Convergent finite difference scheme for the compressible viscous flow

Radim Hošek & Bangwei She

hosek@math.cas.cz, she@math.cas.cz

Institute of Mathematics, Czech Academy of Sciences

INTRODUCTION

Compressible Navier–Stokes system:

$$\begin{cases} \rho_t + \nabla \cdot (\rho \mathbf{u}) = 0 \\ (\rho \mathbf{u})_t + \nabla \cdot (\rho \mathbf{u} \otimes \mathbf{u}) = -\nabla p + \mu \Delta \mathbf{u} \end{cases}$$

density ϱ, velocity $\mathbf{u}$ and pressure $p(\varrho) = a\varrho^\gamma$.

Compressible Navier–Stokes system is not known to possess strong long-time solution even for smooth data. Due to works of Lions [1] and Feireisl [2] the existence of weak solution for Lipschitz domain $\Omega \subset \mathbb{R}^3$ is guaranteed for the exponent $\gamma > 3/2$.

Numerical schemes of such model have been studied by many researchers, most among them are concerned with stability, accuracy or efficiency. However whether the solution of a scheme is reasonable or close to the physical solution, needs to be furthered studied. Motivated by the the work of Karper [3], in which the convergence of a DG-type scheme to the weak solution has been studied, we would like to show the convergence to the weak solution for a finite difference scheme as well as its numerical performance.

THE SCHEME

We consider $\Omega = (0,1)^d$, discretized uniformly into cubes K of size h, each of them represented by its center. These centers build the primary grid, while the faces of the cubes $\partial K_{i,\pm}$, represented by their centers, form the dual grid. We look for density ρ in primary grid and velocity $\mathbf{u}$ in the dual grid; only i-th component of velocity being defined in face with the normal vector $\mathbf{e}_i$. The scheme reads

$$\delta_t \rho_K^n + \mathrm{div}_{\mathrm{up}}[\rho^n, \mathbf{u}^n]_K - h^\alpha (\Delta_h \rho^n)_K = 0, \qquad \delta_t\{\rho^n \hat{\mathbf{u}}_i^n\} + \{\mathrm{div}_{\mathrm{up}}[\rho^n \hat{\mathbf{u}}_i^n, \mathbf{u}^n]\} + \delta_{x_i} p^n - \mu \Delta_h u_i - h^\alpha \sum_{j=1}^d \delta_{x_j}\left(\hat{u}_i^n \delta_{x_j} \rho^n\right) = 0.$$

The averaging operators $\{.\}$ and $\hat{.}$ provide the transition between two grids, $\mathbf{div}_{\mathrm{up}}$ is the standard upwind divergence and δ_t a backward time difference ⇒ implicit scheme. Boundary conditions: homogeneous Dirichlet for velocity and homogeneous Neumann for density.

CONVERGENCE

Let $(\varrho_h, \mathbf{u}_h)_{h\to 0}$ be a sequence of solutions obtained through the numerical scheme on grids with dicretization parameter h and let the exponent $\gamma > 3$. Then there exists a subsequence $(\rho_h, \mathbf{u}_h) \rightharpoonup (\rho, \mathbf{u})$, which is a weak solution to compressible Navier–Stokes equations.

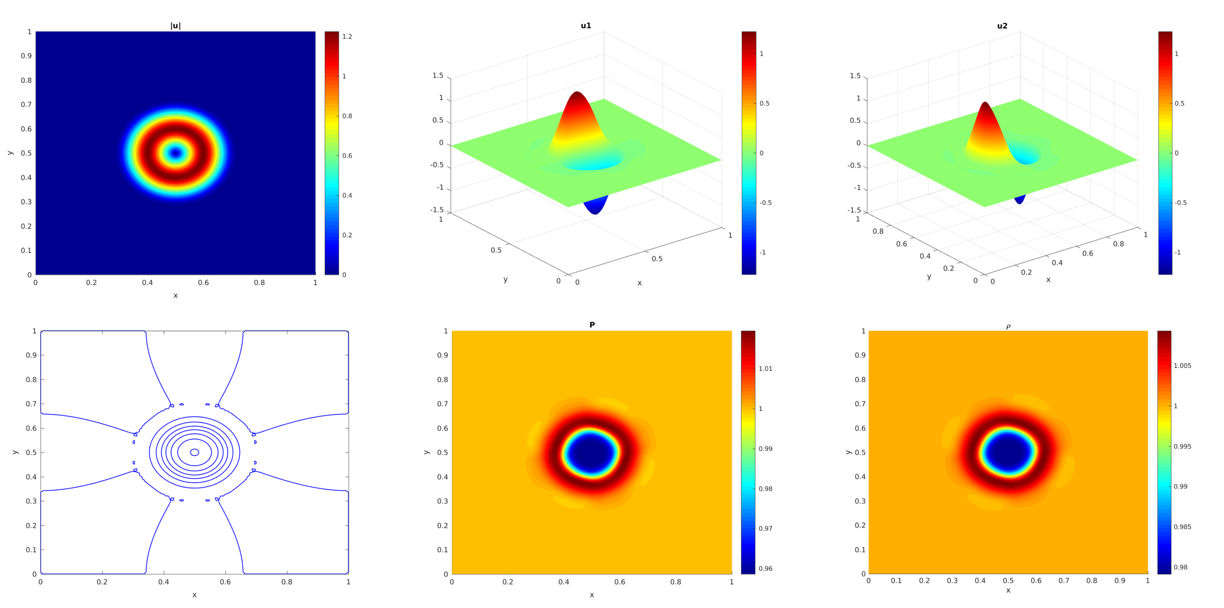
- Stability and existence of numerical solution,
  - renormalized continuity scheme - ingredient for
  - a priori energy estimates,
  - existence of a num sol (Schaeffer fix point),
  - positivity of the density,
  - equiintegrability ⇒ compactness of the terms.
- Consistency formulation - numerical solution is a weak solution plus residual
- Weak convergence
  - Convergence of the terms from compactness,
  - dealing with nonlinear terms,
  - higher integrability of the density (en.est.: $p \in L^\infty(L^1)$ only),
  - strong convergence of the density ⇒ convergence of the pressure term.

NUMERICAL TEST

Gresho Vortex test

Computational domain: $\Omega = [0,1]^2$. Initial condition

$$\rho = 1, \mathbf{u} = \tfrac{u_r}{r}\begin{pmatrix} y - 0.5 \\ 0.5 - x \end{pmatrix}, \; u_r(r) = \sqrt{\gamma}\begin{cases} 2r/R, & 0 \le r < R/2, \\ 2(1 - r/R), & R/2 \le r < R. \end{cases}$$
$$r = \sqrt{(x-0.5)^2 + (y-0.5)^2}, R = 0.2.$$



Upper figures: velocity magnitude and components; bellow: streamlines, pressure and density

$T = 0.01, \mu = 0.01$								
h	$\|\mathbf{u}_h - \mathbf{u}\|_{L^2(L^2)}$	EOC	$\|\nabla \mathbf{u}_h - \nabla \mathbf{u}\|_{L^2(L^2)}$	EOC	$\|\rho_h - \rho\|_{L^1(L^1)}$	EOC	$\|p_h - p\|_{L^\infty(L^1)}$	EOC
1/16	2.23e-01	–	7.84e-03	–	3.19e-06	–	6.66e-03	–
1/32	1.19e-01	0.91	4.09e-03	0.94	1.63e-06	0.97	4.27e-03	0.64
1/64	6.04e-02	0.97	2.01e-03	1.03	5.92e-07	1.46	2.27e-03	0.91
1/128	2.66e-02	1.18	8.98e-04	1.16	2.24e-07	1.40	1.17e-03	0.96

Acknowledgement

The present work has been supported by European Research Council(ERC) Grant Agreement 320078 "Mathematical thermodynamics of fluids" (PI Eduard Feireisl).

References

[1] P.-L. Lions. Mathematical topics in fluid mechanics. In: Compressible Models, vol. 2. Oxford University Press, New York (1998).

[2] E. Feireisl. On compactness of solutions to the comprssible isentropic Navier–Stokes equations when the density is not square integrable. Comment. Math. Univ. Carolinae, 42(1), 83–98 (2001).

[3] T. K. Karper. A convergent FEM-DG method for the compressible Navier—Stokes equations, Numer. Math. 125, 441–510 (2013).