Boundary layer for a non-Newtonian flow over a rough surface
derivation of a wall law

Aneta Wróblewska-Kamińska
Institute of Mathematics, Polish Academy of Sciences
joint work with David Gérard-Varet

Praha, 31st August 2016
The main goal:

- To describe the effect of wall-roughness on fluid flow
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Motivations:

▶ simulations of fluid flow when details of roughness is not known or too small (too expensive) for computational grids
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- Glaciology: the interaction of glaciers with the underlying rocks is unavailable

Idea!

- Try to describe some average effect
- How?: Replace the rough boundary by the artificial but sooth one and prescribed then a homogenised boundary condition: wall law
- Question: What is a proper, a good wall law (in a sense of error estimates)?
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Non-Newtonian fluids

- blood flow, glacier motion, paint, quicksand
- smart fluid, corn starch and water (oobleck), silica and polyethylene glycol (liquid body armour)
Model to consider: domain $\Omega^\varepsilon \subset \mathbb{R}^2$

- $\Omega = (0, 1)^2$ is the flat portion of the channel
- $R^\varepsilon$ is the rough portion of the channel
- $\Gamma^\varepsilon := \{x_2 = \varepsilon \gamma(x_1/\varepsilon)\}$ - a bottom surface parametrized by $\varepsilon \ll 1$
- $\Sigma_0 := (0, 1) \times \{0\}$ is the interface between the rough and flat part
Model to consider: stationary (generalized) p-Stokes system on $\Omega^\varepsilon \subset \mathbb{R}^2$

\[
\left\{
\begin{array}{ll}
-\text{div} S(Du^\varepsilon) + \nabla p^\varepsilon = e_1 & \text{in } \Omega^\varepsilon, \\
\text{div} u^\varepsilon = 0 & \text{in } \Omega^\varepsilon, \\
u^\varepsilon|_{\Gamma^\varepsilon} = 0, & u^\varepsilon|_{x_2=1} = 0, \\
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$e_1 = (1, 0)^t$
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\]

\begin{itemize}
  \item $e_1 = (1, 0)^t$
  \item $S : \mathbb{R}_{\text{sym}}^{2 \times 2} \rightarrow \mathbb{R}_{\text{sym}}^{2 \times 2}$ - the stress tensor of the fluid of power-law type
\end{itemize}

\[S : \mathbb{R}_{\text{sym}}^{2 \times 2} \rightarrow \mathbb{R}_{\text{sym}}^{2 \times 2}, \quad S(Du^\varepsilon) = |Du^\varepsilon|^{p-2}Du^\varepsilon, \quad 1 < p < +\infty\]
Strategy how to understand the asymptotic behaviour of solutions and derive a wall law.
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As for the Newtonian case, one can expect that:

$$u^ε_{app}(x) = u^0(x) + εu^1(x) + \cdots + u^0_{bl}(x/ε) + εu^1_{bl}(x/ε) + \cdots$$
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Then the plan is:

▶ to derive the equations satisfied by the first terms

▶ to show convergence of the boundary layer term to a constant field away from the boundary

▶ The limit wall law is a homogeneous Dirichlet condition.

▶ The \(O(\varepsilon)\) correction to this wall law is a slip condition of Navier type, with \(O(\varepsilon)\) slip length.

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Zero order approximation - the Poiseuille flow

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  u^0\big|_{\Sigma_0} = 0, & u^0\big|_{x_2=1} = 0, \quad u \text{ 1-periodic in } x_1.
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\[
p^0 = 0, \quad u^0 = (U(x_2), 0), \quad U(x_2) = \frac{p - 1}{p} \left( \sqrt{2 - \frac{p}{p-1}} - \sqrt{2 \frac{p}{p-1}} |x_2 - \frac{1}{2} |^\frac{p}{p-1} \right).
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Solution is extended to the whole rough channel by: \( u^0 = 0, \ p^0 = 0 \) in \( \mathbb{R}^\varepsilon \).
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- \(u_0\) is continuous across the interface \(\Sigma_0\), but the associated stress is not!

\[A := D(u^0)|_{\Sigma_0^+} = \frac{1}{2} \begin{pmatrix} 0 & U'(0) \\ U'(0) & 0 \end{pmatrix}, \quad \text{with } U'(0) = \sqrt{2^{\frac{p-2}{p-1}}}.\]
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\]

- \( [S(Du^0)n - p^0 n]|_{\Sigma_0} = |A|^{p-2} An = \left( \begin{smallmatrix} -1/2 \\ 0 \end{smallmatrix} \right), \quad \text{with } [f] := f|_{\Sigma_0^+} - f|_{\Sigma_0^-} \)
Error estimate of zero order approximation

We write

\[ u^\varepsilon = u^0 + w^\varepsilon \]
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Consequently

\[ 1 < p \leq 2 \quad (M > \|Du^0\|_{L^\infty(\Omega)}) \]

\[ \|Dw^\varepsilon\|_{L^p(\Omega \cap \{|Dw^\varepsilon| \geq M\})}^p + \|Dw^\varepsilon\|_{L^2(\Omega \cap \{|Dw^\varepsilon| \leq M\})}^2 + \|Dw^\varepsilon\|_{L^p(R^\varepsilon)}^p \leq C\varepsilon \]
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The jump in the stress tensor generates this quite huge error

\[ | \int_{\Sigma_0} (S(Du^0)n|_{\Sigma_0^+} \cdot w^\varepsilon \ dS | \leq C \| w^\varepsilon \|_{L^p(\Sigma_0)} \leq C' \varepsilon^{\frac{p-1}{p}} \| \nabla w^\varepsilon \|_{L^p(R^\varepsilon)} \]
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- \( 1 < p \leq 2 \) \( (M > \| Du^0 \|_{L^\infty(\Omega)}) \)
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  \[ \left| \int_{\Sigma_0^-} (S(Du^0)n) |w^\varepsilon| dS \right| \leq C \| w^\varepsilon \|_{L^p(\Sigma_0)} \leq C' \varepsilon^{\frac{p-1}{p}} \| \nabla w^\varepsilon \|_{L^p(R^\varepsilon)} \]

- This jump should be corrected by \( u_{bl} \)
  \[ u^\varepsilon \approx u^0(x) + \varepsilon u_{bl}(x/\varepsilon), \quad p^\varepsilon \approx p^0(x) + p_{bl}(x/\varepsilon) \]

s.t. the total approximation has no jump
Boundary layer system (BL)

By Taylor expansion:

\[ U(x^2) = U(\varepsilon y^2) = U(0) + \varepsilon U'(0) y^2 + \ldots \]

Formally

\[ D(u_0 + \varepsilon u_{bl} \cdot /\varepsilon)) \approx A + D u_{bl}, \text{ where } A = D(u_0) \big|_{x^2 = 0} + \ldots \]

Then we derive the following boundary layer system:

\[
\begin{align*}
- \text{div} S(A + Du_{bl}) + \nabla p_{bl} &= 0 \text{ in } \Omega \\
- \text{div} S(Du_{bl}) + \nabla p_{bl} &= 0 \text{ in } \Omega - \Omega_{\Gamma} \\
\text{div} u_{bl} &= 0 \text{ in } \Omega + \Omega_{\Gamma} - \Omega_{\Omega}
\end{align*}
\]

\[ u_{bl} \big|_{\Gamma_{\Omega}} = 0, \quad u_{bl} \big|_{y^2} = 0 - u_{bl} \big|_{y^2} = 0 \]

with (no)jump condition

\[ (S(A + Du_{bl}) n - p_{bl} n) \big|_{\Sigma} = (S(Du_{bl}) n - p_{bl} n) \big|_{\Sigma} = 0, \quad n = (0, \ -1, t) \]
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- Formally \( D(u^0 + \varepsilon u_{bl}(\cdot/\varepsilon)) \approx A + D_y u_{bl} \), where \( A = D(u^0)|_{x_2=0^+} \)
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\text{div} u_{bl} &= 0 \quad \text{in } \Omega^+_{bl} \cup \Omega^-_{bl}, \\
\left. u_{bl}\right|_{\Gamma_{bl}} &= 0, \\
\left. u_{bl}\right|_{y_2=0^+} - \left. u_{bl}\right|_{y_2=0^-} &= 0,
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(BL)
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-\text{div} S(A + D u_{bl}) + \nabla p_{bl} &= 0 & \text{in } \Omega_{bl}^+, \\
-\text{div} S(D u_{bl}) + \nabla p_{bl} &= 0 & \text{in } \Omega_{bl}^-, \\
\text{div} u_{bl} &= 0 & \text{in } \Omega_{bl}^+ \cup \Omega_{bl}^-, \\
u_{bl}|_{\Gamma_{bl}} &= 0, \\
u_{bl}|_{y_2=0^+} - u_{bl}|_{y_2=0^-} &= 0,
\end{align*}
\]

(BL)

▶ with (no)jump condition

\[
(S(A + D u_{bl})n - p_{bl}n)|_{\Sigma_0^+} - (S(D u_{bl})n - p_{bl}n)|_{\Sigma_0^-} = 0, \quad n = (0, -1)^t.
\]
Boundary layer system (BL), existence of solutions
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**Theorem (Existence of solutions)**

For all $1 < p < 2$, system (BL) has a unique solution $(u_{bl}, p_{bl}) \in W^{1,p}_{loc}(\Omega_{bl}) \times L^{p'}_{loc}(\Omega_{bl})/\mathbb{R}$ satisfying for any $M > |A|:

$$Du_{bl} \mathbb{1}_{\{|Du_{bl}| \leq M\}} \in L^2(\Omega_{bl}), \quad Du_{bl} \mathbb{1}_{\{|Du_{bl}| \geq M\}} \in L^p(\Omega_{bl}).$$
Boundary layer system (BL), existence of solutions

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*For all* $p \geq 2$, system (BL) *has a unique solution* $(u_{bl}, p_{bl}) \in W_{loc}^{1,p}(\Omega_{bl}) \times L_{loc}^{p'}(\Omega_{bl})/\mathbb{R}$ *s.t.*

\[
Du_{bl} \in L^p(\Omega_{bl}) \cap L^2(\Omega_{bl}).
\]
The behaviour of $u_{bl}$ – Exponential convergence to the horizontal constant field

**Theorem**

*For any $1 < p < +\infty$ and for some $C, \delta > 0$,*

$$|u_{bl}(y) - u^\infty| \leq C e^{-\delta y^2}, \quad \forall y \in \Omega^+_{bl}$$

*where $u^\infty = (U^\infty, 0)$ is some constant horizontal vector field.*
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**Proof:**

- Regularity properties of $u_{bl}$:
  - $u_{bl}$ has $C^{1,\alpha}$ regularity over $\Omega_{bl} \cap \{y_2 > 1\}$ for some $0 < \alpha < 1$,
  - $\nabla u_{bl}$ is bounded uniformly over $\Omega_{bl} \cap \{y_2 > 1\}$.

Adaptation of local regularity result for power-law fluids in 2-d space: Kaplický, Stará, Málek, Wolf (since 2002)
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Adaptation of local regularity result for power-law fluids in 2-d space: Kaplický, Stará, Málek, Wolf (since 2002)

- The heart of the proof: a Saint-Venant estimates. Quite technical!

$$E(t) := \int_{\{y_2 > t\}} |\nabla u_{bl}|^2 \leq C \exp(-\delta t)$$
New approximation

We introduce the solution \( u_0,\varepsilon \) of

\[
\begin{cases}
-\text{div} S(Du_0,\varepsilon) + \nabla p_0,\varepsilon = e_1, \\
\text{div} u_0,\varepsilon = 0,
\end{cases}
\]

\( x \in \Omega_{\varepsilon} \cap \{ x_2 > N_\varepsilon \} \),

\( u_0,\varepsilon |_{\Sigma_{N}} = 0 \),

\( u_0,\varepsilon |_{\{ x_2 = 1 \}} = 0 \),

and it is periodic in \( x_1 \) with period 1.

\( r_\varepsilon \) satisfies the Bogovski problem

\[
\begin{cases}
\text{div} r_\varepsilon = 0 \text{ in } \Omega_{\varepsilon} \cap \{ x_2 > N_\varepsilon \}, \\
r_\varepsilon |_{\Sigma_{N}} = \varepsilon (u_{bl}(\cdot/\varepsilon) - u_\infty) |_{\Sigma_{N}},
\end{cases}
\]

\( r_\varepsilon |_{\{ x_2 = 1 \}} = 0 \),

and

\[
\| r_\varepsilon \|_{W^{1,p}(\Omega_{\varepsilon} \cap \{ x_2 > N_\varepsilon \})} \lesssim C_\varepsilon \exp(-\delta N |\ln \varepsilon|).
\]
New approximation

\[ \Omega_N^\varepsilon := \Omega^\varepsilon \cap \{ x_2 > N\varepsilon|\ln \varepsilon| \}, \quad \Omega_{0,N}^\varepsilon = \Omega^\varepsilon \cap \{ 0 < x_2 < N\varepsilon|\ln \varepsilon| \}, \]

and \( \Sigma_N = (0, 1) \times \{ x_2 = N\varepsilon|\ln \varepsilon| \}. \)
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\begin{cases}
-\text{div}S(Du^{0,\varepsilon}) + \nabla p^{0,\varepsilon} = e_1, & x \in \Omega_N^\varepsilon, \\
\text{div}u^{0,\varepsilon} = 0, & x \in \Omega_N^\varepsilon, \\
u^{0,\varepsilon}|_{\Sigma_N} = (U'(0)x_2) + \varepsilon u^\infty, \\
u^{0,\varepsilon}|_{\{x_2 = 1\}} = 0,
\end{cases}
\]

\( u^{0,\varepsilon} \) is periodic in \( x_1 \) with period 1.
New approximation

\[ \Omega^\varepsilon_N := \Omega^\varepsilon \cap \{ x_2 > N\varepsilon|\ln \varepsilon| \}, \quad \Omega^\varepsilon_{0,N} = \Omega^\varepsilon \cap \{ 0 < x_2 < N\varepsilon|\ln \varepsilon| \}, \]

and \[ \Sigma_N = (0, 1) \times \{ x_2 = N\varepsilon|\ln \varepsilon| \}. \]

We introduce the solution \( u^{0,\varepsilon} \) of

\[
\begin{cases}
-\text{div} S(Du^{0,\varepsilon}) + \nabla p^{0,\varepsilon} = e_1, & x \in \Omega^\varepsilon_N, \\
\text{div} u^{0,\varepsilon} = 0, & x \in \Omega^\varepsilon_N, \\
u^{0,\varepsilon}|_{\Sigma_N} = \left( u'(0)x_2 \right) + \varepsilon u^\infty, \\
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\end{cases}
\]

\( u^{0,\varepsilon} \) is periodic in \( x_1 \) with period 1.

and let \( r^\varepsilon \) satisfy the Bogovski problem

\[
\begin{cases}
\text{div} r^\varepsilon = 0 & \text{in} \ \Omega^\varepsilon_N, \\
r^\varepsilon |_{\Sigma_N} = \varepsilon(u_{bl}(\cdot/\varepsilon) - u^\infty)|_{\Sigma_N}, \\
r^\varepsilon |_{\{x_2=1\}} = 0.
\end{cases}
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New approximation

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r^\varepsilon|_{\{x_2=1\}} = 0.
\end{cases}
\]

and

\[
\|r^\varepsilon\|_{W^{1,p}(\Omega_N^\varepsilon)} \leq C \varepsilon^\frac{1}{p} \exp(-\delta N|\ln\varepsilon|).
\]
Finally, we define the approximation \((u^\varepsilon_{\text{app}}, p^\varepsilon_{\text{app}})\)

\[
u^\varepsilon_{\text{app}}(x) = \begin{cases} 
  u^{0,\varepsilon}(x) + r^\varepsilon(x) & x \in \Omega_N^\varepsilon \\
  \left( \frac{u'(0)x_2}{0} \right) + \varepsilon u_{bl}(x/\varepsilon), & x \in \Omega_{0,N}^\varepsilon \\
  \varepsilon u_{bl}(x/\varepsilon), & x \in \mathcal{R}^\varepsilon
\end{cases}
\]
Finally, we define the approximation \((u^\varepsilon_{app}, p^\varepsilon_{app})\)

\[
\begin{align*}
    u^\varepsilon_{app}(x) &= \begin{cases} 
        u^{0,\varepsilon}(x) + r^\varepsilon(x), & x \in \Omega^\varepsilon_N \\
        \left( u^{(0)}(0)x_2 \right) + \varepsilon u_{bl}(x/\varepsilon), & x \in \Omega^\varepsilon_{0,N} \\
        \varepsilon u_{bl}(x/\varepsilon), & x \in \mathbb{R}^\varepsilon
    \end{cases}
\end{align*}
\]

With

\[
    u^\varepsilon_{app}|_{\partial \Omega^\varepsilon} = 0, \quad \text{div} \, u^\varepsilon_{app} = 0 \quad \text{over} \ \Omega^\varepsilon_N \cup \Omega^\varepsilon_{0,N} \cup \mathbb{R}^\varepsilon
\]
Finally, we define the approximation \((u_{\text{app}}^\varepsilon, p_{\text{app}}^\varepsilon)\)

\[
 u_{\text{app}}^\varepsilon(x) = \begin{cases} 
 u^{0,\varepsilon}(x) + r^\varepsilon(x) & x \in \Omega_N^\varepsilon \\
 \left( u'(0)x_2 \right) + \varepsilon u_{\text{bl}}(x/\varepsilon) & x \in \Omega_{0,N}^\varepsilon \\
 \varepsilon u_{\text{bl}}(x/\varepsilon) & x \in \mathbb{R}^\varepsilon 
\end{cases}
\]

With \(u_{\text{app}}^\varepsilon|_{\partial\Omega^\varepsilon} = 0\), \(\text{div} u_{\text{app}}^\varepsilon = 0\) over \(\Omega_N^\varepsilon \cup \Omega_{0,N}^\varepsilon \cup \mathbb{R}^\varepsilon\).

\(\bigarrow\) \(u_{\text{app}}^\varepsilon\) has zero jump at the interfaces \(\Sigma_0\) and \(\Sigma_N\)
Finally, we define the approximation \((u_{app}^\varepsilon, p_{app}^\varepsilon)\)

\[
\begin{aligned}
    u_{app}^\varepsilon(x) = \begin{cases}
    u^0,\varepsilon(x) + r^\varepsilon(x) & x \in \Omega_N^\varepsilon \\
    \left( u'(0)x_2 \right) + \varepsilon u_{bl}(x/\varepsilon) & x \in \Omega_{0,N}^\varepsilon \\
    \varepsilon u_{bl}(x/\varepsilon) & x \in R^\varepsilon
    \end{cases}
\end{aligned}
\]

With

\[
    u_{app}^\varepsilon|_{\partial\Omega^\varepsilon} = 0, \quad \text{div} u_{app}^\varepsilon = 0 \quad \text{over} \ \Omega_N^\varepsilon \cup \Omega_{0,N}^\varepsilon \cup R^\varepsilon
\]

- \(u_{app}^\varepsilon \) has zero jump at the interfaces \(\Sigma_0\) and \(\Sigma_N\)
- The stress tensor still has a jump at \(\Sigma_N\)
Theorem
(Error estimates)

For $1 < p \leq 2$, there exists $C$ such that

$$\|u^\varepsilon - u_{\text{app}}^\varepsilon\|_{W^{1,p}(\Omega^\varepsilon)} \leq C(|\varepsilon| \ln \varepsilon)^{1 + \frac{1}{p}}$$

For $p \geq 2$, there exists $C$ such that

$$\|u^\varepsilon - u_{\text{app}}^\varepsilon\|_{W^{1,p}(\Omega^\varepsilon)} \leq C(|\varepsilon| \ln \varepsilon)^{\frac{1}{p-1} + \frac{1}{p}}$$
Theorem
(Error estimates)

- For $1 < p \leq 2$, there exists $C$ such that
  \[ \| u^\varepsilon - u^\varepsilon_{app} \|_{W^{1,p}(\Omega^\varepsilon)} \leq C (\varepsilon |\ln \varepsilon|)^{1 + \frac{1}{p'}} \]

- For $p \geq 2$, there exists $C$ such that
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- This is much better than for the zero order approximation $u^0$ with zero Dirichlet boundary condition on $\Sigma_0$
Theorem
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- This is much better than for the zero order approximation $u^0$ with zero Dirichlet boundary condition on $\Sigma_0$

- $u_{app}^\varepsilon$ involves in a crucial way solutions $u^{0,\varepsilon}$ and the contribution of $r^\varepsilon$ is very small
Instead of considering $u^{0,\varepsilon}$, we could consider the solution $\bar{u}_\varepsilon^0$ of

\[
\begin{cases}
-\text{div} S(\bar{u}_\varepsilon^0) + \nabla p_\varepsilon^0 = e_1, & x \in \Omega, \\
\text{div} \bar{u}_\varepsilon^0 = 0, & x \in \Omega, \\
\bar{u}_\varepsilon^0|_{\Sigma_0} = \varepsilon u^\infty, \\
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We find that

\[
\|u^\varepsilon - \bar{u}_\varepsilon^0\|_{W^{1,p}(\Omega_N^\varepsilon)} = O(\varepsilon|\ln \varepsilon|),
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\bar{u}_{\varepsilon}^0|_{\Sigma_0} = \varepsilon u^\infty, \\
\bar{u}_{\varepsilon}^0|_{\{x_2=1\}} = 0.
\end{cases}
$$

We find that

$$
\|u_{\varepsilon} - \bar{u}_{\varepsilon}^0\|_{W^{1,P}(\Omega_N^\varepsilon)} = O(\varepsilon|\ln \varepsilon|),
$$

while

$$
\|u_{\varepsilon} - u^0\|_{W^{1,P}(\Omega_N^\varepsilon)} \geq c' \varepsilon
$$
Instead of considering $u^{0,\varepsilon}$, we could consider the solution $\bar{u}^{0}_\varepsilon$ of

\[
\begin{cases}
-\text{div} S(\bar{u}^{0}_\varepsilon)) + \nabla p^0_\varepsilon = e_1, & x \in \Omega, \\
\text{div} \bar{u}^{0}_\varepsilon = 0, & x \in \Omega, \\
\bar{u}^{0}_\varepsilon|_{\Sigma_0} = \varepsilon u^\infty, \\
\bar{u}^{0}_\varepsilon|_{\{x_2 = 1\}} = 0.
\end{cases}
\]

We find that

\[
\|u^{\varepsilon} - \bar{u}^{0}_\varepsilon\|_{W^{1,p}(\Omega_\varepsilon^N)} = O(\varepsilon |\ln \varepsilon|),
\]

while

\[
\| u^{\varepsilon} - u^0 \|_{W^{1,p}(\Omega_\varepsilon^N)} \geq c^' \varepsilon
\]

- Approximation $\bar{u}^{0}_\varepsilon$ with $\varepsilon u^\infty$ on $\Sigma_0$ gives smaller error than $u^0$ with 0 on $\Sigma_0$
Let us conclude:

We distinguish between two approximations (outside the boundary layer):

- A crude approximation, involving the generalized Poiseuille flow
- A refined approximation, involving $u_0 \varepsilon$

The first choice corresponding to Dirichlet wall law $u|_{\Sigma_0} = 0$ and neglects the role of the roughness

The second choice takes it into account through the inhomogeneous Dirichlet condition $u|_{\Sigma_0} = \varepsilon u_\infty = \varepsilon (U_\infty, 0)$

One can provide then the following refined wall law:

$$u^n|_{\Sigma_0} = 0, \quad u_\tau|_{\Sigma_0} = \varepsilon F(D(u^n)|_{\Sigma_0})$$
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- One can provide then the following refined wall law:
  \[ u_n|_{\Sigma_0} = 0, \quad u_\tau|_{\Sigma_0} = \varepsilon \mathcal{F}((D(u)n)_\tau|_{\Sigma_0}) \]
How can we see this is a wall law, although slightly abstract?
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$U^\infty$ can be seen as a function of the tangential shear

$$(D(u^0)n)_\tau|_{\Sigma_0} = \partial_2 u_1^0|_{\Sigma_0} = U'(0)$$
How can we see this is a wall law, although slightly abstract?

$U^\infty$ can be seen as a function of the tangential shear

$$ (D(u^0)n)_\tau|_{\Sigma_0} = \partial_2 u^0_1|_{\Sigma_0} = U'(0) $$

through the mapping

$$ U'(0) \rightarrow A := \begin{pmatrix} 0 & u'(0) \\ u'(0) & 0 \end{pmatrix} \rightarrow u_{bl} \text{ solution of (BL)} \rightarrow U^\infty = \lim_{y_2 \rightarrow +\infty} u_{bl,1}. $$
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Denoting by $\mathcal{F}$ this application, we write

$$(\bar{u}_\varepsilon^0)_\tau|_{\Sigma_0} = \varepsilon \mathcal{F}((D(u^0)n)_\tau|_{\Sigma_0}) \approx \varepsilon \mathcal{F}((D(\bar{u}_\varepsilon^0)n)_\tau|_{\Sigma_0})$$

whereas $(\bar{u}_\varepsilon^0)_n = 0.$
Boundary Layer for a Non-Newtonian Flow over a Rough Surface