On Leray's self-similar solutions of the Navier-Stokes equations

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Leray's profile equations

$$(LE) \qquad \begin{cases} \nabla \cdot U = 0 & \text{in} \quad \mathbb{R}^3, \\ (U \cdot \nabla)U - \Delta U + ay \cdot \nabla U + aU = -\nabla P & \text{in} \quad \mathbb{R}^3. \end{cases}$$

Here $a > 0$,

$$U = (U_1, U_2, U_3) = \text{velocity profile}$$
$$P = \text{pressure profile}$$

LERAY (1934): A non trivial solution (U, P) to (LE) gives rise to a self similar backward solution (u, p) to the (NSE) with blow up at $(0, 0)$, where

$$\begin{cases} u(x,t) = \frac{1}{\sqrt{-2at}} U\Big(\frac{x}{\sqrt{-2at}}\Big), \qquad p(x,t) = \frac{1}{-2at} P\Big(\frac{x}{\sqrt{-2at}}\Big) \\ (x,t) \in Q := \mathbb{R}^3 \times (-\infty, 0) \end{cases}$$

- $(U_0, -ay \cdot U_0)$ ($U_0 = \text{const}$) is a solution to (LE)
- $\Pi = \frac{1}{2}|U|^2 + P + ay \cdot U$ satisfies the maximum principle
- If $U \in L^\infty(\mathbb{R}^3)$ then u has Type I singularity at $(0,0)$

$$\|u(t)\|_{L^\infty} = \frac{1}{\sqrt{-2at}}\|U\|_{L^\infty} \quad \forall\, t < 0.$$

- Exclude self similar blow by Liouville properties of (LE).

Theorem (Nečas, Růžička, Šverák (1996))

The only weak solution $U \in L^3(\mathbb{R}^3)$ to (LE) is $U \equiv 0$. In particular, if (u,p) is a self similar Leray solution to (NSE) (satisfying the global energy inequality), then $u \equiv 0$.

Theorem (Tsai (1998))

1. *If* $U \in L^p(\mathbb{R}^3)$, $3 < p < +\infty$ *is a weak solution to (LE) then* $U \equiv 0$.
2. *If* $U \in L^\infty(\mathbb{R}^3)$, *is a weak solution to (LE) then* $U \equiv \text{const}$.
3. *If* (u, p) *is a local self similar Leray solution to (NSE) (satisfying the local energy inequality), then* $u \equiv 0$.

- Proof is based on the maximum principle of Π from the elliptic equation

$$-\Delta\Pi + (U + ay) \cdot \nabla\Pi = -|\nabla \times U|^2 \leq 0 \quad \text{in} \quad \mathbb{R}^3.$$

- The range $1 < p < 3$ not included.
- For local Leray solutions (u, p) it is natural $|U(y)| \leq \frac{C}{1+|y|}$.

Generalization of Tsai's result

Theorem (Chae, W. (2016))

Let $(U,P) \in C^\infty(\mathbb{R}^3)^3 \times C^\infty(\mathbb{R}^3)$ be a solution to (LE), and $\Omega = \nabla \times U$.

1. *Suppose that for some $q > 0$*

$$\|U\|_{L^q(B_1(y_0))} + \|\Omega\|_{L^2(B_1(y_0))} = o(|y_0|^{\frac{1}{2}}) \quad as \quad |y_0| \to +\infty.$$

Then, U is a constant function.

2. *Suppose that for some $\frac{3}{2} < q < +\infty$ and $\alpha > 0$*

$$\int_{B_1(y_0)\cap\{|U|>\alpha\}} |U|^q dx \to 0 \quad as \quad |y_0| \to +\infty.$$

Then U is a constant function.

Local pressure projection

For a ball $B \subset \mathbb{R}^3$ we define $E_B^* : W^{-1,s}(B) \to W^{-1,s}(B)$:

$$\langle E_B^* F, v\rangle = \int_B \pi \nabla \cdot v, \quad F \in W^{-1,s}(B), v \in W_0^{1,s'}(B),$$

where $\pi \in L_0^s(B)$ is the unique pressure of the Stokes system

$$\nabla \cdot w = 0, \quad -\nabla w + \nabla \pi = F \quad \text{in} \quad B$$
$$w = 0 \quad \text{on} \quad \partial B$$

Short: $E_B^*(F) = -\nabla\pi$. Clearly, $(E_B^*)^2 = E_B^*$

Local pressure representation

We have the following pressure representation in (NSE)

$$-\nabla p = -\partial_t \nabla p_h - \nabla p_1 - \nabla p_2 \quad \text{in} \quad B \times (-\infty, 0),$$

where

$$\begin{aligned} \nabla p_h &= -E_B^*(u), \\ \nabla p_1 &= -E_B^*((u \cdot \nabla)u), \\ \nabla p_2 &= E_B^*(\Delta u). \end{aligned}$$

Setting $v_B = u + \nabla p_h$, we see that

$$(NSE)_B \qquad \partial_t v_B + (u \cdot \nabla)u - \Delta u = -\nabla p_1 - \nabla p_2 \quad \text{in} \quad B \times (-\infty, 0).$$

Local suitable weak solution

Definition (local suitable weak solution)

Let $Q = \mathbb{R}^3 \times (-\infty, 0)$. A vector function $u \in L^2_{loc}(Q)$ is called a *local suitable weak solution to (NSE)*, if

1. $u \in L^\infty_{loc}(-\infty, 0; L^2_{loc}(\mathbb{R}^3)) \cap L^2_{loc}(-\infty, 0; W^{1,2}_{loc,\sigma}(\mathbb{R}^3))$.
2. u is a distributional solution to (NSE)
3. $\forall\ B \subset \mathbb{R}^3$ for all $\phi \in C^\infty_c(B \times (-\infty, 0))$, $\phi \geq 0$, for a. e. $t \in (-\infty, 0)$

$$\frac{1}{2}\int_B |v_B(t)|^2 \phi(t) dx + \int_{-\infty}^{t}\int_B |\nabla v_B|^2 \phi dxds$$

$$\leq \frac{1}{2}\int_{-\infty}^{t}\int_B |v_B|^2 \Big(\Delta + \frac{\partial}{\partial t}\Big)\phi + |v_B|^2 u \cdot \nabla \phi dxds$$

$$+ \int_{-\infty}^{t}\int_B (u \otimes v_B) : \nabla^2 p_h \phi dxds + \int_{-\infty}^{t}\int_B (p_1 + p_2) v_B \cdot \nabla \phi dxds$$

Local weak solution

Definition (Local weak and strong solution)

Let $f \in L^1(Q)$. Firstly, $u \in V^p(Q)$, $1 < p < \infty$, is called a *local weak solution* to the generalized NSE, if for all $\varphi \in C_c^\infty(Q)$ with $\nabla \cdot \varphi$

$$\int_Q -u \cdot \partial_t \varphi - u \otimes u : \nabla \varphi + S : D(u) = \int_Q f \cdot \varphi$$

Secondly, u is called a *local strong solution* if for all $Q' \Subset Q$

$$|D(u)|^{\frac{p-2}{2}} D(u) \in L^2(I', W^{1,2}(\Omega')).$$

Lemma 1 (ε criterion)

Let $\frac{3}{2} < q \le 3$. There exist two constants $\varepsilon_q > 0$ and $C_q > 0$ such that: If $u \in V^2_\sigma(Q)$ be a local suitable weak solution to (NSE), and $Q_r(z_0) \subset Q$, $z_0 = (x_0, t_0)$, then the condition

$$A_q(r, z_0) := r^{\frac{q-3}{q}} \operatorname*{ess\,sup}_{t \in (t_0 - r^2, t_0)} \|u(t)\|_{L^q(B_r(x_0))} \le \varepsilon_q. \tag{1}$$

implies $u \in L^\infty(Q_{\frac{r}{2}}(z_0))$, and it holds

$$\operatorname*{ess\,sup}_{Q_{\frac{r}{2}}(z_0)} |u| \le C_q r^{-1} A_q(r, z_0). \tag{2}$$

Step 1 *Decay estimate*: Let $0 < r \le 1$. By Sobolev's inequality,

$$r^{-2} \int_{B_r(y_0)} |U|^2 dy \le C \int_{B_1(y_0)} |U|^2 + |\nabla U|^2 dy$$
$$\le \|U\|^2_{L^q(B_2(y_0))} + \|\Omega\|^2_{L^q(B_2(y_0))} = o(|y_0|).$$

Define

$$\Psi(y_0) := \int_{B_1(y_0)} |U|^2 + |\nabla U|^2 dy.$$

Consequently,

$$r^{-2} \int_{B_r(y_0)} |U|^2 dy \le \Psi(y_0) = o(|y_0|).$$

Step 2 *U has sub linear growth*:

For almost all $t \in \left(-\frac{1}{2a} - 1, -\frac{1}{2a}\right)$:

$$\begin{aligned} \int_{B_r(y_0)} |u(x,t)|^2 dx &= \frac{1}{-2at} \int_{B_r(y_0)} \left| U\left(\frac{x}{\sqrt{-2at}}\right)\right|^2 dx \\ &\leq \sqrt{1+2a} \int_{B_r\left(\frac{y_0}{\sqrt{-2at}}\right)} |U|^2 dy \\ &\leq Cr^2 \tilde{\Psi}(y_0), \end{aligned} \tag{3}$$

where $\tilde{\Psi}(y_0) = \sqrt{1+2a}\,\sup_{1\leq s\leq \sqrt{1+2a}} \Psi\left(\frac{y_0}{s}\right) = o(|y_0|)$.

Proof of main theorem (1.)

We take

$$r = \min\left\{1, \frac{\varepsilon_2^2}{\tilde{\Psi}(y_0)}\right\}.$$

where $\varepsilon_2 > 0$ denotes the constant in Lemma 1 ($q = 2$). Then,

$$A_2(r, z_0) \le \varepsilon_2 \quad \text{for} \quad z_0 = \left(y_0, -\frac{1}{2a}\right).$$

Then Lemma 1 and (**??**) yields

$$|U(y_0)| \le \sup_{Q_{\frac{r}{2}}(z_0)} |u| \le C_2 r^{-1} A_2(r, z_0) \le C_2 r^{-\frac{1}{2}} \tilde{\Psi}(y_0)^{\frac{1}{2}}$$
$$\le C_2(\varepsilon_2^{-1}\tilde{\Psi}(y_0) + \tilde{\Psi}(y_0)^{\frac{1}{2}}).$$

This implies that U has sub linear growth.

Step 3 $|P(y)| = O(|y|^{\frac{9}{2}})$

Step 4: $\Pi = \frac{|U|^2}{2} + P + ay \cdot U =$ const by the maximum principle. Thus,

$$-\nabla P = \nabla \frac{|U|^2}{2} + ay \cdot \nabla U + aU, \quad \Delta\Big(P + \frac{|U|^2}{2}\Big) = 0.$$

Step 5 *U is constant*:

$$-\Delta U + (U \cdot \nabla)U - \nabla \frac{|U|^2}{2} = -\Delta U + \Omega \times U = 0.$$

$$-\Delta \frac{|U|^2}{2} + |\nabla U|^2 = 0 \quad \text{in} \quad \mathbb{R}^3.$$

On the other hand, applying $\nabla \cdot$ to (LE), we get

$$-\Delta P = \sum_{i,j=1}^{3} \frac{\partial U_i}{\partial y_j} \frac{\partial U_j}{\partial y_i j} = |\nabla U|^2 - |\nabla \Omega|^2.$$

Combining the above identities, we find $|\Omega|^2 = 0$.

Thank you !