel's formula to the whole real line:
$$u(t) := S(f)(t) := \int_{-\infty}^{t} U(t,r)\operatorname{div} F(r)dr$$
- Make use of L^p-L^q-type gradient estimates of the evolution families in Lorentz spaces:
$$\|\nabla U^*(t,s)\phi\|_{L^{q,1}} \le C(t-s)^{-\frac{1}{2}-\frac{3}{2}(\frac{1}{p}-\frac{1}{q})}\|\phi\|_{L^{p,1}}$$

Theorem Let $\nu(\cdot)$ and $\omega(\cdot)$ be Lipschitz continuous and $0 < c < \nu(t) < C$. Then, for each $F \in L^\infty(\mathbb{R}, L^{3/2,\infty})$ the linearized equations of (NS) and (OU) have a unique mild solution $u \in C_w(\mathbb{R}, L^{3,\infty})$. Additionally
$\sup_{t\in\mathbb{R}}\|u(t)\|_{L^{3,\infty}} \le C\|F\|_{L^\infty(\mathbb{R},L^{3/2,\infty})}$.

Sketch of the proof:

- Consider the sublinear operator
$$T_t : \phi \mapsto \|\nabla U^*(t,t-\cdot)\phi\|_{L^{3,1}}.$$
Decay property of $\nabla U^*(t,s)$ implies
$$T_t : L^{3/2\pm\epsilon,1} \to L^{1\pm\delta^\pm,\infty}((-\infty,t),\mathbb{R}).$$
- Duality argument yields finiteness of mild solution in a weak sense if $T_t : L^{3/2,1} \to L^1((-\infty,t),\mathbb{R})$ is bounded
- Application of the interpolation theorem of Marcinkiewicz implies the desired boundedness of T_t
- Uniqueness follows also by L^p-L^q-type estimates

Periodic Solutions

T-periodic ($T > 0$) coefficients of the linear part imply $U(t+T,s+T) = U(t,s)$ and $S(F(\cdot))(t+T) = S(F(\cdot+T))(t)$.

$\Rightarrow$ For T-periodic $\nu(\cdot)$ or $\omega(\cdot)$, the solution operator S maps T-periodic forces F to T-periodic solutions u.

Semilinear Systems

- Write semilinear differential equations as a fixed point problem.
- Use Hölder inequality in Lorentz spaces for nonlinearities.
- Smallness condition on the force F permits application of the Banach fixed-point theorem.

Theorem Let $\nu(\cdot)$, $\omega(\cdot)$ be Lipschitz continuous, T-periodic ($T > 0$) and $0 < c < \nu(t) < C$. Then (NS) and (OU) have unique small T-periodic mild solutions $u \in C_w(\mathbb{R}, L^{3,\infty})$ if F is small in $L^\infty(\mathbb{R}, L^{3/2,\infty})$ and T-periodic.

Abstract Generalization

- Consider the abstract Cauchy problem
$$u'(t) - A(t)u(t) = BF(t), \quad t \in \mathbb{R}.$$
- Exchange L^p-L^q-type estimates by abstract polynomial decay
$$\|B^*U^*(t,s)\psi\|_V \le C(t-s)^{-\alpha_i}\|\psi\|_{Z_i}$$
with Banach space V, Banach interpolation couple (Z_1, Z_2) and $\alpha_1 > 1 > \alpha_2 > 0$.
- Transfer of the previous proof yields for each $F \in L^\infty(\mathbb{R}; V')$ a mild solution $u \in C_w(\mathbb{R}; (Z_1', Z_2')_{\theta,\infty})$, where $1 = (1-\theta)\alpha_1 + \theta\alpha_2$.
- Periodic and semilinear problems can be transfered in the same way.

References

[1] G. P. Galdi, A. L. Silvestre, Existence of time-periodic solutions to the Navier-Stokes equations around a moving body. *Pacific J. Math.* **223** (2006), 251-267.

[2] M. Geissert, M. Hieber, H. Nguyen, A general approach to time periodic incompressible viscous fluid flow problems. *Arch. Rational Mech. Anal.* **220** (2016), 1095-1118.

[3] M. Yamazaki, The Navier-Stokes equations in the weak-L^n space with time-dependent external force, *Math. Ann.*, **317** (2000), 635-675.