On a non-stationary fluid flow problem in an infinite periodic pipe

Stephanie Zube

Institute of Mathematics, University of Zurich, Switzerland

stephanie.zube@math.uzh.ch

Abstract

We study a linearized non-stationary incompressible Navier-Stokes problem

$$\partial_t \mathbf{u} - \nu\Delta\mathbf{u} + (\mathbf{U}\cdot\nabla)\mathbf{u} + \nabla p = \mathbf{f}$$

with prescribed flux in a two or three dimensional L-periodic "with respect to the x_n-axis" pipe. We look for the pressure $p(x,t)$ having the following form

$$p(x,t) = -q(t)x_n + p_0(t) + \tilde{p}(x,t),$$

where $p_0(t)$ is an arbitrary function, $\tilde{p}(x,t)$ is a L-periodic function and $q(t)$ is associated to the flux condition. We will also focus on the asymptotic behavior in an infinite periodic pipe.

1 Introduction

Linearized non-stationary Navier-Stokes problem:

$$\begin{cases} \partial_t\mathbf{u} - \nu\Delta\mathbf{u} + (\mathbf{U}\cdot\nabla)\mathbf{u} + \nabla p = \mathbf{f} \quad \text{in } \Pi_L, \\ \operatorname{div}(\mathbf{u}) = 0 \quad \text{in } \Pi_L, \quad \mathbf{u}(x,0) = \mathbf{a}(x), \\ \mathbf{u}|_{S_L} = 0 \text{ and } \mathbf{u}(x',0,t) = \mathbf{u}(x',L,t) \quad t\in(0,T), \\ \int_{\sigma_{x_n}} u_n(x,t)dx' = F(t), \quad \forall x_n \in (0,L),\ n = 2,3 \end{cases} \tag{1.1}$$

where $\mathbf{U}$ satisfies

$$\mathbf{U} \in L_\infty(0,T;W_2^1(\Pi_L)) \cap L_\infty(\Pi_L)), \quad \operatorname{div}(\mathbf{U}) = 0, \quad \mathbf{U}|_{S_L} = 0,$$
$$\mathbf{U} \text{ is } L\text{-periodic in } x_n, \text{ i.e. } \mathbf{U}(x',x_n,t) = \mathbf{U}(x',x_n+L,t) \tag{1.2}$$

and for the domain we have

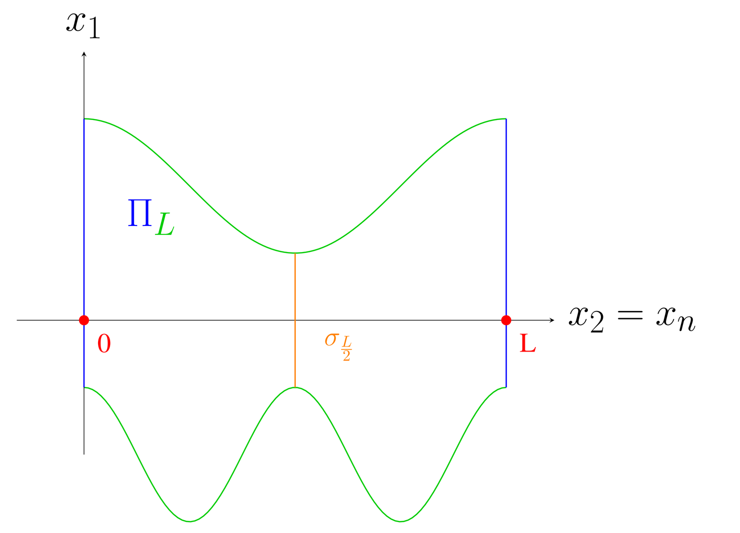
σ_{x_n} = 1 or 2 dim. open and bounded subset at level x_n

D is the largest diameter of σ_{x_n}

$\Pi_L = \{x\in\sigma_{x_n}\times\{x_n\},\ x_n\in(0,L)\}$, $\quad S_L = \partial\Pi_L \backslash (\sigma_0\cup\sigma_L)$

boundary of Π_L is of class $C^{0,1}$

2D Example of a domain:



Remark 1.1. *The case $\mathbf{U}(x,t) = 0$ coincides with the non-stationary Stokes problem.*

Remark 1.2. *Let $0<\alpha\leq L$ and $\boldsymbol{\eta}\in W_2^1(\Pi_L)$ with $\operatorname{div}(\boldsymbol{\eta}) = 0$ and $\boldsymbol{\eta}|_{S_L} = 0$, then we have*

$$0 = \int_{\Pi_\alpha} \operatorname{div}(\boldsymbol{\eta})\,dx = \int_{\sigma_0} \eta_n\,dx' + \int_{S_\alpha} \boldsymbol{\eta}\cdot\boldsymbol{n}\,dS_\alpha - \int_{\sigma_\alpha}\eta_n\,dx',$$

and therefore $\int_{\sigma_0}\eta_n(x',0,t)\,dx' = \int_{\sigma_\alpha}\eta_n(x',\alpha,t)\,dx'$ for every $0<\alpha\leq L$. We will denote in what follows by $\int_\sigma \eta_n\,dx'$ the common value of this integral.

Statement of the problem: We will show that we can take

$$p(x,t) = -q(t)x_n + p_0(t) + \widetilde{p}(x,t) \tag{1.3}$$

with $p_0(t)$ arbitrary function, $\widetilde{p}(x,t)$ L-periodic function w.r.t. x_n, $q(t)$ associated to the flux condition. The weak formulation:

$$\begin{cases} \mathbf{u}\in L_2(0,T;H(\Pi_L)), \quad \partial_t\mathbf{u}\in L_2(0,T;L_2(\Pi_L)), \quad \mathbf{u}\,|_{t=0} = \mathbf{a}, \\ \int_0^t\int_{\Pi_L}\partial_t\mathbf{u}\cdot\boldsymbol{\eta}\,dxd\tau + \nu\int_0^t\int_{\Pi_L}\nabla\mathbf{u}\cdot\nabla\boldsymbol{\eta}\,dxd\tau + \int_0^t\int_{\Pi_L}(\mathbf{U}\cdot\nabla)\mathbf{u}\cdot\boldsymbol{\eta}\,dxd\tau \\ = \int_0^t\int_{\Pi_L}\mathbf{f}\cdot\boldsymbol{\eta}\,dxd\tau + L\int_0^t q(\tau)\int_\sigma\eta_n\,dx'd\tau \quad \forall\boldsymbol{\eta}\in L_2(0,T;H(\Pi_L)), \\ \int_\sigma u_n(x,t)\,dx' = F(t), \quad q\in L_2(0,T) \end{cases} \tag{1.4}$$

$$H(\Pi_L) = \{\boldsymbol{\eta}\in W_2^1(\Pi_L)\mid \operatorname{div}(\boldsymbol{\eta}) = 0,\ \boldsymbol{\eta}(x',0) = \boldsymbol{\eta}(x',L),\ \boldsymbol{\eta}|_{S_L} = 0\}$$

2 Existence

Theorem 2.1

Suppose that $\mathbf{f}\in L_2(0,T;L_2(\Pi_L))$, $\mathbf{a}\in W_2^1(\Pi_L)$, $F\in W_2^1(0,T)$, $T\in(0,\infty)$, $F(0) = \int_\sigma a_n\,dx'$ and $\mathbf{U}$ satisfies (1.2). Then the problem (1.4) admits a unique solution $(\mathbf{u},q)\in L_2(0,T;H(\Pi_L))\times L_2(0,T)$ and the following estimate

$$\sup_{t\in[0,T]}\|\mathbf{u}(\cdot,t)\|^2_{W_2^1(\Pi_L)} + \int_0^t\|\frac{\partial}{\partial\tau}\mathbf{u}\|^2_{L_2(\Pi_L)}\,d\tau + \int_0^t\|\nabla\mathbf{u}\|^2_{L_2(\Pi_L)}\,d\tau$$
$$+ \|q\|^2_{L_2(0,T)} \leq c(\|F\|^2_{W_2^1(0,T)} + \int_0^T\|\mathbf{f}\|^2_{L_2(\Pi_L)}d\tau + \|\mathbf{a}\|^2_{W_2^1(\Pi_L)}) \tag{2.1}$$

is valid. Here the constant c depends on L, D, T, ν and the function $\mathbf{U}$.

Idea of the proof: Since the problem (1.4) is linear, we have

$$(\mathbf{u}(x,t), q(t), \tilde{p}(x,t)) = (\mathbf{V}(x,t), 0, \tilde{p}_1(x,t)) + (\mathbf{v}(x,t), q(t), \tilde{p}_2(x,t))$$

with

$$\begin{cases} \partial_t\mathbf{V} - \nu\Delta\mathbf{V} + (\mathbf{U}\cdot\nabla)\mathbf{V} + \nabla\tilde{p}_1 = \mathbf{f} \quad \text{in } \Pi_L, \\ \operatorname{div}(\mathbf{V}) = 0 \quad \text{in } \Pi_L, \quad \mathbf{V}(x,0) = \mathbf{a}(x), \\ \mathbf{V}|_{S_L} = 0 \text{ and } \mathbf{V}(x',0,t) = \mathbf{V}(x',L,t) \quad t\in(0,T), \end{cases} \tag{2.2}$$

and

$$\begin{cases} \partial_t\mathbf{v} - \nu\Delta\mathbf{v} + (\mathbf{U}\cdot\nabla)\mathbf{v} + \nabla\tilde{p}_2 = \mathbf{q} \quad \text{in } \Pi_L, \\ \operatorname{div}(\mathbf{v}) = 0 \quad \text{in } \Pi_L, \quad \mathbf{v}(x,0) = 0, \\ \mathbf{v}|_{S_L} = 0 \text{ and } \mathbf{v}(x',0,t) = \mathbf{v}(x',L,t) \quad t\in(0,T), \\ \int_\sigma v_n\,dx' = \widetilde{F}(t), \end{cases} \tag{2.3}$$

where $\widetilde{F}(t) = F(t) - \int_\sigma V_n\,dx'$. Then we can show the existence of a unique weak solution and the estimate for each single problem. □

3 Asymptotic behavior

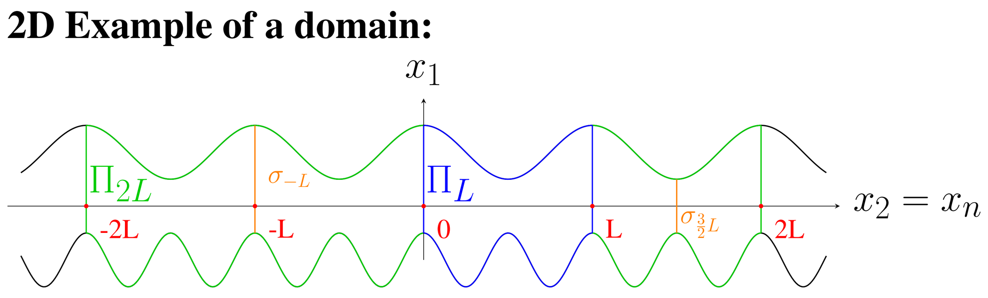
In this section we analyze the asymptotic behavior in space between the problem (1.4), defined in one periodic cell and the analogous problem, defined in a bigger domain:

$$\begin{cases} \mathbf{u}_\ell\in L_2(0,T;H(\Pi_{\ell L})), \quad \partial_t\mathbf{u}_\ell\in L_2(0,T;H(\Pi_{\ell L})), \quad \mathbf{u}_\ell|_{t=0} = \mathbf{a}, \\ q_\ell\in L_2(0,T), \quad \int_\sigma u_{\ell,n}(x',x_n,t)\,dx' = F(t),\ \forall x_n\in(-\ell L,\ell L) \\ \int_0^T\int_{\Pi_{\ell L}}\partial_t\mathbf{u}_\ell\cdot\boldsymbol{\eta}\,dxd\tau + \nu\int_0^T\int_{\Pi_{\ell L}}\nabla\mathbf{u}_\ell\cdot\nabla\boldsymbol{\eta}\,dxd\tau + \int_0^T\int_{\Pi_{\ell L}}(\mathbf{U}\cdot\nabla)\mathbf{u}_\ell\cdot\boldsymbol{\eta}\,dxd\tau \\ = \int_0^T\int_{\Pi_{\ell L}}\mathbf{f}\cdot\boldsymbol{\eta}\,dxd\tau + 2L\ell\int_0^T q_\ell(\tau)\int_\sigma\eta_n\,dx'd\tau \quad \forall\boldsymbol{\eta}\in L_2(0,T;H(\Pi_{\ell L})) \end{cases} \tag{3.1}$$

with $\mathbf{f}$, $\mathbf{a}$, $\mathbf{U}$ being L-periodically extended functions in x_n and

$$\Pi_{\ell L} = \{x\in\sigma_{x_n}\times\{x_n\},\ x_n\in(-\ell L,\ell L)\} = \bigcup_{z=-\ell}^{\ell-1}\{\{\Pi_L\cup\sigma_0\cup\sigma_L\} + zLe_n\},$$
$$S_{\ell L} = \partial\Pi_{\ell L}\backslash(\sigma_{-\ell L}\cup\sigma_{\ell L}),$$
$$H(\Pi_{\ell L}) = \{\boldsymbol{\eta}\in W_2^1(\Pi_{\ell L})\mid\operatorname{div}(\boldsymbol{\eta}) = 0, \boldsymbol{\eta}(x',-\ell L) = \boldsymbol{\eta}(x',\ell L), \boldsymbol{\eta}|_{S_{\ell L}} = 0\}$$

2D Example of a domain:



Remark 3.1. *For the uniqueness and existence of (3.1) compare previous section. For $\ell\in\mathbb{N}^+$ it is obvious and for $\ell\in\mathbb{R}^+\backslash\mathbb{N}$ we need in addition to assume that Π_L is symmetric w.r.t. $x_n = L/2$ and $\boldsymbol{U}$, $\boldsymbol{a}$ are L-periodic symmetric functions w.r.t. $x_n = L/2$.*

3.1 Case $\mathbf{u}_\ell = \mathbf{u}$, $\ell\in\mathbb{N}^+$

An auxiliary result for the solution $(\mathbf{u},q)$ of the problem (1.4):

Lemma 3.1

Let $\ell\in\mathbb{N}^+$, $T\in(0,\infty)$, $\mathbf{f}\in L_2(0,T;L_2(\Pi_L))$ and $\mathbf{U}$ satisfies (1.2). If $\mathbf{u}$, $\mathbf{a}$, $\mathbf{U}$ and $\mathbf{f}$ in the domain $\Pi_{\ell L}$ are extended L-periodically w.r.t. x_n, then for every $\mathbf{v}\in L_2(0,T;H(\Pi_{\ell L}))$ we have that

$$\int_0^t\int_{\Pi_{\ell L}}\partial_t\mathbf{u}\cdot\mathbf{v}\,dxd\tau + \nu\int_0^t\int_{\Pi_{\ell L}}\nabla\mathbf{u}\cdot\nabla\mathbf{v}\,dxd\tau + \int_0^t\int_{\Pi_{\ell L}}(\mathbf{U}\cdot\nabla)\mathbf{u}\cdot\mathbf{v}\,dxd\tau$$
$$= \int_0^t\int_{\Pi_{\ell L}}\mathbf{f}\cdot\mathbf{v}\,dxd\tau + 2L\ell\int_0^t q\int_\sigma v_n\,dx'd\tau. \tag{3.2}$$

From this lemma we derive that $\mathbf{u}_\ell = \mathbf{u}$ for $\ell\in\mathbb{N}^+$.

3.2 Case $\mathbf{u}_\ell$ converges exponentially to $\mathbf{u}$, $\ell\in\mathbb{R}^+\backslash\mathbb{N}$

$\Pi_{\ell L}$ consists not of an integer number of periodicity cells Π_L; we assume that Π_L and $\mathbf{f}$, $\mathbf{a}$ are symmetric w.r.t. $x_n = L/2$

$$\widehat{H}(\Pi_{\ell L}) = \{\boldsymbol{\eta}\in W_2^1(\Pi_{\ell L})\mid\operatorname{div}(\boldsymbol{\eta}) = 0,\ \boldsymbol{\eta}|_{\partial\Pi_{\ell L}} = 0\}$$

Lemma 3.2

Let $\ell\in\mathbb{R}^+\backslash\mathbb{N}$, $T\in(0,\infty)$, $\mathbf{f}\in L_2(0,T;L_2(\Pi_L))$ and $\mathbf{U}$ satisfies (1.2). If $\mathbf{u}$, $\mathbf{a}$, $\mathbf{U}$ and $\mathbf{f}$ are extended L-periodically w.r.t. x_n, then for every $\mathbf{v}\in L_2(0,T;\widehat{H}(\Pi_{\ell L}))$ we have

$$\int_0^t\int_{\Pi_{\ell L}}\partial_t\mathbf{u}\cdot\mathbf{v}\,dxd\tau + \nu\int_0^t\int_{\Pi_{\ell L}}\nabla\mathbf{u}\cdot\nabla\mathbf{v}\,dxd\tau + \int_0^t\int_{\Pi_{\ell L}}(\mathbf{U}\cdot\nabla)\mathbf{u}\cdot\mathbf{v}\,dxd\tau$$
$$= \int_0^t\int_{\Pi_{\ell L}}\mathbf{f}\cdot\mathbf{v}\,dxd\tau. \tag{3.3}$$

This result follows from the previous lemma.

Theorem 3.1

Let $\mathbf{f}\in L_2(0,T;L_2(\Pi_L))$, $\mathbf{a}\in W_2^1(\Pi_L)$, $T\in(0,\infty)$, $\mathbf{U}$ satisfies (1.2) and in addition we assume $|\mathbf{U}|_\infty = \sup_{t\in[0,T]}\sup_{x\in\Pi_L}|\mathbf{U}| < 2\sqrt{\nu}$. Then the following estimate

$$\sup_{t\in[0,T]}\|(\mathbf{u}_\ell-\mathbf{u})(\cdot,t)\|^2_{W_2^1(\Pi_{\ell L/2})} + \int_0^T\|\partial_t(\mathbf{u}_\ell-\mathbf{u})\|^2_{L_2(\Pi_{\ell L/2})}\,d\tau \leq Ce^{-\ell\alpha} \tag{3.4}$$

holds for constants C and α being independent of ℓ.

Idea of the proof:

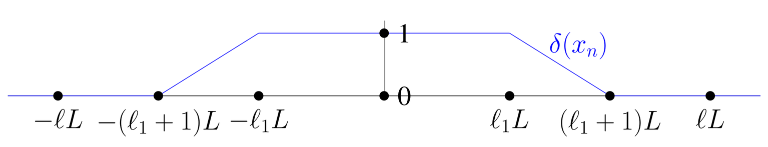
- Using previous lemma, which implies for $\mathbf{w}_\ell = \mathbf{u}_\ell - \mathbf{u}$:

$$\int_0^t\int_{\Pi_{\ell L}}\partial_t\mathbf{w}_\ell\cdot\mathbf{v}\,dxd\tau + \nu\int_0^t\int_{\Pi_{\ell L}}\nabla\mathbf{w}_\ell\cdot\nabla\mathbf{v}\,dxd\tau$$
$$+ \int_0^t\int_{\Pi_{\ell L}}(\mathbf{U}\cdot\nabla)\mathbf{w}_\ell\cdot\mathbf{v}\,dxd\tau = 0 \quad \forall\mathbf{v}\in L_2(0,T;\widehat{H}(\Pi_{\ell L})). \tag{3.5}$$

- Finding a good test function: We choose $\mathbf{v} = \delta\mathbf{w}_\ell - \boldsymbol{\beta} \in L_2(0,T;\widehat{H}(\Pi_{\ell L}))$ with $\boldsymbol{\beta}\in L_2(0,T;\overset{\circ}{W}{}_2^{1,2}(D_{\ell_1}))$ such that

$$\begin{cases} \operatorname{div}(\boldsymbol{\beta}) = \partial_{x_n}\delta w_{\ell,n} \quad \text{in } D_{\ell_1}, \\ \|\nabla\boldsymbol{\beta}\|_{L_2(D_{\ell_1})} \leq \hat{c}\|w_{\ell,n}\|_{L_2(D_{\ell_1})} \end{cases} \tag{3.6}$$

and



for $\ell_1\leq\ell-1$ and $D_{\ell_1} = \Omega_{\ell_1+1}\backslash\Omega_{\ell_1}$.

- Putting $\mathbf{v}$ into (3.5) and showing the convergence.

□

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