Title: Steady-State Navier-Stokes Flow around a Moving Body

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Summary. In this article we present an updated account of the fundamental mathematical results pertaining the steady-state flow of a Navier-Stokes liquid past a rigid body which is allowed to rotate. Precisely, we shall address questions of existence, uniqueness, regularity, asymptotic structure, generic properties, and (steady and unsteady) bifurcation. Moreover, we will perform a rather complete analysis of the long-time behavior of dynamical perturbation to the above flow, thus inferring, in particular, sufficient conditions for their stability and asymptotic stability.

1 Introduction

The motion of a rigid body in a viscous liquid represents one of the most classical and most studied chapters of applied and theoretical fluid mechanics. Actually, the study of this problem, at different scales, is at the foundation of many branches of applied sciences such as biology, medicine, and car, airplane and ship manufacturing, to name a few. The dynamics of the liquid associated to these problems is, of course, of the utmost relevance and, already in very elementary cases, can be quite intricate or even, at times, far from being obvious. For example, consider a rigid sphere of radius R, moving by *constant* translatory motion with speed v_0 and entirely immersed in a surrounding liquid, of kinematic viscosity ν. Then, it is experimentally observed (see [69]) that if $\mathrm{Re} := v_0\mathsf{R}/\nu \lesssim 200$, the flow is steady, stable and axisymmetric. However, if $200 \lesssim \mathrm{Re} \lesssim 270$, this flow loses its stability, and another stable, steady but no longer axisymmetric flow sets in, as evidenced by the shifting of the wake with respect to the direction of motion of the sphere. It is worth emphasizing the loss of symmetry of the flow, in spite of the symmetry of the data. Moreover, if $270 \lesssim \mathrm{Re} \lesssim 300$, the steady flow is unstable, and the liquid regime becomes oscillatory, as shown by the highly organized time-periodic motion of the wake behind the sphere. The remarkable feature of this phenomenon is that the unsteadiness of the flow arises *spontaneously*, even though the imposed conditions are *time-independent* (constant speed of the body). Another significant example is furnished if now the sphere, instead of moving by a translatory motion, rotates with constant angular velocity, ω_0, along one of its diameters. Here, again in view of the symmetry of the data, one would guess that, at least for "small" values of $|\omega_0|$ (more precisely, of the dimensionless number $|\omega_0|\,\mathsf{R}^2/\nu$), the flow of the liquid is steady with streamlines being circles perpendicular to and centered around the axis of rotation. Actually, this is not the case, unless the inertia of the liquid is entirely disregarded. In fact, though the flow is steady, due to inertia the sphere behaves like a "centrifugal fan", receiving the liquid near the poles and throwing it away at the equator; see [12], [87].

Already from these brief considerations one can fairly deduce that a rigorous mathematical study of the motion of a viscous liquid around an obstacle presents a plethora of intriguing problems of considerable difficulties, beginning with the very existence of steady-state solutions under general conditions on the data and their uniqueness, going through more complicated issues such as analysis of steady and time-periodic bifurcation, and long time behavior of time-dependent perturbations. It is objective of this article to address some of these fundamental problems, as well as point out certain outstanding questions that still await for an answer.

In real experiments the liquid occupies, of course, a finite (though "sufficiently large") spatial region. However, "wall effects" are irrelevant for the occurrence of the basic phenomena of the type described above. Therefore, in order not to spoil their underlying causes, it is customary to formulate the mathematical theory of the motion of a body in a viscous liquid as an *exterior* problem. This corresponds to the assumption that the liquid fills the *entire* three-dimensional space outside the body. It should be remarked that this assumption, though simplifying on one hand, on the other hand adds more complication to the mathematical analysis, in that classical and powerful tools valid for bounded flow, are no longer available in this case. As it turns out, most of the questions that we shall analyze require, for their answers, a somewhat detailed analysis of the solutions at large distance from the body.

From a historical viewpoint, the mathematical analysis of the steady flow of a viscous liquid past a rigid body may be traced back to the pioneering contributions of Stokes [107], Kirchhoff [73] and Thomson (Lord Kelvin) & Tait [108] in mid and late 1880's. However, it was only in the 1930's that, thanks to the far-reaching and genuinely new ideas introduced by Jean Leray [83], the investigation of the problem received a substantial impulse. Leray's results, mostly devoted to the existence problem, were further deepened, extended and completed over the years by a number of fundamental researches due, mostly, to O.A. Ladyzhenskaya, H. Fujita, R. Finn, K.I. Babenko, and J.G. Heywood. It is important to observe that the efforts of all these authors were directed to the study of cases where the body is not allowed to spin. The more general and more complicated situation of a rotating body became the object of a systematic study only at the beginning of the third millennium, with the basic contributions, among others, of R. Farwig, T. Hishida, M. Hieber, Y. Shibata, the authors of the present paper and their associates.

Main goal of this review article is to furnish an up-to-date *state of the art* of the fundamental mathematical properties of steady-state flow of a Navier-Stokes liquid past a rigid body, which is also allowed to rotate. Thus, existence, uniqueness, regularity, asymptotic structure, generic properties, and (steady and unsteady) bifurcation issues will be addressed. In addition, a rather complete analysis of the long-time behavior of dynamical perturbation to the above solutions will be performed to deduce, in particular, sufficient conditions for their stability and asymptotic stability as well.

With the exception of part of the last Section 10, this study will be focused on the case when the translational velocity, $\boldsymbol{v}_0$, of the body is not zero, and its angular velocity is either zero or else has a non-vanishing component in the direction of $\boldsymbol{v}_0$. The reason for such a choice lies in the fact that under these assumptions, the mathematical questions listed above have a rather complete answer. On the other side, if one relaxes these assumptions, the picture becomes much less clear. The interested reader is referred to [47, §§ X.9 and XI.7] for all main properties known in this case. Furthermore, for the same reason of incompleteness of results, only consider three-dimensional flow will be considered. An update source of information regarding plane motions can be found, for example, in [47, Chapter XII], [38], and [58]. Finally, other significant investigations are left out of this article, such as the motion of the coupled system body-liquid (i.e., when the motion of the body is no longer prescribed, but becomes part of the problem), as well as the very important case when the body is deformable, for which the reader is referred to [40] and [46; 5], respectively.

The plan of the article is as follows. After collecting in Section 2 the main notation used throughout, in Section 3 it is provided the mathematical formulation of the problem. Section 4 is dedicated to existence questions. There, one begins to recall classical approaches and corresponding results due to Leray, Ladyzhenskaya and Fujita. Successively, improved findings obtained by the function-analytic method introduced by Galdi are presented, based on the degree for proper Fredholm maps of index 0. Regularity and uniqueness questions of solutions are next addressed in Sections 5 and 6, respectively. Section 7 is dedicated to the (spatial) asymptotic behavior, beginning by recalling the original results of Finn and Babenko when the body is not spinning, to the more recent general contributions of Galdi & Kyed and Deuring and their associates, valid also in the case of a rotating body. Successively, in Section 8, one investigates the geometric structure of the solution manifold for data of arbitrary "size". In particular, it is shown that, generically, the number of solutions corresponding to a given (nonzero) translational velocity and angular velocity is finite and odd.

The following Section 9 is devoted to steady and time-periodic (Hopf) bifurcation of steady-state solutions. There, it is provided necessary and sufficient conditions for this type of bifurcation to occur. In the final Section 10 one analyzes the long time behavior of time-dependent perturbations to a given steady-state, providing, as a special case, sufficient conditions for attractivity and asymptotic stability. These results can be, roughly speaking, grouped in two different categories. The first one, where one assumes that the unperturbed steady state is "small in size". In the second one, instead, one makes suitable hypothesis on the location in the complex plane of eigenvalues of the relevant linearized operator (spectral stability).

In conclusion to this introductory section, it is worth emphasizing that throughout this article there have been highlighted a number of intriguing *unsettled questions* that still need an answer and represent as many avenues open to the interested mathematician.

2 Notation

The symbols $\mathbb{N}$, $\mathbb{Z}$, and $\mathbb{R}$, $\mathbb{C}$ stand for the sets of positive and relative integers, and the fields of real and complex numbers, respectively. We also put $\mathbb{N}_+ := \mathbb{N} \cap (0, \infty)$, $\mathbb{R}_+ := \mathbb{R} \cap (0, \infty)$.

Vectors in $\mathbb{R}^3$ will be indicated by boldfaced letters. A base in $\mathbb{R}^3$ is denoted by $\{\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3\} \equiv \{\boldsymbol{e}_i\}$, and the components of a vector $\boldsymbol{v}$ in the given base, by v_1, v_2 and v_3.

Unless stated otherwise, the Greek letter Ω will denote a fixed exterior domain of $\mathbb{R}^3$, namely, the complement of the closure, $\mathscr{B}$, of a bounded, open, and simply connected set of $\mathbb{R}^3$. It will be assumed Ω of class C^2, and the origin O of the coordinate system $\{O, \boldsymbol{e}_i\}$ is taken in the interior of $\mathscr{B}$. Also, d is the diameter of $\mathscr{B}$, so that, setting $B_R := \{\boldsymbol{x} \in \mathbb{R}^3 : (x_1^2 + x_2^2 + x_3^2)^{\frac{1}{2}} < R\}$, $R > 0$, one has $\mathscr{B} \subset B_d$.

For $R \geq d$, the following notation will be adopted

$$\Omega_R = \Omega \cap B_R\,, \quad \Omega^R = \Omega - \overline{\Omega_R}\,,$$

where the bar denotes closure.

One puts $u_t := \partial u/\partial t$, $\partial_1 u := \partial u/\partial x_1$, and, for α a multi-index, one denotes by D^α the usual differential operator of order $|\alpha|$. For $|\alpha| = 2$ one shall simply write D^2.

Given an open and connected set $A \subseteq \mathbb{R}^3$, $L^q(A)$, $L^q_{loc}(A)$, $1 \leq q \leq \infty$, $W^{m,q}(A)$, $W_0^{m,q}(A)$ ($W^{0,q} \equiv W_0^{0,q} \equiv L^q$), $W^{m-1/q,q}(\partial A)$, $m \in \mathbb{N}_+ \cup \{0\}$, stand for the usual Lebesgue, Sobolev, and trace space classes, respectively, of real or complex functions. (The same font style will be used to denote scalar, vector and tensor function spaces.) Norms in $L^q(A)$, $W^{m,q}(A)$, and $W^{m-1/q,q}(\partial A)$ are indicated by $\|.\|_{q,A}$, $\|.\|_{m,q,A}$, and $\|.\|_{m-1/q,q(\partial A)}$. The scalar product of functions $u, v \in L^2(A)$ will be denoted by $(u, v)_A$. In the above notation, the subscript A will be omitted, unless confusion arises.

As customary, for $q \in [1, \infty]$ one lets $q' = q/(q-1)$ be its Hölder conjugate.

By $D^{m,q}(\Omega)$, $1<q<\infty$, $m\in\mathbb{N}_+$, one denotes the space of (equivalence classes of) functions u such that

$$|u|_{m,q} := \sum_{|\alpha|=m}\Big(\int_\Omega |D^\alpha u|^q\Big)^{\frac{1}{q}} < \infty\,,$$

and by $D_0^{m,q}(\Omega)$ the completion of $C_0^\infty(\Omega)$ in the norm $|\cdot|_{m,q}$. Moreover, setting,

$$\mathcal{D}(\Omega) := \{\boldsymbol{u}\in C_0^\infty(\Omega) : \operatorname{div}\boldsymbol{u}=0\}\,,$$

$\mathcal{D}_0^{1,2}(\Omega)$ is the completion of $\mathcal{D}(\Omega)$ in the norm $|\cdot|_{1,2}$. By $\mathcal{D}_0^{-1,2}(\Omega)$ [respectively, $D_0^{-1,2}(\Omega)$] one denotes the normed dual space of $\mathcal{D}_0^{1,2}(\Omega)$ [respectively, $D_0^{1,2}(\Omega)$], and by $\langle\cdot,\cdot\rangle$ [respectively, $[\cdot,\cdot]$] the associated duality pairing.

By $H_q(\Omega)$ it is indicated the completion of $\mathcal{D}(\Omega)$ in the norm $L^q(\Omega)$, and one simply writes $H(\Omega)$ for $q=2$. Further, P is the (Helmholtz-Weyl) projection from $L^q(\Omega)$ onto $H_q(\Omega)$. Notice that, since Ω is a sufficiently smooth exterior domain, P is independent of q.

If M is a map between two spaces, by $\mathsf{D}\,(M)$, $\mathsf{N}\,(M)$ and $\mathsf{R}\,(M)$ one denotes its domain, null space and range, respectively, and by $\mathrm{Sp}\,(M)$ its spectrum.

In the following, B is a real Banach space with associated norm $\|\cdot\|_B$. The complexification of B is denoted by $B_{\mathbb{C}} := B + \mathrm{i}\,B$. Likewise, the complexification of a map M between two Banach spaces will be indicated by $M_{\mathbb{C}}$.

For $q\in[1,\infty]$, $L^q(a,b;B)$ is the space of functions $u:(a,b)\in\mathbb{R}\to B$ such that

$$\Big(\int_a^b \|u(t)\|_B^q\,dt\Big)^{\frac{1}{q}} < \infty,\quad \text{if } q\in[1,\infty)\,;\quad \operatorname*{ess\,sup}_{t\in(a,b)}\|u(t)\|_B<\infty,\quad \text{if } q=\infty.$$

Given a function $u\in L^1(-\pi,\pi;B)$, $\overline{u}$ is its average over $[-\pi,\pi]$, namely,

$$\overline{u} := \frac{1}{2\pi}\int_{-\pi}^{\pi} u(t)\,dt\,.$$

Furthermore, one says that u is 2π-*periodic*, if $u(t+2\pi)=u(t)$, for a.a. $t\in\mathbb{R}$. Set

$$\mathscr{W}^2_{2\pi,0}(\Omega) := \Big\{\boldsymbol{u}\in L^2(-\pi,\pi;W^{2,2}(\Omega)\cap\mathcal{D}_0^{1,2}(\Omega)) \text{ and } \boldsymbol{u}_t\in L^2(-\pi,\pi;H(\Omega)) : \\ \boldsymbol{u} \text{ is } 2\pi\text{-periodic with } \overline{\boldsymbol{u}}=\boldsymbol{0}\Big\}$$

with associated norm

$$\|u\|_{\mathscr{W}^2_{2\pi,0}} := \Big(\int_{-\pi}^{\pi}\|\boldsymbol{u}_t(t)\|_2^2\,dt\Big)^{1/2} + \Big(\int_{-\pi}^{\pi}\|\boldsymbol{u}(t)\|_{2,2}^2\,dt\Big)^{1/2}\,.$$

One also defines

$$\mathscr{H}_{2\pi,0}(\Omega) := \Big\{\boldsymbol{u}\in L^2(-\pi,\pi;H(\Omega)) :\ \boldsymbol{u} \text{ is } 2\pi\text{-periodic with } \overline{\boldsymbol{u}}=\boldsymbol{0}\Big\}\,.$$

Finally, C, C_0, C_1, etc., denote positive constants, whose particular value is unessential to the context. When one wishes to emphasize the dependence of C on some parameter ξ, it will be written $C(\xi)$.

3 Formulation of the Problem

Suppose one has a rigid body, $\mathscr{B}$, moving by *prescribed* motion in an otherwise quiescent viscous liquid, $\mathscr{L}$, filling the entire space outside $\mathscr{B}$. Mathematically, $\mathscr{B}$ will be taken as the closure of a simply connected bounded domain of class C^2. For the sake of generality, a given velocity distribution is allowed on $\partial\mathscr{B}$, due, for example, to a tangential motion of the boundary wall or to an outflow/inflow mechanism, as well as it is assumed that on $\mathscr{L}$ is acting a given body force. (The presence in the model of a body force other than gravity –whose contribution can be always incorporated in the pressure term– could be questionable on physical grounds. However, from the mathematical point of view, it might be useful in consideration of extending the results to more general liquid models, where now the "body force" would represent the contribution to the linear momentum equation of other appropriate fields.) In order to study the motion of $\mathscr{L}$ under these circumstances, it is appropriate to write its governing equations in a *body-fixed* frame, $\mathcal{S}$, so that the region occupied by $\mathscr{L}$ becomes time-independent. One thus gets

$$\left.\begin{aligned}&\boldsymbol{v}_t + (\boldsymbol{v}-\boldsymbol{V})\cdot\nabla\boldsymbol{v} + \boldsymbol{\omega}\times\boldsymbol{v} = \nu\Delta\boldsymbol{v} - \nabla p + \boldsymbol{f}\\ &\operatorname{div}\boldsymbol{v} = 0\end{aligned}\right\}\ \text{ in } \Omega\times(0,\infty)\,. \tag{1}$$

(For the derivation of these equations, we refer to [40, Section 1, eq. (1.15)].) In these equations, $\boldsymbol{v}, \rho p$ are *absolute* velocity and pressure fields of $\mathscr{L}$, respectively, ρ and ν its (constant) density and kinematic viscosity, and $\boldsymbol{f}$ is the body force acting on $\mathscr{L}$. Moreover,

$$\boldsymbol{V} := \boldsymbol{\xi} + \boldsymbol{\omega}\times\boldsymbol{x}\,,$$

with $\boldsymbol{\xi}$ and $\boldsymbol{\omega}$, in the order, velocity of the center of mass and angular velocity of $\mathscr{B}$ in $\mathcal{S}$. Finally, $\Omega := \mathbb{R}^3\backslash\mathscr{B}$ is the *time-independent* region occupied by $\mathscr{L}$ that *will be assumed of class* C^2. (Several peripheral results continue to hold with less or no regularity at all. This will be emphasized in the assumptions occasionally.) The system (1) is endowed with the following boundary condition

$$\boldsymbol{v} = \boldsymbol{v}_* + \boldsymbol{V} \ \text{ at } \partial\Omega\times(0,\infty)\,, \tag{2}$$

with $\boldsymbol{v}_*$ a prescribed field, expressing the adherence of the liquid at the boundary walls of the body, and asymptotic conditions

$$\lim_{|x|\to\infty}\boldsymbol{v}(x,t) = \boldsymbol{0}\,,\ t\in(0,\infty)\,, \tag{3}$$

representative of the property that the liquid is quiescent at large spatial distances from the body.

Throughout this paper it shall be assumed that the vectors $\boldsymbol{\xi}$ *and* $\boldsymbol{\omega}$ *do not depend on time.* This assumption imposes certain limitations on the *type* of motion that $\mathscr{B}$ can execute with respect to a fixed *inertial* frame. Precisely [47], the center of mass of $\mathscr{B}$ must move with constant speed along a circular helix whose axis is parallel to $\boldsymbol{\omega}$. The helix will degenerate into a circle when $\boldsymbol{\xi}\cdot\boldsymbol{\omega} = 0$, in which case the motion of the body reduces to a constant rotation. Without loss of generality, we set $\boldsymbol{\omega} = \omega\,\boldsymbol{e}_1$,

$\omega \geq 0$, and $\boldsymbol{\xi} = v_0\,\boldsymbol{e}$ with $\boldsymbol{e}$ a unit vector. As indicated in the introductory section, one is only interested in the case when the motion of the body does not reduce to a uniform rotation. For this reason, *unless otherwise stated, it will be assumed*

$$v_0 \neq 0 \;\text{ and }\; \boldsymbol{e}\cdot\boldsymbol{e}_1 \neq 0. \tag{4}$$

By shifting the origin of the coordinate system $\mathcal{S}$ suitably (Mozzi-Chasles transformation), and scaling velocity and length by $v_0\,\boldsymbol{e}\cdot\boldsymbol{e}_1$ and d, respectively, one can then show that (1) can be put in the following form in the shifted frame $\mathcal{S}'$ (see [47, pp. 496-497])

$$\left.\begin{aligned} &-\boldsymbol{v}_t + \Delta\boldsymbol{v} + \lambda\,\partial_1\boldsymbol{v} + \mathcal{T}(\boldsymbol{e}_1\times\boldsymbol{x}\cdot\nabla\boldsymbol{v} - \boldsymbol{e}_1\times\boldsymbol{v})\\ &\qquad\qquad\qquad\qquad = \lambda\boldsymbol{v}\cdot\nabla\boldsymbol{v} + \nabla p + \boldsymbol{f}\\ &\operatorname{div}\boldsymbol{v} = 0\end{aligned}\right\}\ \text{in } \Omega\times(0,\infty)$$
$$\boldsymbol{v} = \boldsymbol{v}_* + \mathsf{V} \text{ at } \partial\Omega\times(0,\infty)\,;\quad \lim_{|x|\to\infty}\boldsymbol{v}(x,t) = \boldsymbol{0}\,,\quad \text{all } t\in(0,\infty)\,, \tag{5}$$

where $\mathcal{T} := \dfrac{\omega\,d^2}{\nu}$,

$$\lambda := \begin{cases} \dfrac{v_0\,d}{\nu}\,\boldsymbol{e}\cdot\boldsymbol{e}_1\,, & \text{if } \boldsymbol{\omega}\neq\boldsymbol{0}\,,\\[2mm] \dfrac{v_0\,d}{\nu}\,, & \text{if } \boldsymbol{\omega}=\boldsymbol{0}\quad(\boldsymbol{e}\equiv\boldsymbol{e}_1)\,,\end{cases} \tag{6}$$

and

$$\mathsf{V} := \boldsymbol{e}_1 + \tfrac{\mathcal{T}}{\lambda}\,\boldsymbol{e}_1\times\boldsymbol{x}\,. \tag{7}$$

Of course, all fields entering the equations in (5) are regarded as non-dimensional. Observe also that, in the rescaled length variables, the diameter of $\mathscr{B}$ becomes 1. In order to simplify the presentation, the origin of the coordinate system $\mathcal{S}'$ will be supposed to lie in the interior of $\mathscr{B}$. Finally, we notice that, in view of (4), it follows $\lambda \neq 0$. Since all results presented in this article are independent of whether $\lambda \gtrless 0$, *it will assumed throughout* $\lambda > 0$.

Of particular relevance to this article are time-independent solutions (steady-state flow) of problem (5)–(7), which may occur only when $\boldsymbol{f}$ and $\boldsymbol{v}_*$ are also time-independent. From (5)one thus infers that these solutions must satisfy the following boundary-value problem

$$\left.\begin{aligned} &\Delta\boldsymbol{v} + \lambda\,\partial_1\boldsymbol{v} + \mathcal{T}(\boldsymbol{e}_1\times\boldsymbol{x}\cdot\nabla\boldsymbol{v} - \boldsymbol{e}_1\times\boldsymbol{v}) = \lambda\boldsymbol{v}\cdot\nabla\boldsymbol{v} + \nabla p + \boldsymbol{f}\\ &\operatorname{div}\boldsymbol{v} = 0\end{aligned}\right\}\ \text{in } \Omega$$
$$\boldsymbol{v} = \boldsymbol{v}_* + \mathsf{V} \text{ at } \partial\Omega\,;\quad \lim_{|x|\to\infty}\boldsymbol{v}(x) = \boldsymbol{0}\,. \tag{8}$$

The primary objective of this article is to provide an updated review of some fundamental properties of solutions to (6)–(8). The latter include existence, uniqueness, regularity, asymptotic structure, generic properties, and steady and unsteady bifurcation issues. Moreover, a rather complete analysis of the long-time behavior of dynamical perturbation to these solutions will be performed that will lead, in particular, to a number of stability and asymptotic stability results, under various assumptions.

4 Existence

The starting point is the following general definition of *weak (or generalized) solution* for problem (6)–(8) [81].

Definition 1. Let $\boldsymbol{f} \in \mathcal{D}_0^{-1,2}(\Omega)$, $\boldsymbol{v}_* \in W^{1/2,2}(\partial\Omega)$. A vector field $\boldsymbol{v} : \Omega \to \mathbb{R}^3$ is a *weak solution* to problem (8)–(7) if the following conditions hold

(a) $\boldsymbol{v} \in D^{1,2}(\Omega)$ with $\operatorname{div} \boldsymbol{v} = 0$;
(b) $\boldsymbol{v}$ satisfies the equation

$$\begin{aligned} -(\nabla \boldsymbol{v}, \nabla \boldsymbol{\varphi}) + \lambda\,(\partial_1 \boldsymbol{v}, \boldsymbol{\varphi}) + \mathcal{T}\,(\boldsymbol{e}_1 \times \boldsymbol{x} \cdot \nabla \boldsymbol{v} - \boldsymbol{e}_1 \times \boldsymbol{v}, \boldsymbol{\varphi}) + \lambda\,(\boldsymbol{v} \cdot \nabla \boldsymbol{\varphi}, \boldsymbol{v}) \\ = \langle \boldsymbol{f}, \boldsymbol{\varphi} \rangle\,, \quad \text{for all } \boldsymbol{\varphi} \in \mathscr{D}(\Omega)\,. \end{aligned} \tag{9}$$

(c) $\boldsymbol{v} = \boldsymbol{v}_* + \mathsf{V}$ at $\partial\Omega$ in the trace sense.
(d) $\displaystyle\lim_{R\to\infty} R^{-2} \int_{\partial B_R} |\boldsymbol{v}| = 0\,.$

(Formally, (9) is obtained by taking the scalar product of both sides of $(8)_1$ by $\boldsymbol{\varphi}$, and integrating by parts over Ω. Since $D^{1,2}(\Omega) \subset W^{1,2}(\Omega_R)$, $R > 1$, condition (c) is meaningful.)

Remark 1. If $\boldsymbol{f} \in W_0^{-1,2}(\Omega')$, for all bounded Ω' with $\overline{\Omega'} \subset \Omega$, then to every weak solution one can associate a suitable corresponding pressure field. More precisely, there exists $p \in L^2_{\text{loc}}(\Omega)$ such that

$$\begin{aligned} -(\nabla \boldsymbol{v}, \nabla \boldsymbol{\psi}) + \lambda\,(\partial_1 \boldsymbol{v}, \boldsymbol{\psi}) + \mathcal{T}\,(\boldsymbol{e}_1 \times \boldsymbol{x} \cdot \nabla \boldsymbol{v} - \boldsymbol{e}_1 \times \boldsymbol{v}, \boldsymbol{\psi}) + \lambda\,(\boldsymbol{v} \cdot \nabla \boldsymbol{\psi}, \boldsymbol{v}) \\ = -(p, \operatorname{div} \boldsymbol{\psi}) + [\boldsymbol{f}, \boldsymbol{\psi}]\,, \quad \text{for all } \boldsymbol{\psi} \in C_0^\infty(\Omega)\,, \end{aligned}$$

where $[\cdot,\cdot]$ stands for the duality pairing $D_0^{-1,2} \leftrightarrow D_0^{1,2}$. Notice that this equation is formally obtained by dot-multiplying both sides of $(8)_1$ by $\boldsymbol{\psi}$ and integrating by parts over Ω. The proof of this property, based on the representation of elements of $D_0^{-1,2}$ vanishing on $\mathcal{D}_0^{1,2}$, is given in [47, Lemma XI.1.1].

The next step is the construction of a suitable extension, $\boldsymbol{U}$, of the boundary data. The crucial property of such extension is condition (10) given below. In fact, as will become clear later on, this allows one to obtain the fundamental *a priori* estimate for the existence result. Now, the validity of (10) is related to the magnitude of the flux through the boundary $\partial\Omega$, Φ, of the field $\boldsymbol{v}_*$:

$$\Phi := \int_{\partial\Omega} \boldsymbol{v}_* \cdot \boldsymbol{n}\,,$$

with $\boldsymbol{n}$ unit outer normal to $\partial\Omega$. For simplicity, it will be assumed $\Phi = 0$, even though all main results continue to hold also when $|\Phi|$ is sufficiently "small". Th reader is referred to Open Problem 1 for further considerations about this issue.

The existence of the appropriate extension of the boundary data is provided by the following result whose proof can be found in [47, Lemma X.4.1]

Lemma 1 *Let*

$$\boldsymbol{v}_* \in W^{1/2,2}(\partial\Omega), \quad \int_{\partial\Omega} \boldsymbol{v}_* \cdot \boldsymbol{n} = 0\,.$$

Then, for any $\eta > 0$, there exists $\boldsymbol{U} = \boldsymbol{U}(\eta, \boldsymbol{v}_*, \mathsf{V}, \Omega) : \Omega \to \mathbb{R}^3$ with bounded support such that

(i) $\boldsymbol{U} \in W^{1,2}(\Omega)$;
(ii) $\boldsymbol{U} = \boldsymbol{v}_* + \mathsf{V}$ *at* $\partial\Omega$;
(iii) $\operatorname{div} \boldsymbol{U} = 0$ *in* Ω.

Furthermore, for all $\boldsymbol{u} \in \mathcal{D}_0^{1,2}(\Omega)$*, it holds that*

$$|(\boldsymbol{u} \cdot \nabla \boldsymbol{U}, \boldsymbol{u})| \leq \eta\, |\boldsymbol{u}|_{1,2}^2. \tag{10}$$

Finally, if $\|\boldsymbol{v}_*\|_{1/2,2(\partial\Omega)} \leq M$*, for some* $M > 0$*, then*

$$\|\boldsymbol{U}\|_{1,2} \leq C_1 \|\boldsymbol{v}_* + \mathsf{V}\|_{1/2,2(\partial\Omega)} \tag{11}$$

where $C_1 = C_1(\eta, M, \Omega)$.

Remark 2. In view of the above result, it easily follows that the existence of a weak solution is secured if there is $\boldsymbol{u} \in \mathcal{D}_0^{1,2}(\Omega)$ satisfying

$$\begin{aligned} -(\nabla \boldsymbol{u}, \nabla \boldsymbol{\varphi}) + \lambda\, (\partial_1 \boldsymbol{u}, \boldsymbol{\varphi}) + \mathcal{T}\, (\boldsymbol{e}_1 \times \boldsymbol{x} \cdot \nabla \boldsymbol{u} - \boldsymbol{e}_1 \times \boldsymbol{u}, \boldsymbol{\varphi}) + \lambda\, (\boldsymbol{u} \cdot \nabla \boldsymbol{\varphi}, \boldsymbol{u}) \\ + \lambda [(\boldsymbol{U} \cdot \nabla \boldsymbol{\varphi}, \boldsymbol{u}) - (\boldsymbol{u} \cdot \nabla \boldsymbol{U}, \boldsymbol{\varphi})] - (\nabla \boldsymbol{U}, \nabla \boldsymbol{\varphi}) + \lambda\, (\partial_1 \boldsymbol{U} - \boldsymbol{U} \cdot \nabla \boldsymbol{U}, \boldsymbol{\varphi}) \\ + \mathcal{T}\, (\boldsymbol{e}_1 \times \boldsymbol{x} \cdot \nabla \boldsymbol{U} - \boldsymbol{e}_1 \times \boldsymbol{U}, \boldsymbol{\varphi}) = \langle \boldsymbol{f}, \boldsymbol{\varphi} \rangle\,, \quad \text{for all } \boldsymbol{\varphi} \in \mathscr{D}(\Omega)\,. \end{aligned} \tag{12}$$

In fact, setting, $\boldsymbol{v} := \boldsymbol{u} + \boldsymbol{U}$ one gets at once that conditions (a)–(c) of Definition 1 are met. Moreover, since, by Sobolev theorem $\mathcal{D}_0^{1,2}(\Omega) \subset L^6(\Omega)$ (e.g. [47, Theorem II.7.5]), from [47, Lemma II.6.3] it follows

$$\lim_{R \to \infty} \frac{1}{R^{\frac{3}{2}}} \int_{\partial B_R} |\boldsymbol{u}| = 0\,, \quad \text{for all } \boldsymbol{u} \in D_0^{1,2}(\Omega)\,,$$

so that also requirement (d) is met, even with a better order of decay. In view of all the above, we may equally refer to both $\boldsymbol{v}$ and $\boldsymbol{u}$ as "weak solution".

4.1 Early Contributions

It will be now presented and summarized classical approaches and results to the existence of weak solutions due, basically, to Jean Leray [83], Olga A. Ladyzhenskaya [81] and Hiroshi Fujita [31]. Besides their historical relevance and intrinsic interest, these results will also provide a further motivation for the entirely distinct approach –recently introduced in [43; 49]–that will be describd in Section 5.

4.1.1 Leray's Contribution

In his famous pioneering work on the steady-state Navier-Stokes equations [83, Chapitres II & III] Leray shows that for *any* sufficiently regular $\boldsymbol{f}$ and $\boldsymbol{v}_*$, with $\Phi = 0$, there is at least one corresponding solution $(\boldsymbol{v}, p)$ to $(8)_{1,2,3}$–(7), which, in addition, satisfies

$\boldsymbol{v} \in D^{1,2}(\Omega)$. (As a matter of fact, Leray requires $\boldsymbol{f} \equiv \boldsymbol{0}$; [83, §3 at p. 32]. However, for his method to go through, the weaker assumption of a "smooth" $\boldsymbol{f}$ would suffice.) It is just in this weak sense that Leray interprets the condition at infinity $(8)_4$. (As noticed earlier on, this condition can be expressed in a sharper, though still weak, way; see Remark 2.) Leray's construction, basically, consists in solving the original problem $(8)_{1,2,3}$–(7) on each element of a sequence of *bounded* domains $\{\Omega_k\}_{k>1}$ with $\Omega = \cup_{k=1}^{\infty} \Omega_k$, under the further condition $\boldsymbol{v} = 0$ on the "fictitious" boundary ∂B_k ("invading domains" technique). In turn, on every Ω_k, a sufficiently smooth solution, $\boldsymbol{v}_k$, to the system (8) is determined by combining Leray-Schauder degree theory with a *uniform* bound on the Dirichlet integral $|\boldsymbol{v}_k|_{1,2}^2$. (It should be observed that, even though the demonstration provided by Leray is presented in the language of Leray-Schauder fixed-point theorem, such a result was not yet available at that time; see [85; 86].) The latter is crucial, in that it allows Leray to select a subsequence that, uniformly on compact sets, converges to a solution of the original problem, meant in a suitable integral sense. It must be emphasized that in order to obtain the above bound, the property (10) of the extension is crucial. (Notice that a bound on $\boldsymbol{v}_k$ can also be obtained by an alternative method, based on a contradiction argument; see [83, Chapitre II, §III]. Even though the latter is more general than the one based on the existence of an extension satisfying (10) (see [50, Introduction]), however, it does not necessarily provide a *uniform* bound independent of k, and, therefore, is of no use in the context of the "invading domains" technique.) An important feature of the solution constructed by Leray is that it could be shown to satisfy the so called "generalized energy inequality"

$$\begin{aligned} -|\boldsymbol{u}|_{1,2}^2 - \lambda(\boldsymbol{u}\cdot\nabla\boldsymbol{U},\boldsymbol{u}) - (\nabla\boldsymbol{U},\nabla\boldsymbol{u}) + \lambda\,(\partial_1\boldsymbol{U} - \boldsymbol{U}\cdot\nabla\boldsymbol{U},\boldsymbol{u}) \\ +\mathcal{T}\,(\boldsymbol{e}_1\times\boldsymbol{x}\cdot\nabla\boldsymbol{U} - \boldsymbol{e}_1\times\boldsymbol{U},\boldsymbol{u}) - \langle\boldsymbol{f},\boldsymbol{u}\rangle \le 0\,, \end{aligned} \tag{13}$$

formally obtained by setting $\boldsymbol{\varphi} \equiv \boldsymbol{u}$ in (12) and replacing " $=$ " with " $\le$ ". A more familiar form of (13) can be obtained if $\boldsymbol{f}$ and $\boldsymbol{v}_*$ have some more regularity. For example, if, in addition $\boldsymbol{f} \in L^2(\Omega)$ and $\boldsymbol{v}_* \in W^{3/2,2}(\partial\Omega)$ then it can be shown that (13) is equivalent to the following one (see [47, Theorem XI.3.1(i)])

$$-2\|\boldsymbol{D}(\boldsymbol{v})\|_2^2 + \int_{\partial\Omega} \{(\boldsymbol{v}_* + \mathsf{V})\cdot\boldsymbol{T}(\boldsymbol{v},p) - \frac{\lambda}{2}(\boldsymbol{v}_* + \mathsf{V})^2\boldsymbol{v}_*\}\cdot\boldsymbol{n} - \langle\boldsymbol{f},\boldsymbol{v}\rangle \le 0\,, \tag{14}$$

where $\boldsymbol{T}(\boldsymbol{v},p) = \nabla\boldsymbol{v} + (\nabla\boldsymbol{v})^\top - p\,\boldsymbol{I}$, $\boldsymbol{I}$ identity matrix, is the Cauchy stress tensor. It is worth emphasizing that (14) *would* represent the *energy balance* for the motion $(\boldsymbol{v},p)$, *provided* one could replace " $\le$ " with " $=$ ". The inequality sign in the above formulas is, again, a consequence of the little information that this solution carries at large spatial distances. For the same reason, the uniqueness question is *left out*.

4.1.2 Ladyzhenskaya's Contribution

Ladyzhenskaya was the first to introduce the definition and the use of the term "generalized (or weak) solution" as currently used, for steady-state Navier-Stokes problems [81, p. 78]. Her construction still employs the "invading domains" technique utilized by

Leray, but the way in which she proves the existence of the solution on each bounded domain Ω_k of the sequence is somewhat simpler and more direct. More precisely, Ladyzenskaya considers (12) with $\mathcal{T}=0$ and $\boldsymbol{v}_*\equiv\mathbf{0}$, and shows that it can be equivalently rewritten as a nonlinear equation in the Hilbert space $\mathcal{D}_0^{1,2}(\Omega_k)$:

$$\mathcal{M}(\boldsymbol{u}) := \boldsymbol{u} + \lambda\,\boldsymbol{A}(\boldsymbol{u}) = \boldsymbol{F} \tag{15}$$

where $\boldsymbol{F}$ is prescribed $\mathcal{D}_0^{1,2}(\Omega_k)$ and $\boldsymbol{A}$ is a (nonlinear) compact operator. (The extension to the case $\mathcal{T}\neq 0$ would be straightforward.) Therefore, the operator $\mathcal{M}$, defined on the whole of $\mathcal{D}_0^{1,2}(\Omega_k)$, is a compact perturbation of a homeomorphism. Moreover, using arguments similar to Leray's, one can show that every solution to (15) is uniformly bounded in $\mathcal{D}_0^{1,2}(\Omega_k)$, for all $\lambda\in[0,\lambda_0]$, arbitrary fixed $\lambda_0>0$. Then, by the Leray-Schauder degree theory it follows that (15) has a weak solution, $\boldsymbol{u}_k\in\mathcal{D}_0^{1,2}(\Omega_k)$, for the given $\boldsymbol{F}$. Since $|\boldsymbol{u}_k|_{1,2}$ is uniformly bounded in k, Ladyzhenskaya shows that a subsequence can be selected converging to a weak solution in the sense of Definition 1; see Remark 2. It is worth emphasizing that, if Ω is an *exterior* domain, the operator $\boldsymbol{A}$ is *not* compact, see [49, Proposition 80], so that the "invading domain technique" is indeed necessary for the argument to work. Moreover, if $\boldsymbol{u}$ is *merely* in $\mathcal{D}_0^{1,2}(\Omega)$, with Ω *exterior* domain, the very equation (22) would not be meaningful in such a case. Finally, it is important to observe that Ladyzhenskaya's solution, as Leray's, satisfies only the generalized energy *inequality* (13), and, again, the uniqueness question is *left open* because of the little asymptotic information carried by functions from $D_0^{1,2}(\Omega)$.

4.1.3 Fujita's Contribution

Fujita's approach to the existence of a weak solution [31] is entirely different from those previously mentioned. In fact, it consists in adapting to the time-independent case the method introduced by Eberhard Hopf for the *initial-value* problem [66]. The method referred to above is the by now classical Faedo-Galerkin method. (Also, strictly speaking, Fujita considers the case $\mathcal{T}=0$, even though the extension of his method to the more general case presents no conceptual difficulty.) As is well known, the idea is to look first for an "approximate solution" to (12), $\boldsymbol{u}_N$, in the manifold $\mathscr{M}(N)$ spanned by the first N elements of a basis of $\mathcal{D}_0^{1,2}(\Omega)$. This is a finite-dimensional problem whose solution, at the N-th step, is found by solving a suitable nonlinear equation. Fujita solves the latter by means of Brouwer fixed-point theorem [31, Lemma 3.1], provided $|\Phi|$ is "small enough", and then shows that $|\boldsymbol{u}_N|_{1,2}$ is uniformly bounded in N. With this information in hand, one can then select a subsequence $\{\boldsymbol{u}_{N'}\}$ that in the limit $N'\to\infty$ converges (in a suitable sense) to a vector $\boldsymbol{u}\in\mathcal{D}_0^{1,2}(\Omega)$ satisfying (12); see also [47, Theorem X.4.1]. The advantage of Fujita's approach, besides being more elementary, resides also in the fact that the solution is constructed directly in the whole domain Ω. However, also in this case, solutions satisfy only the generalized energy *inequality*, and their uniqueness is also *left out*.

4.2 A Function-Analytic Approach

The most significant aspect of solutions constructed by the above authors is that their existence is ensured for data of *arbitrary* "size", provided only the mass flux through the boundary is not too large. (Notice that, of course, $\|\boldsymbol{v}_*\|_{1/2,2,\partial\Omega}$ arbitrarily "large", and $|\Phi|$ "small" are not, in general, at odds.) However, as emphasized already a few times, these solutions possess no further asymptotic information at large distances other than that deriving from the fact that $\boldsymbol{v} \in L^6(\Omega)$, consequence of the of the property $\boldsymbol{v} \in D^{1,2}(\Omega)$ and Sobolev inequality; see Remark 2. With such a little information, it is, basically, hopeless to show fundamental properties of the solution that are yet expected on physical grounds, such as (i) balance of energy equation, namely, (14) with the *equality sign*, and (ii) uniqueness for "small" data.

The main objective of this subsection is to show that, in fact, this undesired feature can be removed by using another and completely different approach. The approach, introduced in [43; 49], consists in formulating the original problem as a nonlinear equation in a suitable Banach space, and then use the mod 2 degree for proper Fredholm maps of index 0 to show *just under the conditions on the data stated in* Definition 1, the existence of a corresponding weak solution possessing "better" properties at "large" distances. As a consequence, one proves that these weak solutions satisfy, in addition, the requirements (i) and (ii) above. Moreover, this abstract setting shows itself appropriate for the study of other important properties of solutions, including generic properties, and steady and time-periodic bifurcation; see Sections 8 and 9.

In order to give a precise statement of the main results, it is appropriate to introduce the necessary functional setting. To this end, let

$$\mathscr{R}(\boldsymbol{u}) := \boldsymbol{e}_1 \times \boldsymbol{x} \cdot \nabla \boldsymbol{u} - \boldsymbol{e}_1 \times \boldsymbol{u}$$

and set

$$X(\Omega) = \{\boldsymbol{u} \in \mathcal{D}_0^{1,2}(\Omega) : \ \partial_1 \boldsymbol{u}, \ \mathscr{R}(\boldsymbol{u}) \in \mathcal{D}_0^{-1,2}(\Omega)\}\,, \tag{16}$$

where $\partial_1 \boldsymbol{u} \in \mathcal{D}_0^{-1,2}(\Omega)$ means that there is $C > 0$ such that

$$|(\partial_1 \boldsymbol{u}, \boldsymbol{\varphi})| \le C\,|\boldsymbol{\varphi}|_{1,2}\,, \quad \text{for all } \boldsymbol{\varphi} \in \mathscr{D}(\Omega)\,,$$

and, therefore, by the Hahn-Banach theorem $\partial_1 \boldsymbol{u}$ can be uniquely extended to an element of $\mathcal{D}_0^{-1,2}(\Omega)$ that will still be denoted by $\partial_1 \boldsymbol{u}$. Analogous considerations hold for $\mathscr{R}(\boldsymbol{u})$. It can be shown [49, Proposition 65] that when endowed with its "natural" norm

$$\|\boldsymbol{u}\|_X := |\boldsymbol{u}|_{1,2} + |\partial_1 \boldsymbol{u}|_{-1,2} + |\mathscr{R}(\boldsymbol{u})|_{-1,2}\,,$$

$X(\Omega)$ becomes a reflexive, separable Banach space. Obviously, $X(\Omega)$ is a *strict* subspace of $\mathcal{D}_0^{1,2}(\Omega)$.

The primary objective is to prove existence of weak solution in the space $X(\Omega)$. In this respect, one observes that all classical approaches mentioned earlier on, furnish weak solutions in $\mathcal{D}_0^{1,2}(\Omega)$ which embeds only in $L^6(\Omega)$; see Remark 2. The fundamental property of $X(\Omega)$, expressed in the following lemma, is that it embeds in a much "better" space.

Lemma 2 *Let $\Omega \subset \mathbb{R}^3$ be an exterior domain and assume $\boldsymbol{u} \in \mathcal{D}_0^{1,2}(\Omega)$ with $\partial_1 \boldsymbol{u} \in \mathcal{D}_0^{-1,2}(\Omega)$. Then, $\boldsymbol{u} \in L^4(\Omega)$, and there is $C_1 = C_1(\Omega) > 0$ such that*

$$\|\boldsymbol{u}\|_4 \le C_1 \, |\partial_1 \boldsymbol{u}|_{-1,2}^{\frac{1}{4}} |\boldsymbol{u}|_{1,2}^{\frac{3}{4}} . \tag{17}$$

Thus, in particular, $X(\Omega) \subset L^4(\Omega)$.

Proof. Obviously, if $\boldsymbol{u} \equiv \boldsymbol{0}$ there is nothing to prove, so one shall assume $\boldsymbol{u} \not\equiv \boldsymbol{0}$. The proof for an arbitrary exterior domain is somewhat complicated by several technical issues; see [43, Proposition 1.1 with proof on pp. 8–13]. However, if $\Omega \equiv \mathbb{R}^3$ it becomes simpler and will be sketched here. (The inequality proved in [43, Proposition 1.1] is, in fact, weaker than (17). However, one can apply to eq. (1.31) of [43] almost *verbatim* the argument given in the current proof after (23), and show the stronger form (17).) For a given $\boldsymbol{g} \in C_0^\infty(\mathbb{R}^3)$, consider the following problem

$$\Delta \boldsymbol{\varphi} - \mu \, \partial_1 \boldsymbol{\varphi} = \boldsymbol{g} + \nabla p , \quad \operatorname{div} \boldsymbol{\varphi} = 0 , \quad \text{in } \mathbb{R}^3 , \tag{18}$$

where $\mu > 0$. By [47, Theorem VII.4.1], problem (18) has at least one solution such that

$$\begin{aligned} &\boldsymbol{\varphi} \in L^{s_1}(\mathbb{R}^3) \cap D^{1,s_2}(\mathbb{R}^3) \cap D^{2,s_3}(\mathbb{R}^3) \, , \partial_1 \boldsymbol{\varphi} \in L^{s_3}(\mathbb{R}^3) \\ &p \in L^{s_4}(\mathbb{R}^3) \cap D^{1,s_3}(\mathbb{R}^3) \end{aligned} \tag{19}$$

for all $s_1 > 2$, $s_2 > 4/3$, $s_3 > 1$, $s_4 > 3/2$, which satisfies the estimate

$$\mu^{1/4} |\boldsymbol{\varphi}|_{1,2} \le C \|\boldsymbol{g}\|_{4/3} , \tag{20}$$

with $C = C(s_1, \ldots, s_4)$. Using (18)–(19), and recalling that by the Sobolev inequality $\mathcal{D}_0^{1,2}(\mathbb{R}^3) \subset L^6(\mathbb{R}^3)$, one shows after integration by parts

$$(\boldsymbol{u}, \boldsymbol{g}) = (\boldsymbol{u}, \Delta \boldsymbol{\varphi} - \mu \, \partial_1 \boldsymbol{\varphi} - \nabla p) = -(\nabla \boldsymbol{u}, \nabla \boldsymbol{\varphi}) - \mu (\boldsymbol{u}, \partial_1 \boldsymbol{\varphi}) . \tag{21}$$

The following identity is valid for all $\boldsymbol{u} \in \mathcal{D}_0^{1,2}(\mathbb{R}^3)$ with $\partial_1 \boldsymbol{u} \in \mathcal{D}_0^{-1,2}(\mathbb{R}^3)$, and $\boldsymbol{\psi} \in \mathcal{D}_0^{1,2}(\mathbb{R}^3)$ with $\partial_1 \boldsymbol{\psi} \in L^{\frac{6}{5}}(\mathbb{R}^3)$, and can be shown by the arguments from [43, p. 12–13]

$$(\boldsymbol{u}, \partial_1 \boldsymbol{\psi}) = -\langle \partial_1 \boldsymbol{u}, \boldsymbol{\psi} \rangle .$$

In view of (19) we may use the latter in (21) to get

$$(\boldsymbol{u}, \boldsymbol{g}) = -(\nabla \boldsymbol{u}, \nabla \boldsymbol{\varphi}) + \mu \langle \partial_1 \boldsymbol{u}, \boldsymbol{\varphi} \rangle .$$

which implies

$$|(\boldsymbol{u}, \boldsymbol{g})| \le (|\boldsymbol{u}|_{1,2} + \mu \, |\partial_1 \boldsymbol{u}|_{-1,2}) \; |\boldsymbol{\varphi}|_{1,2} . \tag{22}$$

Replacing (20) into this latter inequality, one finds

$$|(\boldsymbol{u}, \boldsymbol{g})| \le C \left(\mu^{\frac{3}{4}} \; |\partial_1 \boldsymbol{u}|_{-1,2} + \mu^{-\frac{1}{4}} |\boldsymbol{u}|_{1,2} \right) \|\boldsymbol{g}\|_{\frac{4}{3}} .$$

Since $\boldsymbol{g}$ is arbitrary in $C_0^\infty(\mathbb{R}^3)$, it follows that $\boldsymbol{u} \in L^4(\mathbb{R}^3)$ and, furthermore,

$$\|\boldsymbol{u}\|_4 \le C \left(\mu^{\frac{3}{4}} \; |\partial_1 \boldsymbol{u}|_{-1,2} + \mu^{-\frac{1}{4}} |\boldsymbol{u}|_{1,2} \right) , \quad \text{for all } \mu > 0. \tag{23}$$

By a simple calculation we show that the right-hand side of (23) as a function of μ attains its minimum at

$$\mu = |\boldsymbol{u}|_{1,2}/(3\,|\partial_1\boldsymbol{u}|_{-1,2})\,,$$

which, once replaced in (23) proves (17), provided $|\partial_1\boldsymbol{u}|_{-1,2}\neq 0$. To show that this is indeed the case, suppose the contrary. Then $(\partial_1\boldsymbol{u},\boldsymbol{\varphi})=0$ for all $\boldsymbol{\varphi}\in\mathscr{D}(\mathbb{R}^3)$, so that there is $\mathsf{p}\in D^{1,2}(\mathbb{R}^3)$ such that $\partial_1\boldsymbol{u}=\nabla\mathsf{p}$ in $\mathbb{R}^3$; see, e.g., [47, Lemma III.3.1]. From $\operatorname{div}\boldsymbol{u}=0$, we deduce $\Delta\mathsf{p}=0$ in $\mathbb{R}^3$ in the sense of distributions, which, by the property of p in turn furnishes $\partial_1\boldsymbol{u}=\nabla p\equiv\boldsymbol{0}$, and this contradicts the fact that $\boldsymbol{u}\in L^6(\mathbb{R}^3)$. The proof is thus completed. □

One is now in a position to state the following general existence result.

Theorem 1 *For any* $\lambda\neq 0$, $\mathcal{T}\geq 0$, $\boldsymbol{f}\in\mathcal{D}_0^{-1,2}(\Omega)$ *and* $\boldsymbol{v}_*\in W^{1/2,2}(\partial\Omega)$ *with* $\Phi=0$ *there exists at least one weak solution,* $\boldsymbol{v}$*, to* (8)–(7) *that in addition satisfies* $\boldsymbol{v}-\boldsymbol{U}\in X(\Omega)$*, with* $\boldsymbol{U}$ *given in* Lemma 1*. Moreover,* $\boldsymbol{v}$ *obeys the estimate*

$$\|\boldsymbol{v}-\boldsymbol{U}\|_X\leq C_1\left(|\boldsymbol{f}|_{-1,2}+|\boldsymbol{f}|_{-1,2}^3\right)+C_2\left(\|\boldsymbol{v}_*+\mathsf{V}\|_{1/2,2,\partial\Omega)}+\|\boldsymbol{v}_*+\mathsf{V}\|_{1/2,2(\partial\Omega)}^3\right)\quad(24)$$

where $C_1=C_1(\lambda,\mathcal{T},\Omega)$*, and* $C_2=C_2(\lambda,\mathcal{T},\Omega,M)$*, whenever* $\|\boldsymbol{v}_*\|_{1/2,2(\partial\Omega)}\leq M$

A full proof of Theorem 1 is given in [49, Theorem 86(i)]. Here it shall be reproduced the main ideas leading to the result, referring the reader to the cited reference for all missing details.

The first step is to write (12) as a nonlinear equation in the space $\mathcal{D}_0^{-1,2}(\Omega)$. To reach this goal, for fixed $\lambda,\mathcal{T}$ one defines the *generalized Oseen operator*

$$\mathscr{O}:\boldsymbol{u}\in X(\Omega)\mapsto\mathscr{O}(\boldsymbol{u})\in\mathcal{D}_0^{-1,2}(\Omega)\tag{25}$$

where

$$\langle\mathscr{O}(\boldsymbol{u}),\boldsymbol{\varphi}\rangle:=-(\nabla\boldsymbol{u},\nabla\boldsymbol{\varphi})+\lambda\langle\partial_1\boldsymbol{u},\boldsymbol{\varphi}\rangle+\mathcal{T}\,\langle\mathscr{R}(\boldsymbol{u}),\boldsymbol{\varphi}\rangle\,,\quad\boldsymbol{\varphi}\in\mathcal{D}_0^{1,2}(\Omega)\,.\tag{26}$$

Likewise, one introduces the operators $\mathscr{N}$, and $\mathscr{K}$ from $X(\Omega)$ to $\mathcal{D}_0^{-1,2}(\Omega)$ as follows:

$$\begin{aligned}&\langle\mathscr{N}(\boldsymbol{u}),\boldsymbol{\varphi}\rangle:=\lambda(\boldsymbol{u}\cdot\nabla\boldsymbol{\varphi},\boldsymbol{u})\\&\langle\mathscr{K}(\boldsymbol{u}),\boldsymbol{\varphi}\rangle:=-\lambda[(\boldsymbol{U}\cdot\nabla\boldsymbol{u},\boldsymbol{\varphi})+(\boldsymbol{u}\cdot\nabla\boldsymbol{U},\boldsymbol{\varphi})]\end{aligned}\qquad\boldsymbol{\varphi}\in\mathcal{D}_0^{1,2}(\Omega)\,.\tag{27}$$

(The dependence of the relevant operators on the parameters λ and $\mathcal{T}$ will be emphasized only when needed; see Sections 8 and 9.) Finally, let $\boldsymbol{F}$ denote the uniquely determined element of $\mathcal{D}_0^{-1,2}(\Omega)$ such that, for all $\boldsymbol{\varphi}\in\mathcal{D}_0^{1,2}(\Omega)$,

$$\langle\boldsymbol{F},\boldsymbol{\varphi}\rangle:=(\nabla\boldsymbol{U},\nabla\boldsymbol{\varphi})-\lambda(\partial_1\boldsymbol{U}-\boldsymbol{U}\cdot\nabla\boldsymbol{U},\boldsymbol{\varphi})-\mathcal{T}\,(\mathscr{R}(\boldsymbol{U}),\boldsymbol{\varphi})+\langle\boldsymbol{f},\boldsymbol{\varphi}\rangle\,.\tag{28}$$

In view of Lemma 1 and Lemma 2, and with the help of Hölder inequality it is easy to show that the operators $\mathscr{O}$, $\mathscr{N}$, and $\mathscr{K}$, and the element $\boldsymbol{F}$ are well defined.

Setting

$$\mathscr{L}:=\mathscr{O}+\mathscr{K}\,,\tag{29}$$

the objective is to solve the following problem: *For any* $\boldsymbol{F}\in\mathcal{D}_0^{-1,2}(\Omega)$*, find* $\boldsymbol{u}\in X(\Omega)$ *such that*

$$\mathscr{L}(\boldsymbol{u})+\mathscr{N}(\boldsymbol{u})=\boldsymbol{F}\,.\tag{30}$$

It is plain that, if this problem is solvable, then $\boldsymbol{v}=\boldsymbol{u}+\boldsymbol{U}$ is a weak solution satisfying the statement of Theorem 1.

The strategy to solve the above problem consists in showing that the map $\mathcal{M} := \mathscr{L} + \mathscr{N} : X(\Omega) \mapsto \mathcal{D}_0^{-1,2}(\Omega)$ is *surjective*. To reach this goal, one may use a very general result furnished in [49], based on the mod 2 degree of proper C^2 Fredholm maps of index 0 due to Smale [105]. More precisely, from [49, Theorem 59(a)] it follows, in particular, the following.

Proposition 1. *Let Z, Y be Banach spaces with Z reflexive. Let $L : Z \mapsto Y$, $N : Z \mapsto Y$ and set $M = L + N$. Suppose*

(i) *M is weakly sequentially continuous (that is, if $z_n \to z$ weakly in Z, then $M(z_n) \to M(z)$ weakly in Y);*
(ii) *N is quadratic (that is, there is a bilinear bounded operator $B : Z \times Z \mapsto Y$ such that $N(z) = B(z, z)$ for all $z \in Z$);*
(iii) *L maps homeomorphically Z onto Y;*
(iv) *The Fréchet derivative of N is compact at every $z \in Z$;*
(v) *There is $\phi : \mathbb{R}_+ \mapsto \mathbb{R}_+$ mapping bounded set into bounded set, with $\phi(s) \to 0$ as $s \to 0$, such that*
$$\|z\|_Z \leq \phi\left(\|M(z)\|_Y\right).$$

Then M is surjective.

This proposition will be applied with $Z \equiv X(\Omega)$, $Y \equiv \mathcal{D}_0^{-1,2}(\Omega)$, $L \equiv \mathscr{L}$, and $N \equiv \mathscr{N}$. With this in mind, one begins to show the following.

Lemma 3 *The operator $\mathscr{N}$ is quadratic, and $\mathcal{M} := \mathscr{L} + \mathscr{N}$ is weakly sequentially continuous.*

Proof. The first property is obvious, since

$$\mathscr{N}(\boldsymbol{u}) = \lambda \mathcal{B}(\boldsymbol{u}, \boldsymbol{u}) \tag{31}$$

where, for $\boldsymbol{w}_i \in X(\Omega)$, $i = 1, 2$,

$$\langle \mathcal{B}(\boldsymbol{w}_1, \boldsymbol{w}_2), \boldsymbol{\varphi} \rangle := (\boldsymbol{w}_1 \cdot \nabla \boldsymbol{\varphi}, \boldsymbol{w}_2)\,, \text{ all } \boldsymbol{\varphi} \in \mathcal{D}_0^{1,2}(\Omega). \tag{32}$$

Suppose next $\boldsymbol{u}_k \to \boldsymbol{u}$ weakly in $X(\Omega)$; one has to show that $\mathcal{M}(\boldsymbol{u}_k) \to \mathcal{M}(\boldsymbol{u})$ weakly in $\mathcal{D}_0^{-1,2}(\Omega)$. This amounts to prove that

$$\lim_{k\to\infty} \langle \mathcal{M}(\boldsymbol{u}_k), \boldsymbol{\varphi} \rangle = \langle \mathcal{M}(\boldsymbol{u}), \boldsymbol{\varphi} \rangle\,, \quad \text{for all } \boldsymbol{\varphi} \in \mathcal{D}(\Omega). \tag{33}$$

In fact, on one hand, being $\mathcal{D}_0^{1,2}(\Omega)$ reflexive [47, Exercise II.6.2], the generic linear functional acting on $\boldsymbol{W} \in \mathcal{D}_0^{-1,2}(\Omega)$ is of the form $\langle \boldsymbol{W}, \boldsymbol{\varphi} \rangle$, for some $\boldsymbol{\varphi} \in \mathcal{D}_0^{1,2}(\Omega)$. On the other hand, it is

$$|\mathcal{M}(\boldsymbol{u}_k)|_{-1,2} \leq C_0\,,$$

$C_0 > 0$ independent of k, as is at once established from (26)–(27) and the uniform boundedness of $\|\boldsymbol{u}_k\|_X$. Now, to show (33) it is observed that the latter implies that there is $M_1 > 0$ independent of k, such that

$$|\boldsymbol{u}_k|_{1,2} \leq C\,.$$

Thus, along a subsequence $\{\boldsymbol{u}_{k'}\}$,

$$\lim_{k'\to\infty}(\nabla \boldsymbol{u}_{k'}, \nabla \boldsymbol{\varphi}) = (\nabla \boldsymbol{u}, \nabla \boldsymbol{\varphi}); \quad \lim_{k'\to\infty}(\partial_1 \boldsymbol{u}_{k'}, \boldsymbol{\varphi}) = (\partial_1 \boldsymbol{u}, \boldsymbol{\varphi});$$
$$\lim_{k'\to\infty}(\mathscr{R}(\boldsymbol{u}_{k'}), \boldsymbol{\varphi}) = (\mathscr{R}(\boldsymbol{u}), \boldsymbol{\varphi}), \quad \text{for all } \boldsymbol{\varphi} \in \mathscr{D}(\Omega). \tag{34}$$

Moreover, by the embedding $\mathcal{D}_0^{1,2}(\Omega) \subset W^{1,2}(\Omega_R)$, $R > 1$, Rellich compactness theorem, and Cantor diagonalization method one can also show

$$\lim_{k'\to\infty} \|\boldsymbol{u}_{k'} - \boldsymbol{u}\|_{4,\Omega_R} = 0, \quad \text{for all } R > 1, \tag{35}$$

see [47, Proposition 66] for details. The desired property (33) is then a simple consequence of (26), (27), (34), (35), and Hölder inequality. □

The following result also holds.

Lemma 4 *Let $\boldsymbol{u} \in X(\Omega)$. Then,*

$$\langle \partial_1 \boldsymbol{u}, \boldsymbol{u} \rangle = \langle \mathscr{R}(\boldsymbol{u}), \boldsymbol{u} \rangle = 0$$

Proof. If $\boldsymbol{u} \in \mathscr{D}(\Omega)$ the proof is trivial, being a consequence of simple integration by parts. However, if $\boldsymbol{u}$ is *just* in $X(\Omega)$ the claim is not obvious since it is not known whether $\mathscr{D}(\Omega)$ is dense in $X(\Omega)$. As a consequence, one has to argue in a different and more complicated way, especially to show the property for $\mathscr{R}$. The proof becomes then lengthy, technical and tricky. For this reason it will be omitted and the reader is referred to [43, pp. 12–13] for the first property and to [49, Proposition 70] for the second one. □

The above lemma is crucial for the next result –a particular case of that shown in [49, Proposition 78]– ensuring the validity of condition (iii) in Proposition 1.

Lemma 5 *The operator $\mathscr{L} := \mathscr{O} + \mathscr{K}$ is a linear homeomorphism of $X(\Omega)$ onto $\mathcal{D}_0^{-1,2}(\Omega)$. Moreover, there is a constant $C = C(\lambda, \mathcal{T}, \Omega)$ such that*

$$\|\boldsymbol{u}\|_X \leq C\, |\mathscr{L}(\boldsymbol{u})|_{-1,2}\,. \tag{36}$$

Proof. Referring to the cited reference for a full proof, here only the leading ideas will be sketched. As shown in [78, Theorem 2.1] and [49] the generalized Oseen operator $\mathscr{O}$ is a homeomorphism of $X(\Omega)$ onto $\mathcal{D}_0^{-1,2}(\Omega)$, and, moreover,

$$|\boldsymbol{u}|_{1,2} + |\partial_1 \boldsymbol{u}|_{-1,2} + |\mathscr{R}(\boldsymbol{u})|_{-1,2} \leq C\, |\mathscr{O}(\boldsymbol{u})|_{-1,2}\,.$$

Therefore, by classical results on Fredholm operators, it is enough to show that (i) $\mathscr{K}$ is compact, and (ii) $\mathsf{N}(\mathscr{L}) = \{0\}$. Let $\{\boldsymbol{u}_k\} \subset X(\Omega)$ be a bounded sequence and let Ω_R contain the support of $\boldsymbol{U}$. Observing that $X(\Omega) \subset W^{1,2}(\Omega_R)$, the Rellich compactness theorem implies that there is a subsequence of $\{\boldsymbol{u}_k\}$ that is Cauchy in $L^4(\Omega_R)$. Since by $(27)_1$ and Hölder inequality, for all $\boldsymbol{\varphi} \in \mathcal{D}_0^{1,2}(\Omega)$

$$|\langle \mathscr{K}(\boldsymbol{u}_{k'}), \boldsymbol{\varphi}\rangle - \langle \mathscr{K}(\boldsymbol{u}_{k''}), \boldsymbol{\varphi}\rangle| \leq 2\lambda\, \|\boldsymbol{U}\|_4 \|\boldsymbol{u}_{k'} - \boldsymbol{u}_{k''}\|_{4,\Omega_R} |\boldsymbol{\varphi}|_{1,2}\,,$$

from Lemma 1(i) one infers (along a subsequence)

$$\lim_{k',k''\to\infty} |\mathscr{K}(\boldsymbol{u}_{k'}) - \mathscr{K}(\boldsymbol{u}_{k''})|_{-1,2} = 0\,,$$

which proves (i). To show (ii), it must shown that

$$\langle \mathscr{O}(\boldsymbol{u}) + \mathscr{K}(\boldsymbol{u}), \boldsymbol{\varphi}\rangle = 0 \ \text{ for all } \boldsymbol{\varphi} \in \mathcal{D}_0^{1,2}(\Omega) \quad \Longrightarrow \quad \boldsymbol{u} = \boldsymbol{0}\,. \tag{37}$$

Since $\boldsymbol{U} \in W^{1,2}(\Omega)$ is of bounded support with $\operatorname{div} \mathbf{U} = \mathbf{0}$, and $\boldsymbol{u} \in \mathcal{D}_0^{-1,2}(\Omega)$, by an easily justified integration by parts we show $(\boldsymbol{U} \cdot \nabla \boldsymbol{u}, \boldsymbol{u}) = 0$. So that by replacing $\boldsymbol{u}$ for $\boldsymbol{\varphi}$ in (37) and using this property along with (26), $(27)_2$ and Lemma 4, one deduces

$$|\boldsymbol{u}|_{1,2}^2 - \lambda(\boldsymbol{u} \cdot \nabla \boldsymbol{U}, \boldsymbol{u}) = 0\,.$$

As a result, (37) is a consequence of the latter and of (10) in Lemma 1. □

The following lemma guarantees condition (iv) in Proposition 1; see [49, Propositions 79].

Lemma 6 *The Fréchet derivative, $\mathscr{N}'(\boldsymbol{u})$, of $\mathscr{N}$ is compact at each $\boldsymbol{u} \in X(\Omega)$.*

Proof. From $(27)_1$ it follows that

$$\lambda^{-1}[\mathscr{N}'(\boldsymbol{u})]\boldsymbol{w} = \mathcal{B}(\boldsymbol{u}, \boldsymbol{w}) + \mathcal{B}(\boldsymbol{w}, \boldsymbol{u})\,,$$

with $\mathcal{B}$ defined in (32). Let $\{\boldsymbol{v}_k\} \subset X(\Omega)$ be such that

$$\|\boldsymbol{v}_k\|_X \le C\,,$$

with C independent of $k \in \mathbb{N}$ and so, by Lemma 2, one gets, in particular

$$\|\boldsymbol{v}_k\|_4 + |\boldsymbol{v}_k|_{1,2} \le C_1\,, \tag{38}$$

with $C_1 = C_1(\Omega) > 0$. Since $X(\Omega)$ is reflexive, there exist $\boldsymbol{v} \in X(\Omega)$ and a subsequence $\{\boldsymbol{v}_{k'}\} \subset X(\Omega)$ converging weakly in $X(\Omega)$ to $\boldsymbol{v}$. As in the proof of Lemma 3 it can also be shown from (38) that (possibly, along another subsequence)

$$\lim_{k'} \|\boldsymbol{v}_k - \boldsymbol{v}\|_{4,\Omega_R} = 0, \ \text{ for all sufficiently large } R\,; \tag{39}$$

see also [49, Proposition 66]. From (32) and Hölder inequality one finds

$$\begin{aligned}|\langle \mathcal{B}(\boldsymbol{u}, \boldsymbol{v}_{k'}) - \mathcal{B}(\boldsymbol{u}, \boldsymbol{v}), \boldsymbol{\varphi}\rangle| &= |\langle \mathcal{B}(\boldsymbol{u}, \boldsymbol{v}_{k'} - \boldsymbol{v}), \boldsymbol{\varphi}\rangle| \\ &\le \left(\|\boldsymbol{u}\|_{4,\Omega_R}\|\boldsymbol{v} - \boldsymbol{v}_{k'}\|_{4,\Omega_R} + \|\boldsymbol{u}\|_{4,\Omega^R}\|\boldsymbol{v} - \boldsymbol{v}_{k'}\|_{4,\Omega^R}\right)|\boldsymbol{\varphi}|_{1,2}\,,\end{aligned}$$

for all sufficiently large R. Using (38) and (39) into this relation gives

$$\lim_{k' \to \infty} |\mathcal{B}(\boldsymbol{u}, \boldsymbol{v}_{k'}) - \mathcal{B}(\boldsymbol{u}, \boldsymbol{v})|_{-1,2} \le C_1\|\boldsymbol{u}\|_{4,\Omega^R}\,,$$

where $C_1 > 0$ is independent of k'. However, R is arbitrarily large and so, by the absolute continuity of the Lebesgue integral, it may be concluded that

$$\lim_{k' \to \infty} |\mathcal{B}(\boldsymbol{u}, \boldsymbol{v}_{k'}) - \mathcal{B}(\boldsymbol{u}, \boldsymbol{v})|_{-1,2} = 0\,. \tag{40}$$

In a completely analogous way, one shows

$$\lim_{k' \to \infty} |\mathcal{B}(\boldsymbol{v}_{k'}, \boldsymbol{u}) - \mathcal{B}(\boldsymbol{v}, \boldsymbol{u})|_{-1,2} = 0\,. \tag{41}$$

From (40) and (41) it then follows that the operator $\mathcal{B}(\boldsymbol{u}, \cdot)$, and hence $\mathscr{N}'(\boldsymbol{u})$, is compact at each $\boldsymbol{u} \in X(\Omega)$. □

In order to apply Proposition 1 to the operator $\mathcal{M}$, it remains to show condition (v), which amounts, basically, to find "good" *a priori* estimates for the equation $\mathcal{M}(\boldsymbol{u}) = \boldsymbol{F}$.

Lemma 7 *There is a constant $C > 0$ such that all solution $\boldsymbol{u} \in X(\Omega)$ to* (30) *satisfy*

$$\|\boldsymbol{u}\|_X \le C\,(|\boldsymbol{F}|_{-1,2} + |\boldsymbol{F}|^3_{-1,2})\,. \tag{42}$$

Proof. Also using (26), $(27)_2$, (10) and Lemma 4, one deduces

$$\langle \mathscr{L}(\boldsymbol{u}), \boldsymbol{u}\rangle = |\boldsymbol{u}|^2_{1,2} - \lambda(\boldsymbol{u}\cdot\nabla\boldsymbol{U}, \boldsymbol{u}) \ge \tfrac{1}{2}\,|\boldsymbol{u}|^2_{1,2}\,, \quad \langle \boldsymbol{F}, \boldsymbol{u}\rangle \le |\boldsymbol{F}|_{-1,2}|\boldsymbol{u}|_{1,2}. \tag{43}$$

Moreover, it is easily checked that for all $\boldsymbol{u} \in \mathscr{D}(\Omega)$,

$$(\boldsymbol{u}\cdot \operatorname{grad} \boldsymbol{u}, \boldsymbol{u}) = 0\,. \tag{44}$$

Now, by Lemma 4, $X(\Omega) \subset L^4(\Omega)$ and so, by [47, Theorem III.6.2], one can find a sequence $\{\boldsymbol{u}_k\} \subset \mathcal{D}(\Omega)$ converging to $\boldsymbol{u}$ in $\mathcal{D}^{1,2}_0(\Omega)\cap L^4(\Omega)$. Since, by Hölder inequality, the trilinear form $(\boldsymbol{u}\cdot \operatorname{grad}\boldsymbol{w}, \boldsymbol{v})$ is continuous in $L^4(\Omega)\times D^{1,2}(\Omega)\times L^4(\Omega)$ one may conclude that (44) continues to hold for all $\boldsymbol{u} \in X(\Omega)$, which gives

$$\langle \mathscr{N}(\boldsymbol{u}), \boldsymbol{u}\rangle = 0\,. \tag{45}$$

Thus, from this and (43) one obtains

$$|\boldsymbol{u}|_{1,2} \le 2|\boldsymbol{F}|_{-1,2}\,. \tag{46}$$

Since $\mathscr{L}(\boldsymbol{u}) = \boldsymbol{F} - \mathscr{N}(\boldsymbol{u})$, from Lemma 5 follows that

$$\|\boldsymbol{u}\|_X \le C\,(|\boldsymbol{F}|_{-1,2} + |\mathscr{N}(\boldsymbol{u})|_{-1,2})\,. \tag{47}$$

Moreover, by Lemma 2

$$\lambda^{-1}|\langle \mathscr{N}(\boldsymbol{u}), \boldsymbol{\varphi}\rangle| = |(\boldsymbol{u}\cdot\nabla\boldsymbol{\varphi}, \boldsymbol{u})| \le \|\boldsymbol{u}\|^2_4|\,|\boldsymbol{\varphi}|_{1,2} \le C_1\,|\partial_1\boldsymbol{u}|^{\frac{1}{2}}_{-1,2}|\boldsymbol{u}|^{\frac{3}{2}}_{1,2}\,|\boldsymbol{\varphi}|_{1,2}\,,$$

so that, by virtue of (46) and (47), one finds

$$\|\boldsymbol{u}\|_X \le C_2\,(|\boldsymbol{F}|_{-1,2} + |\boldsymbol{F}|^{\frac{3}{2}}_{-1,2}\|\boldsymbol{u}\|^{\frac{1}{2}}_X)\,.$$

Using Young's inequality in the latter allows one to deduce the validity of (42), and the proof of the lemma is completed. □

Proof of Theorem 1. The proof of the first statement follows from Proposition 1, and Lemmaa 3, and Lemmas 5–7. Furthermore, by property (11) of $\boldsymbol{U}$ and (28), one finds

$$|\boldsymbol{F}|_{-1,2} \le\ |\boldsymbol{f}|_{-1,2} + C_1\,\|\boldsymbol{v}_* + \mathsf{V}\|_{1/2,2(\partial\Omega)}\,, \tag{48}$$

where $C_1 = C_1(\lambda, \tau, \Omega, M)$ whenever $\|\boldsymbol{v}_*\|_{1/2,2(\partial\Omega)} \le M$. Estimate (24) is then a consequence of Lemma 7 and (48). □

Open Problem 1 *Property* (10) *of the extension* $\boldsymbol{U}$ *is fundamental for the estimate* (46)*. As mentioned earlier on,* (10) *is only known if the flux* Φ *is of "small" magnitude. While by a procedure similar to [25; 63; 50] it is probably possible to show that such a condition on* Φ *is also necessary for the existence of an extension with the above property, one may nevertheless wonder if a small* $|\Phi|$ *would indeed be necessary if the existence problem is approached by other methods. In this respect, by combining a contradiction argument of Leray with properties of the Bernoulli's function in spaces of low regularity, in their deep work [74] Korobkov, Pileckas & Russo have shown existence without restrictions on* $|\Phi|$*, at least for flow (and data) that are axisymmetric along the direction of* $\boldsymbol{\xi}$*. Whether such a result is true in general remains open.*

The following result shows an important property of weak solutions in the class $X(\Omega)$ and so, in particular, applies to those constructed in Theorem 1.

Theorem 2 *Let* $\boldsymbol{f} \in \mathcal{D}_0^{-1,2}(\Omega)$ *and* $\boldsymbol{v}_* \in W^{1/2,2}(\partial\Omega)$*, and let* $\boldsymbol{v}$ *be a corresponding weak solution with* $\boldsymbol{v} - \boldsymbol{U} \in X(\Omega)$*. Then,* $\boldsymbol{v}$ *satisfies the energy equality, namely,* (13) *with the equality sign. If, in addition,* $\boldsymbol{f} \in L^2(\Omega)$ *and* $\boldsymbol{v}_* \in W^{3/2,2}(\partial\Omega)$ *then the latter takes the form of the classical equation of energy balance:*

$$-2\|\boldsymbol{D}(\boldsymbol{v})\|_2^2 + \int_{\partial\Omega} \{(\boldsymbol{v}_* + \mathsf{V}) \cdot \boldsymbol{T}(\boldsymbol{v}, p) - \frac{\lambda}{2}(\boldsymbol{v}_* + \mathsf{V})^2 \boldsymbol{v}_*\} \cdot \boldsymbol{n} = \langle \boldsymbol{f}, \boldsymbol{v} \rangle \tag{49}$$

Proof. From (30), with $\boldsymbol{F}$ given in (28), one deduces

$$\mathscr{L}(\boldsymbol{u}), \boldsymbol{u}\rangle + \langle \mathscr{N}(\boldsymbol{u}), \boldsymbol{u}\rangle = \langle \boldsymbol{F}, \boldsymbol{u}\rangle \,.$$

Employing in this equation (43) and (45), one obtains

$$|\boldsymbol{u}|_{1,2}^2 - \lambda\,(\boldsymbol{u} \cdot \nabla \boldsymbol{U}, \boldsymbol{u}) - \langle \boldsymbol{F}, \boldsymbol{u}\rangle = 0\,,$$

which, recalling the definition of $\boldsymbol{F}$ in (28), shows that $\boldsymbol{u}$ obeys (13) with the equality sign. The second part of the theorem is shown exactly like in [47, pp. 770–771] and will be omitted. □

Open Problem 2 *The natural question arises whether* any *weak solution,* $\boldsymbol{v}$*, corresponding to data satisfying merely the "natural" minimal conditions of* Theorem 1*, is such that* $\boldsymbol{v} - \boldsymbol{U} \in X(\Omega)$*, and, in particular, obeys the equation of energy balance. In a remarkable work [59] Heck, Kim & Kozono have shown that this is indeed the case, at least when* $\mathcal{T} = 0$ *(the body is not spinning),* $\boldsymbol{v}_* \equiv \boldsymbol{0}$*, and* $\boldsymbol{f}$ *is assumed slightly more regular, namely,* $\boldsymbol{f} \in D_0^{-1,2}(\Omega)$*. Whether this result continues to hold for* $\mathcal{T} \neq 0$ *is not known.*

5 Regularity

It is expected that if the data $\boldsymbol{f}$, $\boldsymbol{v}_*$ and the boundary $\partial\Omega$ are sufficiently smooth, then the corresponding weak solution is smooth as well. In this respect, one has the following very general result about interior and boundary regularity.

Theorem 3 *Let $\boldsymbol{v}$ be a weak solution to* (8)–(7)*. Then, if*

$$\boldsymbol{f} \in W^{m,q}_{loc}(\Omega), \quad m \geq 0\,,$$

where $q \in (1,\infty)$ if $m = 0$, while $q \in [3/2,\infty)$ if $m > 0$, it follows that

$$\boldsymbol{v} \in W^{m+2,q}_{loc}(\Omega), \;\; p \in W^{m+1,q}_{loc}(\Omega),$$

where p is the pressure associated to $\boldsymbol{v}$ in Remark 1*. Thus, in particular, if*

$$\boldsymbol{f} \in C^\infty(\Omega), \tag{50}$$

then

$$\boldsymbol{v}, p \in C^\infty(\Omega). \tag{51}$$

Assume, further, Ω of class C^{m+2} and

$$\boldsymbol{v}_* \in W^{m+2-1/q,q}(\partial\Omega), \;\; \boldsymbol{f} \in W^{m,q}(\Omega_R)\,,$$

for some $R > 1$ and with the values of m and q specified earlier. Then,

$$\boldsymbol{v} \in W^{m+2,q}(\Omega_R), \;\; p \in W^{m+1,q}(\Omega_R).$$

Therefore, in particular, if Ω is of class C^∞ and

$$\boldsymbol{v}_* \in C^\infty(\partial\Omega), \;\; \boldsymbol{f} \in C^\infty(\overline{\Omega}_R), \tag{52}$$

it follows that

$$\boldsymbol{v},\; p \in C^\infty(\overline{\Omega}_R). \tag{53}$$

The proof of this result is rather complicated and the interested reader is referred to [47, Theorems X.1.1 and XI.1.2]. However, if one assumes (50) [respectively, (52) and Ω of class C^∞] then the proof of (51) [respectively, (53)] can be obtained by classical results for the Stokes problem in conjunction with a simple boot-strap argument and will be reproduced here.

To show this, one needs the following classical regularity results for weak solutions to the Stokes problem, a particular case of those furnished in [47, Theorems IV.4.1 and IV.5.1], to which the reader is also referred for their proofs.

Lemma 8 *Let $(\boldsymbol{w},\tau) \in W^{1,q}_{\rm loc}(\Omega) \times L^q_{\rm loc}(\Omega)$, $1 < q < \infty$, with $\operatorname{div}\boldsymbol{w} = 0$ in Ω, satisfy*

$$-(\nabla\boldsymbol{w}, \nabla\boldsymbol{\psi}) = [\mathsf{F}, \boldsymbol{\psi}] - (\tau, \operatorname{div}\boldsymbol{\psi})\,, \quad \textit{for all } \boldsymbol{\psi} \in C_0^\infty(\Omega) \tag{54}$$

Then, if $\mathsf{F} \in W^{m,q}_{\rm loc}(\Omega)$, $m \geq 0$, necessarily $(\boldsymbol{w},\tau) \in W^{m+2,q}_{\rm loc}(\Omega) \times W^{m+1,q}_{\rm loc}(\Omega)$. Moreover, assume $\boldsymbol{w} \in W^{1,q}(\Omega_R)$ for some $R > 1$, and $\boldsymbol{w} = \boldsymbol{w}_*$ at $\partial\Omega$. Then, if $\mathsf{F} \in W^{m,q}(\Omega_R)$, $\boldsymbol{w}_* \in W^{m+2-1/q,q}(\partial\Omega)$, necessarily $(\boldsymbol{w},\tau) \in W^{m+2,q}(\Omega_r) \times W^{m+1,q}_{\rm loc}(\Omega_r)$, for any $r \in (1,R)$.

With this result in hand, it can be proved that (50) implies (51). From Remark 1, the weak solution $\boldsymbol{v}$ and the associated pressure field p satisfy (54) with

$$\mathsf{F} := -\lambda\,\partial_1\boldsymbol{v} - \mathcal{T}\,(\boldsymbol{e}_1\times\boldsymbol{x}\cdot\nabla\boldsymbol{v} - \boldsymbol{e}_1\times\boldsymbol{v}) + \lambda\,\boldsymbol{v}\cdot\nabla\boldsymbol{v} + \boldsymbol{f}\,.$$

Then, by assumption, the embedding

$$\mathcal{D}_0^{1,2}(\Omega)\subset W^{1,2}_{\rm loc}(\Omega)\subset L^6_{\rm loc}(\Omega)\,,$$

and the Hölder inequality one has that $\mathsf{F}\in L^{3/2}_{\rm loc}(\Omega)$. From the first statement in Lemma 8, it can then be deduced $\boldsymbol{v}\in W^{2,3/2}(\Omega)_{\rm loc}$, $p\in W^{1,3/2}_{\rm loc}(\Omega)$, and, moreover, that $(\boldsymbol{v},p)$ satisfy $(8)_1$ a.e. in Ω. Next, because of the embedding $W^{2,3/2}_{\rm loc}(\Omega)\subset W^{1,3}_{\rm loc}(\Omega)\subset L^r_{\rm loc}(\Omega)$, arbitrary $r\in[1,\infty)$, one obtains the improved regularity property $\mathsf{F}\in W^{1,s}_{\rm loc}(\Omega)$, for all $s\in[1,3/2)$. Using once again Lemma 8, one infers $\boldsymbol{v}\in W^{3,s}_{\rm loc}(\Omega)$ and $p\in W^{2,s}_{\rm loc}(\Omega)$ which, in particular, gives further regularity for F. By induction, one then proves the desired property $\boldsymbol{v},p\in C^\infty(\Omega)$. The proof of the boundary regularity is performed by an entirely similar argument and, therefore, will be omitted.

6 Uniqueness

This section is dedicated to the investigation of the uniqueness property of weak solutions. Basically, the main known results depend on the summability and regularity assumptions made at the outset on the data $\boldsymbol{f}$ and $\boldsymbol{v}_*$. The following theorem shows, in particular, that every solution in Theorem 1 is unique in its own class of existence, provided the size of the data is sufficiently restricted.

Theorem 4 *Assume $\boldsymbol{v}_i$, $i=1,2$, are weak solutions with $\boldsymbol{v}_i-\boldsymbol{U}\in X(\Omega)$, corresponding to the same $\boldsymbol{f}\in\mathcal{D}_0^{-1,2}(\Omega)$, $\boldsymbol{v}_*\in W^{1/2,2}(\partial\Omega)$. Then, there is $C=C(\lambda,\mathcal{T},\Omega)$ such that if*

$$|\boldsymbol{f}|_{-1,2}+\|\boldsymbol{v}_*\|_{1/2,2(\partial\Omega)}<C\,, \tag{55}$$

necessarily $\boldsymbol{v}_1\equiv\boldsymbol{v}_2$.

Proof. Setting $\boldsymbol{u}_i=\boldsymbol{v}_i-\boldsymbol{U}$, $i=1,2$, with $\boldsymbol{U}$ given in Lemma 1, from (25)–(28) and the assumption one finds

$$\mathscr{L}(\boldsymbol{u}_i)+\mathscr{N}(\boldsymbol{u}_i)=\boldsymbol{F}\,,\ i=1,2\,. \tag{56}$$

Therefore, from Lemma 2, Lemma 7 and (48), one infers in particular

$$\|\boldsymbol{u}_i\|_4\le C_1(|\boldsymbol{f}|_{-1,2}+|\boldsymbol{f}|^3_{-1,2})+C_2(\|\boldsymbol{v}_*+\mathsf{V}\|_{1/2,2(\partial\Omega)}+\|\boldsymbol{v}_*+\mathsf{V}\|^3_{1/2,2(\partial\Omega)})\,, \tag{57}$$

where $C_1=C_1(\lambda,\mathcal{T},\Omega)$, and $C_2=C_2(\lambda,\mathcal{T},\Omega,M)$, whenever $\|\boldsymbol{v}_*\|_{1/2,2(\partial\Omega)}\le M$. Arbitrarily fix the number M once and for all. Setting $\boldsymbol{u}:=\boldsymbol{u}_1-\boldsymbol{u}_2$, from (56) one obtains

$$\mathscr{L}(\boldsymbol{u})=-\mathcal{B}(\boldsymbol{u},\boldsymbol{u}_1)-\mathcal{B}(\boldsymbol{u}_2,\boldsymbol{u})\,, \tag{58}$$

where $\mathcal{B}$ is defined in (32). Thus, observing that

$$|\mathcal{B}(\boldsymbol{u},\boldsymbol{u}_1)+\mathcal{B}(\boldsymbol{u}_2,\boldsymbol{u})|_{-1,2}\le\|\boldsymbol{u}\|_4(\|\boldsymbol{u}_1\|_4+\|\boldsymbol{u}_2\|_4)$$

from (58), Lemma 2 and Lemma 6 one has, in particular,

$$\|\boldsymbol{u}\|_4\,[1-C_3\,(\|\boldsymbol{u}_1\|_4+\|\boldsymbol{u}_2\|_4)]\le 0\,,$$

with $C_3=C_3(\lambda,\mathcal{T},\Omega)$. The result then follows from this inequality and (57). □

The natural question arises of whether the solutions constructed in Theorem 1 are unique in the class of weak solutions, that is, obeying *just* the requirements stated in Definition 1. The answer to this question is positive if $\boldsymbol{f}$ is assumed to possess "good" summability properties at large distances, and $\boldsymbol{v}$ is sufficiently regular. Actually, this property is a particular consequence of the following result for whose proof the reader is referred to [47, Theorem XI.5.3], once one takes into account that, by Sobolev inequality, $L^{\frac{6}{5}}(\Omega)\subset D_0^{-1,2}(\Omega)$, and that $D_0^{-1,2}(\Omega)\subset\mathcal{D}_0^{-1,2}(\Omega)$.

Theorem 5 *Let*

$$\boldsymbol{f}\in L^{6/5}(\Omega)\cap L^{4/3}(\Omega),\quad \boldsymbol{v}_*\in W^{5/4,4/3}(\partial\Omega)\,.$$

Then, there exists $C=C(\Omega,\lambda,\mathcal{T})$ such that, if

$$\|\boldsymbol{f}\|_{6/5}+\|\boldsymbol{v}_*+\mathsf{V}\|_{7/6,6/5(\partial\Omega)}<C\,, \tag{59}$$

$\boldsymbol{v}$ is the only weak solution corresponding to the above data.

Open Problem 3 *In general, it is not known whether solutions of* Theorem 1 *are unique in the class of weak solutions, when $\boldsymbol{f}$ and $\boldsymbol{v}_*$* merely *satisfy the assumptions of that theorem (and are sufficiently small).*

In connection with this problem, it is worth remarking that in the special case $\mathcal{T}=0$, $\boldsymbol{v}_*\equiv\boldsymbol{0}$, and with $\boldsymbol{f}$ slightly more regular (namely, $\boldsymbol{f}\in D_0^{-1,2}(\Omega)$) the result is shown in [59, Theorem 2.3].

7 Asymptotic Behavior

As shown in previous sections, some fundamental attributes of weak solution expected on physical grounds, such as verifying the energy balance and being unique for small data, can be established if one has enough information on their summability properties in a neighborhood of infinity, like the one provided by Theorem 1. However, there are other significant aspects that require a sharp *pointwise* knowledge of the solution at large distances, which *in principle* is not necessarily guaranteed *just* by the mild asymptotic information furnished in that theorem. These aspects include, for instance, the presence of a stationary, unbounded wake region "behind" the body, and a "fast" decay of the vorticity outside the wake region, in support of boundary layer theory. Proving (or disproving) these properties constituted one of the most challenging questions in mathematical fluid dynamics since the pioneering article of Leray.

The case $\mathcal{T} = 0$ was eventually settled in the mid seveties (about forty years after Leray's work) thanks to the effort of Robert Finn, Konstantin I. Babenko and their collaborators. Their contributions will be briefly summarized in the following two subsections. The case $\mathcal{T} \neq 0$ presents much more difficulties, and will be treated successively in Subsection 7.3, by means of a different approach, originally introduced in [41], that allows for a rather complete description of the pointwise asymptotic flow behavior also in that more general situation.

7.1 Finn's Contribution

In the late fifties/mid sixties, in a series of remarkable papers [26; 27; 28; 29; 30], Robert Finn proved the following fundamental results. Let $(\boldsymbol{v}, p)$ be any (sufficiently smooth) solution to (8)–(7) with $\mathcal{T} = 0$ and with $\boldsymbol{f}$ of bounded support, such that $|\boldsymbol{v}(x)| \leq C\,|x|^{-\alpha}$, some $\alpha > 1/2$ and all "large" $|x|$. Then the pointwise asymptotic structure of $(\boldsymbol{v}, p)$ can be sharply evaluated. In particular, combining the integral representation of the solution, obtained via the (time-independent) Oseen fundamental tensor, $\mathbb{E}$, along with a careful estimate of the latter, Finn showed that these solutions exhibit a paraboloidal "wake region", $\mathcal{R}$, with the property that the velocity field, $\boldsymbol{v}$, inside $\mathcal{R}$ decays pointwise slower than it does outside $\mathcal{R}$. More precisely, he proved that $\boldsymbol{v}$ [respectively, $\nabla\boldsymbol{v}$] admits an asymptotic expansion with $\mathbb{E}$ [respectively, $\nabla\mathbb{E}$] being the leading term. (Finn left open the question of the asymptotic behavior of the *second* derivatives of $\boldsymbol{v}$ [26], a problem that was finally solved another forty year later by Deuring [13].) Finn called such solutions "Physically Reasonable" (PR) [29, Definition 5.1], and demonstrated their existence on condition that the magnitude of the data is sufficiently restricted [29, Theorem 4.1]. Later on, one of his students, David Clark, showed that the vorticity field of any PR solution decays exponentially fast outside $\mathcal{R}$ and far from the body [10]. Thanks to its sharp asymptotic (and local regularity) properties, it is easy to show that any PR solution (regardless of the size of the data) is also weak, namely, $\boldsymbol{v} \in D^{1,2}(\Omega)$. However, given that the latter is the only information that weak solutions carry in a neighborhood of infinity, the converse property is by no means obvious, to the point that some author even questioned its validity [61, p. 12]. All this seemed to cast profound doubts about the physical relevance of Leray's weak solutions.

7.2 Babenko's Contribution

The relation between weak and PR solutions was eventually addressed by Babenko [2]. Combining Lizorkin's multipliers theory with anisotropic Sobolev-like inequalities and the representation formula for the solution employed by Finn, he was able to show that, if the boidy force $\boldsymbol{f}$ is of bounded support, every weak solution is, in fact, physically reasonable in the sense of Finn. Babenko's paper can be divided into two main parts. In the first one, he shows, by a very elegant and straightforward argument, that any weak solution, $\boldsymbol{v}$, corresponding to the given data must be in $L^4(\Omega)$, with corresponding pressure field in $L^2(\Omega)$. The second part of the paper is aimed to show that, actually, $\boldsymbol{v} \in L^q(\Omega)$, for *any* $q \in (2, 4]$. Once this property is established, then it is relatively simple to prove that the weak solution decays like $|x|^{-\alpha}$ for *some* $\alpha > 1/2$ and therefore is also PR. It must be noted that Babenko's proof of these further summability properties has aspects that are not fully transparent. Also for this reason, a distinct and more direct proof of Babenko's result was given later on by Galdi [37] and, successively and independently, by Farwig & Sohr [17].

7.3 A General Approach

It must be emphasized one more time that the results reported in the previous two subsections refer to the case $\mathcal{T} = 0$, that is, *the body is not spinning.* If one allows $\mathcal{T} \neq 0$, then the detailed study of the asymptotic properties of a weak solution becomes even more complicated, for several reasons. In the first place, the linear momentum equation $(8)_1$ contains a term that *grows linearly fast* at large spatial distances. As a consequence, the fundamental tensor of the linearized equations, $\mathbb{T}$, is no longer the classical Oseen tensor $\mathbb{E}$ mentioned above, let alone a "perturbation" of it, but, rather, a much more complicated one; see [23, Section 2]. Thus, the representation of the solution that in both contributions of Finn and Babenko plays a fundamental role in the determination of the pointwise behavior, becomes much more involved and, actually, useless for that matter. In fact, as shown in [23, Proposition 2.1], unlike $\mathbb{E}$, the tensor $\mathbb{T}$ does *not* satisfy uniform estimates at large spatial distances.

In view of these issues, in [41] Galdi introduced a completely different approach to the study of the asymptotic structure of a weak solution, that was further generalized and improved in [42; 45; 48]. In this approach, the weak solution $\boldsymbol{v}$ is viewed as limit as $t \to \infty$ along sequences of the (unique) solution, $\boldsymbol{w}(x, t)$, to a suitable *initial value problem.* It can be shown that, in turn, $\boldsymbol{w}$ admits a somewhat simple space-time representation in terms of the Oseen fundamental solution to the *time-dependent* Oseen equation. This fact allows one to obtain a number of sharp spatial estimates for $\boldsymbol{w}$ *uniformly in time*, which are thus preserved in the limit $t \to \infty$, and therefore continue to hold for the weak solution $\boldsymbol{v}$.

Referring to [47, §§X.6, X.8, XI.4, XI.6] for a full account of the (technically complicated and lengthy) proofs of all the above results, here it will only be provided an outline of the main steps of the procedure used in establishing them in the case $\mathcal{T} \neq 0$.

The **first step** consists in determining *sharp summability properties* of a weak solution in a neighborhood of infinity, under appropriate hypothesis on the data. To this end, one can show the following result [47, Theorem XI.6.4].

Lemma 9 *Assume, for some $q_0 > 3$ and all $q \in (1, q_0]$, that*

$$\boldsymbol{f} \in L^q(\Omega) \cap L^{3/2}(\Omega)\,, \quad \boldsymbol{v}_* \in W^{2-1/q_0,q_0}(\partial\Omega) \cap W^{4/3,3/2}(\partial\Omega)\,.$$

Then, every weak solution $\boldsymbol{v}$ to problem (8)–(7) corresponding to $\boldsymbol{f}$, $\boldsymbol{v}_*$, and the associated pressure field p (possibly modified by the addition of a constant, see also Remark 1*), satisfy the following summability properties*

$$\boldsymbol{v} \in L^r(\Omega) \cap D^{1,s}(\Omega)\,,\ \frac{\partial \boldsymbol{v}}{\partial x_1} \in L^t(\Omega)\,,\ p \in L^\sigma(\Omega)\,,$$

for all $r \in (2, \infty]$, $s \in (4/3, \infty]$, $t \in (1, \infty]$, $\sigma \in (3/2, \infty]$. If, in addition, $\boldsymbol{f} \in W^{1,q_0}(\Omega)$, $\boldsymbol{v}_* \in W^{3-1/q_0,q_0}(\partial\Omega)$, then we have also

$$\boldsymbol{v} \in D^{2,\tau}(\Omega)\,, \quad p \in D^{1,\tau}(\Omega)\,,$$

for all $\tau \in (1, \infty]$.

The next objective is to “translate” the above global asymptotic information into a pointwise one. For simplicity, it shall be assumed that $\boldsymbol{f}$ is of bounded support, which also implies, with the help of Theorem 3 that $(\boldsymbol{v}, p) \in C^\infty(\Omega^\rho)$ for sufficiently large ρ.

Thus, in the **second step**, one uses a standard “cut-off” procedure to rewrite (suitably) (8) in the whole space $\mathbb{R}^3$. More specifically, let ψ be a smooth function that is 0 in the neighborhood of $\partial\Omega$ that contains the support of $\boldsymbol{f}$, and 1 sufficiently far from it. Moreover, let $\boldsymbol{Z} \in C_0^\infty(\Omega)$ such that $\operatorname{div} \boldsymbol{Z} = -\nabla\psi \cdot \boldsymbol{v}$ in Ω. (Such a field $\boldsymbol{Z}$ exists, as shown in [47, Theorem III.3.3].) From (8) one can deduce that $\boldsymbol{u} := \psi\, \boldsymbol{v} - \boldsymbol{Z}$, and $\widetilde{p} := \psi\, p$ obey the following problem

$$\left.\begin{array}{r}\Delta \boldsymbol{u} + \lambda\, \partial_1 \boldsymbol{u} + \mathcal{T}(\boldsymbol{e}_1 \times \boldsymbol{x} \cdot \nabla \boldsymbol{u} - \boldsymbol{e}_1 \times \boldsymbol{u}) \qquad\qquad \\ = \lambda \operatorname{div}(\psi\, \boldsymbol{v} \otimes \psi\, \boldsymbol{v}) + \nabla \widetilde{p} + \boldsymbol{F}_{\rm c} \\ \operatorname{div} \boldsymbol{u} = 0 \qquad\qquad\qquad \end{array}\right\} \text{ in } \mathbb{R}^3 \tag{60}$$

where $\boldsymbol{F}_{\rm c}$ is smooth and of bounded support. At this point, the “classical” procedure would be to write the solution $\boldsymbol{u}$ in terms of the fundamental tensor solutions, $\mathbb{T}$, associated with problem (60). However, as remarked earlier on, this would not lead anywhere due to the poor properties of $\mathbb{T}$. Therefore, we argue differently.

In the **third step** one performs a *time-dependent* change of coordinates which transforms (60) into a suitable initial-value problem. To this end, for $t \geq 0$, let

$$\boldsymbol{Q}(t) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\mathcal{T}t) & -\sin(\mathcal{T}t) \\ 0 & \sin(\mathcal{T}t) & \cos(\mathcal{T}t) \end{bmatrix},$$

and define

$$\begin{aligned}
&\boldsymbol{y} := \boldsymbol{Q}(t)\cdot\boldsymbol{x}\,,\\
&\boldsymbol{w}(y,t) := \boldsymbol{Q}(t)\cdot\boldsymbol{u}(\boldsymbol{Q}^\top(t)\cdot\boldsymbol{y})\,,\quad \pi(y,t) := \widetilde{p}(\boldsymbol{Q}^\top(t)\cdot\boldsymbol{y})\,,\\
&\boldsymbol{V}(y,t) := \boldsymbol{Q}(t)\cdot[\psi_\rho\boldsymbol{v}](\boldsymbol{Q}^\top(t)\cdot\boldsymbol{y})\,,\quad \boldsymbol{H}(y,t) := \boldsymbol{Q}(t)\cdot\boldsymbol{F}_{\rm c}(\boldsymbol{Q}^\top(t)\cdot\boldsymbol{y})\,.
\end{aligned}$$

From (60) and Lemma 9 it then follows that

$$\left.\begin{array}{r}
\dfrac{\partial\boldsymbol{w}}{\partial t} = \Delta\boldsymbol{w} + \lambda\,\dfrac{\partial\boldsymbol{w}}{\partial y_1} - \nabla\pi - \lambda{\rm div}\,[\boldsymbol{V}\otimes\boldsymbol{V}] - \boldsymbol{H}\\[2mm]
\nabla\cdot\boldsymbol{w} = 0
\end{array}\right\}\ \text{in } \mathbb{R}^3\times(0,\infty)\,, \tag{61}$$

$$\lim_{t\to0^+}\|\boldsymbol{w}(\cdot,t)-\boldsymbol{u}\|_r = 0\,,\quad \text{all } r\in(2,\infty)\,.$$

Notice that equation $(61)_1$ does not contain the linearly growing term. The solution to the Cauchy problem (61) has the following representation:

$$\begin{aligned}
\boldsymbol{w}(y,t) = (4\pi t)^{-3/2}&\int_{\mathbb{R}^3} {\rm e}^{-|y-z+\lambda\,t\boldsymbol{e}_1|^2/4t}\boldsymbol{u}(z)\,dz\\
&-\int_0^t\int_{\mathbb{R}^3}\boldsymbol{\Gamma}(y-z,t-\tau)\cdot(\mathcal{R}\nabla\cdot[\boldsymbol{V}\otimes\boldsymbol{V}](z,\tau)+\boldsymbol{H}(z,\tau))\,dz\,d\tau\,.
\end{aligned} \tag{62}$$

where $\boldsymbol{\Gamma}(\xi,s)$, $(\xi,s)\in\mathbb{R}^3\times(0,\infty)$ is the well-known Oseen fundamental solution to the time-dependent Stokes system [47, §VIII.3].

In the **final step** one utilizes into (62) the summability properties for $\boldsymbol{u}$ and $\boldsymbol{V}$ obtained from Lemma 9 along with the classical pointwise estimates of $\boldsymbol{\Gamma}$, to produce a *pointwise* estimate for $\boldsymbol{w}(x,t)$, [respectively, $\nabla\boldsymbol{w}(x,t)$] *uniformly in t*. As a result, the latter can be shown to provide analogous bounds for $\boldsymbol{u}(x)$ [respectively, $\nabla\boldsymbol{u}(x)$], which means for the weak solution $\boldsymbol{v}(x)$ [respectively, $\nabla\boldsymbol{v}(x)$] for all "large" $|x|$.

Once the necessary asymptotic information on $\boldsymbol{v}$ is obtained, analogous estimates on the pressure field can be proved observing that from (8) it follows, for sufficiently large ρ, that

$$\begin{aligned}
&\Delta p = \nabla\cdot\boldsymbol{G}\ \text{ in } \Omega^\rho\,,\\
&\frac{\partial p}{\partial n} = g\ \text{ on } \partial\Omega^\rho\,,
\end{aligned}$$

where

$$\begin{aligned}
&\boldsymbol{G} := \lambda\,\boldsymbol{v}\cdot\nabla\boldsymbol{v}\,,\\
&g := [\Delta\boldsymbol{v} + \lambda(\partial_1\boldsymbol{v} - \boldsymbol{v}\cdot\nabla\boldsymbol{v}) + \mathcal{T}(\boldsymbol{e}_1\times\boldsymbol{x}\cdot\nabla\boldsymbol{v} - \boldsymbol{e}_1\times\boldsymbol{v})]\cdot\boldsymbol{n}\,|_{\partial\Omega^\rho}\ .
\end{aligned}$$

The procedure just outlined is at the basis of the following result whose full proof is found in [47, Theorems XI.6.1–XI.6.3].

Theorem 6 *Let $\boldsymbol{v}$ be a weak solution, corresponding to $\boldsymbol{f}$ of bounded support and let p be the corresponding pressure field associated to $\boldsymbol{v}$ by* Remark 1. *Then, for any $\delta,\eta>0$ and all sufficiently large $|x|$,*

$$\begin{aligned}
\boldsymbol{v}(x) &= O\Big(|x|^{-1}(1+\lambda\,s(x))^{-1} + |x|^{-3/2+\delta}\Big)\,,\\
\nabla\boldsymbol{v}(x) &= O\Big(|x|^{-3/2}(1+\lambda\,s(x))^{-3/2} + |x|^{-2+\eta}\Big)\,,\\
p(x) &= p_0 + O(|x|^{-2}\ln|x|)\,,\ \text{for some } p_0\in\mathbb{R}.
\end{aligned} \tag{63}$$

where $s(x) := |x| + x_1$.

Remark 3. This theorem suggests, in particular, that outside any semi-infinite cone, $\mathcal{C}$, whose axis coincides with the negative x_1 axis, the decay is faster than inside $\mathcal{C}$. This is the mathematical explanation of the existence of the wake "behind" the body, once one takes into account that the velocity of the center of mass of the body ($v_0\,\boldsymbol{e}$) is directed along the positive axis x_1 ($\lambda > 0$).

Remark 4. The fundamental tensor solution $\mathbb{E}(x,y) \equiv \{\mathbb{E}_{ij}(x,y)\}$ of the Oseen system (which is obtained by setting $\mathcal{T} = 0$ and disregarding the nonlinear term $\boldsymbol{v}\cdot\nabla\boldsymbol{v}$ in $(8)_1$) is defined through the relations

$$\mathbb{E}_{ij}(x,y) = \left(\delta_{ij}\Delta - \frac{\partial^2}{\partial y_i \partial y_j}\right)\Phi(x-y)\,, \quad \Phi(\xi) := \frac{1}{4\pi\lambda}\int_0^{\frac{\lambda}{2}(|\xi|+\xi_1)} \frac{1-\mathrm{e}^{-\tau}}{\tau}\,d\tau\,.$$

Now, the first terms on the right-hand side of $(63)_1$ and $(63)_2$, are just (sufficiently sharp) bounds for $\mathbb{E}$ and $\nabla\mathbb{E}$ at large $|x|$, respectively; see [47, Section VIII.3]. This is suggestive of the property that $\boldsymbol{a}\cdot\mathbb{E}$ and $\boldsymbol{a}\cdot\nabla\mathbb{E}$, for some suitable vector $\boldsymbol{a}$ could be the leading terms in corresponding asymptotic expansions. Actually, if $\mathcal{T} = 0$, this property is true, and one can show that, in such a case, the following formula holds for all sufficiently large $|x|$ [47, Theorem X.8.1]:

$$\boldsymbol{v}(x) = \boldsymbol{m}\cdot\mathbb{E}(x) + \boldsymbol{\mathcal{V}}(x) \tag{64}$$

where $\boldsymbol{m}$ is a constant vector coinciding with the total force, $\boldsymbol{F}$, exerted by the liquid on the body and

$$\boldsymbol{\mathcal{V}}(x) = O\Big(|x|^{-3/2+\delta}\Big)\,, \quad \text{arbitrary } \delta > 0.$$

Analogous estimate can be proven for $\nabla\boldsymbol{v}(x)$ [47, Theorem X.8.2].

If $\mathcal{T} \neq 0$, in [80, Theorem 1.1], Kyed has shown an asymptotic formula similar to (64) (and an analogous one for $\nabla\boldsymbol{v}$) where now $\boldsymbol{m} = (\boldsymbol{F}\cdot\boldsymbol{e}_1)\boldsymbol{e}_1$, and the quantity $\boldsymbol{\mathcal{V}}$ is a "higher order term" in the sense of Lebesgue integrability at large distances. A pointwise estimate (probably not optimal) for $\boldsymbol{\mathcal{V}}$ is shown in [79, Theorem 5.3.1]. (An even more detailed asymptotic structure was first shown in [23] for solutions to the *linearized* (Stokes) problem and in absence of translational motion.)

This section ends with some important considerations concerning the asymptotic behavior of the vorticity field, $\boldsymbol{\varpi} := \operatorname{curl}\boldsymbol{v}$, of a weak solution, $\boldsymbol{v}$. In this regard, one can prove the following theorem, due to Deuring & Galdi [14], that ensures that $\boldsymbol{\varpi}$ decays exponentially fast outside the wake region and sufficiently far from the body.

Theorem 7 *Under the same assumptions of* Theorem 6 *there are constants* $C, R > 0$ *such that*

$$|\boldsymbol{\varpi}(x)| \le C\,|x|^{-3/2}\mathrm{e}^{-(\lambda/4)\,(|x|+x_1)/(1+R)} \quad \textit{for all } x \in \Omega^R\,.$$

It is worth stressing the importance of this estimate that agrees with the necessary condition supporting the boundary layer assumption, namely, that sufficiently far from the body and the wake, the flow is "basically potential". As a matter of fact, in the case $\mathcal{T} = 0$ one can prove a sharper result that provides a more accurate description of the asymptotic structure of the vorticity field. Precisely, in that case, one has, for all sufficiently large $|x|$,

$$\boldsymbol{\varpi}(x) = \nabla\Phi \times \boldsymbol{m} + O(|x|^{-2}\,\mathrm{e}^{-\frac{\lambda}{2}(|x|+x_1)}) \tag{65}$$

where

$$\Phi(x) = -\frac{\lambda}{4\pi\,|x|}\,\mathrm{e}^{-\frac{\lambda}{2}(|x|+x_1)}$$

and $\boldsymbol{m}$ is a constant vector denoting the total force exerted by the liquid on the body [10; 3].

Open Problem 4 *In the case $\mathcal{T} \neq 0$, it is not known whether the vorticity admits, an asymptotic expansion of the type* (65)*, with an appropriate choice of the leading term.*

8 Geometric and Functional Properties for Large Data

Theorem 1 shows that, for *any* set of data $\mathsf{D} := (\lambda, \mathcal{T}, \boldsymbol{v}_*, \boldsymbol{f})$, in the specified spaces, there exists at least one corresponding weak solution $\boldsymbol{v}$ with the further property that $\boldsymbol{u} := \boldsymbol{v} - \boldsymbol{U} \in X(\Omega)$, for a suitable extension field $\boldsymbol{U}$. Also, Theorem 4 shows that this is, in fact, the only weak solution in that class, provided the data are suitably *restricted*, according to (55). Objective of this section is to analyze the geometric and functional properties of the *solution manifold* in the space $X(\Omega)$, corresponding to data of *arbitrary* magnitude in the class specified in Theorem 1. In order to make the presentation simpler, throughout this section it is set $\boldsymbol{v}_* \equiv \mathbf{0}$.

To reach this goal, one begins to rewrite equation (30) in an equivalent way that emphasizes the dependence of the operator involved on the parameter $\mathfrak{p} := (\lambda, \mathcal{T})$. One thus writes $\mathscr{L}(\mathfrak{p}, \boldsymbol{u})$ for $\mathscr{L}(\boldsymbol{u})$, $\mathscr{N}(\mathfrak{p}, \boldsymbol{u})$ for $\mathscr{N}(\boldsymbol{u})$ with $\mathscr{L}$, $\mathscr{N}$ defined in (25)–(27) and (29). Moreover, let $\mathscr{H} = \mathscr{H}(\mathfrak{p})$ denote the uniquely determined member of $\mathcal{D}_0^{-1,2}(\Omega)$ such that

$$\langle \mathscr{H}, \boldsymbol{\varphi}\rangle := (\nabla \boldsymbol{U}, \nabla\boldsymbol{\varphi}) - \lambda(\partial_1\boldsymbol{U} - \boldsymbol{U}\cdot\nabla\boldsymbol{U}, \boldsymbol{\varphi}) - \mathcal{T}\,(\mathscr{R}(\boldsymbol{U}), \boldsymbol{\varphi})\,, \quad \boldsymbol{\varphi} \in \mathcal{D}_0^{1,2}(\Omega). \tag{66}$$

Thus, for a given $\boldsymbol{f} \in \mathcal{D}_0^{-1,2}(\Omega)$, (30) can be written as

$$\mathscr{M}(\mathfrak{p}, \boldsymbol{u}) = \boldsymbol{f} \ \text{ in } \mathcal{D}_0^{-1,2}(\Omega) \tag{67}$$

where

$$\mathscr{M} : (\mathfrak{p}, \boldsymbol{u}) \in \mathbb{R}^2_+ \times X(\Omega) \mapsto \mathscr{L}(\mathfrak{p}, \boldsymbol{u}) + \mathscr{N}(\mathfrak{p}, \boldsymbol{u}) + \mathscr{H}(\mathfrak{p}) \in \mathcal{D}_0^{-1,2}(\Omega)\,. \tag{68}$$

The *solution manifold* associated to (68) is defined next:

$$\mathfrak{M}(\boldsymbol{f}) = \{(\mathfrak{p}, \boldsymbol{u}) \in \mathbb{R}^2_+ \times X(\Omega) \text{ satisfying (67)–(68) for a given } \boldsymbol{f} \in \mathcal{D}_0^{-1,2}(\Omega)\}$$

The main goal is then to address the following questions.

(a) Geometric structure of the manifold $\mathfrak{M} = \mathfrak{M}(\boldsymbol{f})$;

(b) Topological properties of the associated level set

$$\mathfrak{S}(\mathfrak{p}_0, \boldsymbol{f}) := \{(\mathfrak{p}, \boldsymbol{u}) \in \mathfrak{M}(\boldsymbol{f}),\ \mathfrak{p} = \mathfrak{p}_0\}\,,$$

obtained by fixing also Reynolds and Taylor numbers.

Clearly, "points" in $\mathfrak{S}(\mathfrak{p}_0, \boldsymbol{f})$ are solutions to equation (67) with a prescribed $\mathfrak{p} = \mathfrak{p}_0$, or, equivalently, to equation (30).

The following theorem collects the principal properties of the set $\mathfrak{S}(\mathfrak{p}_0, \boldsymbol{f})$.

Theorem 8 *The following properties hold.*

(i) $\mathfrak{S}(\mathfrak{p}_0, \boldsymbol{f})$ *is not empty;*

(ii) *For any* $(\mathfrak{p}_0, \boldsymbol{f}) \in \mathbb{R}^2_+ \times \mathcal{D}_0^{-1,2}(\Omega)$, $\mathfrak{S}(\mathfrak{p}_0, \boldsymbol{f})$ *is compact. Moreover, there is* $N = N(\mathfrak{p}_0, \boldsymbol{f}) \in \mathbb{N}$ *such that* $\mathfrak{S}(\mathfrak{p}_0, \boldsymbol{f})$ *is homeomorphic to a compact set of* $\mathbb{R}^N$*;*

(iii) *For any* $\mathfrak{p}_0 \in \mathbb{R}^2_+$ *there is an open residual set* $\mathcal{O} = \mathcal{O}(\mathfrak{p}_0) \subset \mathcal{D}_0^{-1,2}(\Omega)$ *such that, for every* $\boldsymbol{f} \in \mathcal{O}$, $\mathfrak{S}(\mathfrak{p}_0, \boldsymbol{f})$ *is constituted by a number of points,* $\kappa = \kappa(\mathfrak{p}_0 \boldsymbol{f})$, *that is finite and odd;*

(iv) *The number* κ *is constant on every connected component of* $\mathcal{O}$.

Proof. As usual, only a sketch of some of the proofs of the above statements will be given, while referring to the appropriate reference for whatever missing. The statement (i) is a consequence of Theorem 1. The proofs of the other statements are based on two fundamental properties of the operator $\mathcal{M} := \mathscr{L}(\mathfrak{p}_0, \cdot) + \mathscr{N}(\mathfrak{p}_0, \cdot)$, namely, being (1) proper, and (2) Fredholm of index 0. Now, Lemma 5 and Lemma 6 guarantee that the Fréchet derivative of $\mathcal{M}$ at every $\boldsymbol{u} \in X(\Omega)$ is a compact perturbation of a homeomorphism, which proves the Fredholm property. Properness means that if $\boldsymbol{F}$ ranges in a compact set, K, of $\mathcal{D}_0^{-1,2}(\Omega)$, all possible corresponding solutions $\boldsymbol{u}$ to $\mathcal{M}(\boldsymbol{u}) = \boldsymbol{F}$ belong to a compact set, K^*, of $X(\Omega)$. To show that this is indeed the case, one observes that in view of the continuity of $\mathcal{M}$, K^* is closed, so that it is enough to show that from any sequence $\{\boldsymbol{u}_n\} \subset K^*$, there is a subsequence (still denoted by $\{\boldsymbol{u}_n\}$) and $\boldsymbol{u} \in X(\Omega)$ such that $\boldsymbol{u}_n \to \boldsymbol{u}$ in $X(\Omega)$. Let $\boldsymbol{F}_n = \mathcal{M}(\boldsymbol{u}_n)$. Since $\{\boldsymbol{F}_n\} \subset K$, one deduces (along a subsequence)

$$\boldsymbol{F}_n \to \boldsymbol{F} \ \text{ in } \mathcal{D}_0^{-1,2}(\Omega)\,, \quad \text{for some } \boldsymbol{F} \in K. \tag{69}$$

Moreover, being $\{\boldsymbol{F}_n\}$ bounded, by Lemma 7 it follows that $\{\boldsymbol{u}_n\}$ is bounded, and so there exists $\boldsymbol{u} \in X$ such that $\boldsymbol{u}_n \to \boldsymbol{u}$ weakly in $X(\Omega)$. By Lemma 3 and (69), the latter implies

$$\mathcal{M}(\boldsymbol{u}) = \boldsymbol{F}\,, \quad \mathcal{M}(\boldsymbol{u}_n) \to \mathcal{M}(\boldsymbol{u}) \ \text{ in } \mathcal{D}_0^{-1,2}(\Omega)\,. \tag{70}$$

One next observes that

$$\mathcal{M}(\boldsymbol{u}_n) - \mathcal{M}(\boldsymbol{u}) = \mathscr{L}(\boldsymbol{u}_n - \boldsymbol{u}) + \mathscr{N}(\boldsymbol{u}_n) - \mathscr{N}(\boldsymbol{u}) \tag{71}$$

and also, since $\mathscr{N}$ is quadratic (Lemma 3),

$$\begin{aligned}\mathscr{N}(\boldsymbol{u}_n)-\mathscr{N}(\boldsymbol{u}) &\equiv \mathcal{B}(\boldsymbol{u}_n,\boldsymbol{u}_n)-\mathcal{B}(\boldsymbol{u},\boldsymbol{u})\\ &=\mathcal{B}(\boldsymbol{u}_n-\boldsymbol{u},\boldsymbol{u})+\mathcal{B}(\boldsymbol{u},\boldsymbol{u}_n-\boldsymbol{u})+\mathcal{B}(\boldsymbol{u}_n-\boldsymbol{u},\boldsymbol{u}_n-\boldsymbol{u})\\ &\equiv [\mathscr{N}'(\boldsymbol{u})](\boldsymbol{u}_n-\boldsymbol{u})+\mathscr{N}(\boldsymbol{u}_n-\boldsymbol{u})\,.\end{aligned}$$

Replacing this identity in (71) one finds

$$\mathcal{M}(\boldsymbol{u}_n)-\mathcal{M}(\boldsymbol{u}))-[\mathscr{N}'(\boldsymbol{u})](\boldsymbol{u}_n-\boldsymbol{u})=\mathcal{M}(\boldsymbol{u}_n-\boldsymbol{u})\,. \tag{72}$$

In view of (70), the first term on the left-hand side of (72) tends to 0 as $n\to\infty$. Likewise, since $\mathscr{N}'(\boldsymbol{u})$ is compact, and hence completely continuous, also the second term on the left-hand side of (72) tends to 0 as $n\to\infty$, so that properness follows from this and Lemma 7. The first property in (ii) is then a corollary of what just proven. As for the second one, we refer to [49, Theorem 93] for a proof. We next come to show statements (iii) and (iv). In this regard, since $\mathcal{M}$ is proper and Fredholm of index 0, by the mod 2 degree of Smale [105] it is enough to show that there exists $\boldsymbol{F}_0\in\mathcal{D}_0^{1,2}(\Omega)$ with the following properties: (a) the equation $\mathcal{M}(\boldsymbol{u})=\boldsymbol{F}_0$ has one and only one solution, $\boldsymbol{u}_0$, and (b) $\mathsf{N}(\mathcal{M}'(\boldsymbol{u}_0))=\{\boldsymbol{0}\}$; see [43, Lemma 6.1]. Now, set $\boldsymbol{F}_0=\boldsymbol{0}$. From Lemma 7 it follows that the only solution to $\mathcal{M}(\boldsymbol{u})=\boldsymbol{0}$ is $\boldsymbol{u}_0=\boldsymbol{0}$. Moreover, by Lemma 3, one finds $\mathcal{N}'(\boldsymbol{0})\equiv\boldsymbol{0}$, so that $\mathcal{M}'(\boldsymbol{0})=\mathscr{L}$ and condition (b) is a consequence of Lemma 5. □

Remark 5. Taking into account that the set $\mathcal{O}$ in Theorem 8 is dense in $\mathcal{D}_0^{-1,2}(\Omega)$, from Theorem 8(iii) one deduces the following interesting property of weak solutions. *Let $\lambda\neq 0$ and $\mathcal{T}\geq 0$ be arbitrarily fixed. Given $\boldsymbol{f}\in\mathcal{D}_0^{-1,2}(\Omega)$ and $\varepsilon>0$, there is $\boldsymbol{g}\in\mathcal{D}_0^{-1,2}(\Omega)$ with $|\boldsymbol{f}-\boldsymbol{g}|_{-1,2}<\varepsilon$ such that the number of weak solutions given in* Theorem 1 *corresponding to the body force $\boldsymbol{g}$ (and $\boldsymbol{v}_*\equiv\boldsymbol{0}$) is finite and odd.*

The next result furnishes a complete generic characterization of the manifold $\mathfrak{M}(\boldsymbol{f})$. Its proof, based on an infinite-dimensional version of the so called "parametrized Sard theorem" [109, Theorem 4.L], is technically involved and lengthy. The interested reader is referred to [49, Theorem 88].

Theorem 9 *The following properties hold.*

(i) *There exists a dense, residual set $\mathcal{Z}\subseteq\mathcal{D}_0^{-1,2}(\Omega)$ such that, for any $\boldsymbol{f}\in\mathcal{Z}$ the solution manifold $\mathfrak{M}(\boldsymbol{f})$ is a 2-dimensional (not necessarily connected) manifold of class C^∞;*

(ii) *For any $\boldsymbol{f}\in\mathcal{Z}$ there exists an open, dense set $\mathfrak{P}=\mathfrak{P}(\boldsymbol{f})\subset\mathbb{R}^2_+$ such that, for each $\mathfrak{p}\in\mathfrak{P}$, equation (67) has a finite number of solutions, $n=n(\mathfrak{p},\boldsymbol{f})$;*

(iii) *The integer $n=n(\mathfrak{p},\boldsymbol{f})$ is independent of $\mathfrak{p}$ on every interval contained in $\mathfrak{P}$.*

Open Problem 5 *It is not known whether, in the physically significant case of vanishing body force $\boldsymbol{f}$ and boundary velocity $\boldsymbol{v}_*$, the number of corresponding steady-state solutions is generically finite.*

9 Bifurcation

As pointed out in the introductory section, if the speed of the center of mass of the body, v_0, reaches a critical value, it is experimentally observed, already in absence of rotation, that the characteristic features of the original steady-state flow of the liquid may change dramatically. The outcome could be either the onset of an entirely different steady-state flow, or even of a time-periodic regime. Objective of this section is to provide necessary conditions and sufficient conditions for the occurrence of this phenomenon. More precisely, Subsection 9.1 will be concerned with time-independent problems, while Subsection 9.2 will deal with the time-periodic case. For the sake of simplicity, it will be assumed throughout $\boldsymbol{v}_* \equiv \boldsymbol{0}$.

9.1 Steady Bifurcation

One is mainly interested in situations where bifurcation is generated by the "combined" action of translation and rotation of the body (provided the latter is not zero). To this end, it is convenient to use a different non-dimensionalization for the equations (8)–(7), in order to introduce an appropriate bifurcation parameter. Precisely rescaling velocity with v_0 and length with v_0/ω, equations (8)–(7) become

$$\left.\begin{array}{l}\Delta \boldsymbol{v} + \lambda\left(\partial_1 \boldsymbol{v} + \boldsymbol{e}_1 \times \boldsymbol{x}\cdot\nabla \boldsymbol{v} - \boldsymbol{e}_1 \times \boldsymbol{v} - \boldsymbol{v}\cdot\nabla \boldsymbol{v}\right) = \nabla p + \boldsymbol{f} \\ \operatorname{div} \boldsymbol{v} = 0\end{array}\right\} \text{ in } \Omega$$
$$\boldsymbol{v} = \boldsymbol{e}_1 + \boldsymbol{e}_1 \times \boldsymbol{x} \text{ at } \partial\Omega\,; \quad \lim_{|x|\to\infty} \boldsymbol{v}(x) = \boldsymbol{0}\,, \tag{73}$$

where now $\lambda := v_0^2(\boldsymbol{e}\cdot\boldsymbol{e}_1)/(\nu\omega)$.

Remark 6. Of course, the above non-dimensionalization requires $\omega \neq 0$. However, for future reference it is important to emphasize that all main results presented in this section continue to hold *in exactly the same form* also when $\omega = 0$.

With the notation introduced in the previous section (see (67) and (68) with $\mathfrak{p} \equiv \lambda$), the original equation (30) is equivalent to the following nonlinear equation

$$\mathscr{M}(\lambda, \boldsymbol{u}) := \mathscr{L}(\lambda, \boldsymbol{u}) + \mathscr{N}(\lambda, \boldsymbol{u}) + \mathscr{H}(\lambda) = \boldsymbol{f} \ \text{ in } \mathcal{D}_0^{-1,2}(\Omega),\ \boldsymbol{u} \in X(\Omega) \tag{74}$$

Definition 2. Let $\boldsymbol{u}_0 \in X(\Omega)$ be a solution to (74) with $\lambda = \lambda_0$. The pair $(\lambda_0, \boldsymbol{u}_0)$ is called a *steady bifurcation point* for (74), if there are two sequences $\{\lambda_k, \boldsymbol{u}_k^{(1)}\}$ and $\{\lambda_k, \boldsymbol{u}_k^{(2)}\}$ with the following properties

(i) $\{\lambda_k, \boldsymbol{u}_k^{(i)}\}$, $i = 1, 2$ solve (74) for all $k \in \mathbb{N}$;
(ii) $\{\lambda_k, \boldsymbol{u}_k^{(i)}\} \to (\lambda_0, \boldsymbol{u}_0)$ in $\mathbb{R} \times X(\Omega)$ as $k \to \infty$, $i = 1, 2$;
(iii) $\boldsymbol{u}_k^{(1)} \not\equiv \boldsymbol{u}_k^{(2)}$, for all $k \in \mathbb{N}$.

One of the main achievements of this section is the proof that, under certain conditions that may be satisfied in problems of physical interest, bifurcation is reduced to the study of a suitable linear eigenvalue problem, *formally analogous* to that occurring in the study of bifurcation for flow in a *bounded* domain; see Theorem 11 and Remark 8.

A necessary condition in order for $(\lambda_0, \boldsymbol{u}_0)$ to be a bifurcation point, is obtained as a corollary to the following result.

Lemma 10 *Let* $\boldsymbol{u}_0 \in X(\Omega)$ *be a solution to* (74) *with* $\lambda = \lambda_0$ *and fixed* $\boldsymbol{f} \in \mathcal{D}_0^{-1,2}(\Omega)$. *If*

$$\mathsf{N}(\mathscr{M}'(\lambda_0, \boldsymbol{u}_0)) = \{\boldsymbol{0}\}\,, \tag{75}$$

namely, the (linear) equation

$$\mathscr{L}(\lambda_0, \boldsymbol{w}) + \lambda_0[\mathcal{B}(\boldsymbol{u}_0, \boldsymbol{w}) + \mathcal{B}(\boldsymbol{w}, \boldsymbol{u}_0)] = \boldsymbol{0} \quad \text{in } \mathcal{D}_0^{-1,2}(\Omega) \tag{76}$$

has only the solution $\boldsymbol{w} = \boldsymbol{0}$ *in* $X(\Omega)$, *then there exists a neighborhood* $U(\lambda_0)$, *such that for each* $\lambda \in U(\lambda_0)$ *there is one and only one* $\boldsymbol{u}(\lambda)$ *solution to* (74). *Moreover, the map* $\lambda \in U \to \boldsymbol{u}(\lambda) \in X(\Omega)$ *is analytic at* $\lambda = \lambda_0$.

(The prime means Fréchet differentiation with respect to $\boldsymbol{u}$.)

Proof. Consider the map

$$F : (\lambda, \boldsymbol{u}) \in U(\lambda_0) \times X(\Omega) \mapsto \mathscr{M}(\lambda, \boldsymbol{u}) - \boldsymbol{f}\,.$$

Also using the fact that $\mathcal{N}(\lambda, \cdot)$ is quadratic (see (31)–(32)), it easily follows that F is analytic (polynomial, in fact) at each $(\lambda, \boldsymbol{u})$. Moreover, by assumption, $F(\lambda_0, \boldsymbol{u}_0) = \boldsymbol{0}$. Thus, the claimed property will follow from the analytic version of the implicit function theorem provided one shows that $F'(\lambda_0, \boldsymbol{u}_0)$ is a bijection. Now, from (31)–(32),

$$F'(\lambda_0, \boldsymbol{u}_0) = \mathscr{M}'(\lambda_0, \boldsymbol{u}_0) \equiv \mathscr{L}(\lambda_0, \cdot) + \lambda_0\, [\mathcal{B}(\boldsymbol{u}_0, \cdot) + \mathcal{B}(\cdot, \boldsymbol{u}_0)]\,,$$

so that by Lemma 5 and Lemma 6 we infer that $F'(\lambda_0, \boldsymbol{u}_0)$ is Fredholm of index 0, and the bijectivity property follows from the assumption (75). □

From this result the following one follows at once.

Corollary 1. *Necessary condition for* $(\lambda_0, \boldsymbol{u}_0)$ *to be a bifurcation point is that*

$$\dim \mathsf{N}(\mathscr{M}'(\lambda_0, \boldsymbol{u}_0)) > 0\,, \tag{77}$$

namely, the (linear) equation

$$\mathscr{L}(\lambda_0, \boldsymbol{w}) + \lambda_0[\mathcal{B}(\boldsymbol{u}_0, \boldsymbol{w}) + \mathcal{B}(\boldsymbol{w}, \boldsymbol{u}_0)] = \boldsymbol{0} \quad \text{in } \mathcal{D}_0^{-1,2}(\Omega) \tag{78}$$

has a non-zero solution $\boldsymbol{w} \in X(\Omega)$.

Remark 7. One can show that (77) is equivalent to the requirement that the linearization of (73) around $(\lambda_0, \boldsymbol{v}_0 \equiv \boldsymbol{u}_0 + \boldsymbol{U})$ corresponding to homogeneous data, namely,

$$\left.\begin{array}{l} \Delta \boldsymbol{w} + \lambda_0\,(\partial_1 \boldsymbol{w} + \boldsymbol{e}_1 \times \boldsymbol{x} \cdot \nabla \boldsymbol{w} - \boldsymbol{e}_1 \times \boldsymbol{w} - \boldsymbol{v}_0 \cdot \nabla \boldsymbol{w} - \boldsymbol{w} \cdot \nabla \boldsymbol{v}_0) = \nabla p \\ \operatorname{div} \boldsymbol{w} = 0 \end{array}\right\} \text{ in } \Omega$$
$$\boldsymbol{w} = \boldsymbol{0} \text{ at } \partial\Omega\,; \quad \lim_{|x|\to\infty} \boldsymbol{w}(x) = \boldsymbol{0}\,, \tag{79}$$

has a non-trivial solution $(\boldsymbol{w},p)\in[D^{2,2}(\Omega)\cap X(\Omega)]\times D^{1,2}(\Omega)$. In fact, saying that (78) has a nonzero solution $\boldsymbol{w}\in X(\Omega)$ means that there exists $\boldsymbol{w}\in X(\Omega)-\{\boldsymbol{0}\}$ such that (see (25)–(30) with $\lambda=\mathcal{T}$))

$$-(\nabla\boldsymbol{w},\nabla\boldsymbol{\varphi})+\lambda_0[\langle\partial_1\boldsymbol{w}+\mathscr{R}(\boldsymbol{w}),\boldsymbol{\varphi}\rangle+(\boldsymbol{w}\cdot\nabla\boldsymbol{v}_0+\boldsymbol{v}_0\cdot\nabla\boldsymbol{w},\boldsymbol{\varphi})]=0\,, \tag{80}$$

for all $\boldsymbol{\varphi}\in\mathcal{D}^{1,2}_0(\Omega)$. However, by the properties of $\boldsymbol{v}_0$ and $\boldsymbol{w}$ combined with the Hölder inequality one shows $\boldsymbol{G}:=(\boldsymbol{w}\cdot\nabla\boldsymbol{v}_0+\boldsymbol{v}_0\cdot\nabla\boldsymbol{w})\in L^{4/3}(\Omega)$, which, in turn, by classical results on the generalized Oseen equation [47, Theorem VIII.8.1] furnishes, in particular, $\boldsymbol{w}\in D^{2,4/3}(\Omega)$. By embedding, the latter implies $\boldsymbol{w}\in D^{1,12/5}(\Omega)\cap L^{12}(\Omega)$, so that $\boldsymbol{G}\in L^{12/7}(\Omega)$ which, again by [47, Theorem VIII.8.1], delivers $\boldsymbol{w}\in D^{2,12/7}(\Omega)\cap D^{1,4}(\Omega)\cap L^{\infty}(\Omega)$. Thus, $\boldsymbol{G}\in L^2(\Omega)$ and the property follows by another application of [47, Theorem VIII.8.1]. Notice that the asymptotic condition in (79) is achieved uniformly pointwise.

Next objective is to provide *sufficient* conditions for $(\lambda_0,\boldsymbol{u}_0)$ to be a bifurcation point. To this end, it will be assumed that, in the neighborhood of $(\lambda_0,\boldsymbol{u}_0)$ there exists a sufficiently smooth solution curve, that is, there is a map $\lambda\in U(\lambda_0)\mapsto\overline{\boldsymbol{u}}(\lambda)\in X(\Omega)$ of class C^2 (say), with $\overline{\boldsymbol{u}}(\lambda_0)=\boldsymbol{u}_0$ and satisfying (74) for the given $\boldsymbol{f}$. Setting $\boldsymbol{w}:=\boldsymbol{u}-\overline{\boldsymbol{u}}$ one thus gets that $\boldsymbol{w}$ satisfies the equation

$$\mathscr{F}(\lambda,\boldsymbol{w}):=\mathscr{L}(\lambda,\boldsymbol{w})+\lambda[\mathcal{B}(\boldsymbol{w},\overline{\boldsymbol{u}}(\lambda))+\mathcal{B}(\overline{\boldsymbol{u}}(\lambda),\boldsymbol{w})]+\mathscr{N}(\lambda,\boldsymbol{w})=\boldsymbol{0}\,. \tag{81}$$

Clearly, $(\lambda_0,\boldsymbol{u}_0)$ is a bifurcation point for (74) if and only if $(\lambda_0,\boldsymbol{0})$ is a bifurcation point for (81). Since, as showed earlier on, $\mathscr{F}'(\lambda_0,\boldsymbol{0})\equiv\mathscr{L}(\lambda_0,\cdot)+\lambda_0[\mathcal{B}(\cdot,\boldsymbol{u}_0(\lambda))+\mathcal{B}(\boldsymbol{u}_0,\cdot)]$ is Fredholm of index 0, a classical result [109, Theorem 8.A] ensures that $(\lambda_0,\boldsymbol{0})$ is a bifurcation point provided the following conditions hold:

(i) $\dim\mathsf{N}\,(\mathscr{F}'(\lambda_0,\boldsymbol{0}))=1\,;$

(ii) $[\mathscr{F}_{\lambda\boldsymbol{w}}(\lambda_0,\boldsymbol{0})](\boldsymbol{w}_1)\notin\mathsf{R}\,(\mathscr{F}'(\lambda_0,\boldsymbol{0}))\,,\quad\boldsymbol{w}_1\in\mathsf{N}\,(\mathscr{F}'(\lambda_0,\boldsymbol{0}))\,,$

where the double subscript denotes differentiation with respect to the indicated variable. Condition (i) specifies in which sense the requirement of Corollary 1 must be met. In order to give a more explicit form to condition (ii), it is convenient to introduce the *Stokes operator*:

$$\widetilde{\Delta}:\boldsymbol{u}\in\mathcal{D}^{1,2}_0(\Omega)\mapsto\widetilde{\Delta}\boldsymbol{u}\in\mathcal{D}^{-1,2}_0(\Omega)\,, \tag{82}$$

with

$$\langle\widetilde{\Delta}\boldsymbol{u},\boldsymbol{\varphi}\rangle=-(\nabla\boldsymbol{u},\nabla\boldsymbol{\varphi})\,,\quad\boldsymbol{\varphi}\in\mathcal{D}^{1,2}_0(\Omega)\,. \tag{83}$$

As is well known, $\widetilde{\Delta}$ is a homeomorphism [47, Theorem V.2.1]. By a straightforward computation one then shows that

$$[\mathscr{F}_{\lambda\boldsymbol{w}}(\lambda_0,\boldsymbol{0})](\boldsymbol{w}_1)=-\frac{1}{\lambda_0}\widetilde{\Delta}\boldsymbol{w}_1+\lambda_0\,[\mathcal{B}(\boldsymbol{w}_1,\dot{\overline{\boldsymbol{u}}}(\lambda_0))+B(\dot{\overline{\boldsymbol{u}}}(\lambda_0),\boldsymbol{w}_1)]\,,$$

(with “ $\cdot$ ” denoting differentiation with respect to λ) and therefore condition (ii) is equivalent to the request that the equation

$$\begin{aligned}&\mathscr{L}(\lambda_0,\boldsymbol{w})+\lambda_0[\mathcal{B}(\boldsymbol{u}_0,\boldsymbol{w})+\mathcal{B}(\boldsymbol{w},\boldsymbol{u}_0)]\\&\qquad=-\frac{1}{\lambda_0}\widetilde{\Delta}\boldsymbol{w}_1+\lambda_0\,[\mathcal{B}(\boldsymbol{w}_1,\dot{\overline{\boldsymbol{u}}}(\lambda_0))+B(\dot{\overline{\boldsymbol{u}}}(\lambda_0),\boldsymbol{w}_1)]\,,\end{aligned} \tag{84}$$

has no solution. All the above is summarized in the following.

Theorem 10 *Suppose the solution set of the equation*

$$\mathscr{L}(\lambda_0, \boldsymbol{w}) + \lambda_0 \left[\mathcal{B}(\boldsymbol{u}_0, \boldsymbol{w}) + \mathcal{B}(\boldsymbol{w}, \boldsymbol{u}_0)\right] = \boldsymbol{0} \tag{85}$$

is a one-dimensional subspace of $X(\Omega)$ and let $\boldsymbol{w}_1$ be a corresponding normalized element. If, in addition, equation (84) *has no solution $\boldsymbol{w} \in X(\Omega)$, then $(\lambda_0, \boldsymbol{u}_0)$ is a bifurcation point for* (74).

The assumptions of the result just proven admits a noteworthy conceptual interpretation in the case when $\dot{\overline{\boldsymbol{u}}}(\lambda_0) = 0$. This happens, in particular, if $\overline{\boldsymbol{u}}(\lambda)$ is constant in a neighborhood of λ_0, a circumstance that may occur by a suitable non-dimensionalization of the original equation [36, Section VI]. To show the above, consider the operator

$$\begin{aligned} L : \boldsymbol{w} \in X(\Omega) \subset \mathcal{D}_0^{1,2}(\Omega) \mapsto L(\boldsymbol{w}) \\ = -\widetilde{\Delta}^{-1}[\partial_1 \boldsymbol{w} + \mathscr{R}(\boldsymbol{w}) + \mathcal{B}(\boldsymbol{v}_0, \boldsymbol{w}) + \mathcal{B}(\boldsymbol{w}, \boldsymbol{v}_0)] \in \mathcal{D}_0^{1,2}(\Omega)\,, \end{aligned}$$

where, as before, $\boldsymbol{v}_0 := \boldsymbol{u}_0 + \boldsymbol{U}$.

The following lemma shows the fundamental properties of L. The proof is quite involved and, for it, the reader is referred to [49, Lemma 111].

Lemma 11 *Assume $\boldsymbol{u}_0 \in L^3(\Omega) \cap L^4_{\rm loc}(\overline{\Omega})$. Then, the operator L is (graph) closed. Moreover, $\mathrm{Sp}(L_{\mathbb{C}}) \cap (0, \infty)$ consists, at most, of a finite or countable number of eigenvalues, each of which is isolated and of finite algebraic and geometric multiplicities, that can only accumulate at 0.*

(Of course, the assumption $\boldsymbol{u}_0 \in L^4_{\rm loc}(\overline{\Omega})$ is redundant if $\boldsymbol{u}_0 \in X(\Omega)$. Also $\boldsymbol{u}_0 \in L^3(\Omega)$ is assured by Lemma 9 if $\boldsymbol{f}$ suitably summable at large distances.)

Combining Corollary 1, Lemma 11 and Theorem 10, one can then show the following.

Theorem 11 *Assume $\dot{\overline{\boldsymbol{u}}}(\lambda_0) = 0$, with $\boldsymbol{u}_0 \in L^3(\Omega) \cap L^4_{\rm loc}(\overline{\Omega})$. Then, a necessary condition for $(\lambda_0, \boldsymbol{u}_0)$ to be a bifurcation point for* (74) *is that $\mu_0 := 1/\lambda_0$ is an eigenvalue for the operator $L_{\mathbb{C}}$. This condition is also sufficient if μ_0 is simple.*

Proof. With the help of (25)–(27), and (30) one sees that condition (77) is equivalent to assuming that the following equation has a non-zero solution $\boldsymbol{w}_1 \in X(\Omega)$

$$\widetilde{\Delta} \boldsymbol{w}_1 + \lambda_0 [\partial_1 \boldsymbol{w}_1 + \mathscr{R}(\boldsymbol{w}_1) + \mathcal{B}(\boldsymbol{v}_0, \boldsymbol{w}_1) + \mathcal{B}(\boldsymbol{w}_1, \boldsymbol{v}_0)] = \boldsymbol{0}$$

Operating with $\widetilde{\Delta}^{-1}$ on both sides of the latter, one concludes that μ_0 must be an eigenvalue of $L_{\rm c}$, which provides the first statement. Performing the same procedure on (85) it can be next shown that the first assumption in Theorem 10 is satisfied if and only if there is a unique (normalized) $\boldsymbol{w}_1 \in X(\Omega)$ such that

$$L(\boldsymbol{w}_1) = \mu_0 \, \boldsymbol{w}_1\,,$$

that is, μ_0 is an eigenvalue of $L_{\rm c}$ of geometric multiplicity 1. Furthermore, operating again with $\widetilde{\Delta}^{-1}$ on both sides of (84) with $\dot{\overline{\boldsymbol{u}}}(\lambda_0) = 0$, one gets

$$\mu_0 \, \boldsymbol{w} - L(\boldsymbol{w}) = -\mu_0^2 \boldsymbol{w}_1$$

which, by the second assumption in Theorem 10, should have no solution, which means that the algebraic multiplicity of μ_0 must be 1 as well, and the proof of the claimed property is completed. □

Another interesting and immediate consequence of Lemma 11 and Theorem 11 is the following one.

Corollary 2. *Let $\boldsymbol{u}_0$ be a solution branch to* (74) *independent of $\lambda \in J$, where J is a bounded interval with $\overline{J} \subset (0,\infty)$. Suppose, furher, that $\boldsymbol{u}_0$ satisfies the assumption of the preceding theorem. Then, there is at most a finite number, m, of bifurcation points to* (74) $(\lambda_k, \boldsymbol{u}_0)$, $\lambda_k \in J$, $k = 1, \cdots, m$.

Remark 8. It is significant to observe that the statements of Theorem 11 and Corollary 2 *formally* coincide with those of analogous theorems for steady bifurcation from steady solution to the Navier-Stokes equation in a *bounded* domain; see, e.g., [6, Section 4.3C]. However, in the latter case L is a *compact* operator defined on the whole of $\mathcal{D}_0^{1,2}(\Omega)$, whereas in the present case, L is a densely defined *unbounded* operator.

Remark 9. Arguing as in Remark 7, one deduces that, under the assumption of Theorem 11, a sufficient condition for $(\lambda_0, \boldsymbol{u}_0)$ to be a bifurcation point is that the eigenvalue problem (78) with $(\boldsymbol{w}, p)$ in the specified function class, has λ_0 as a simple eigenvalue.

9.2 Time-Periodic Bifurcation

In spite of its great relevance and frequent occurrence in experimental fluid mechanics, time-periodic bifurcation in a flow past an obstacle has represented a long-standing and intriguing problem from a rigorous mathematical viewpoint. This situation should be contrasted with flow in a *bounded* domain where, thanks to the pioneering and fundamental contributions of Iudovich [68], Joseph and Sattinger [70], and Iooss [67], complicated time-periodic bifurcation phenomena, like those occurring in the classical Taylor-Couette experiment, could be framed in a rigorous mathematical setting.

In order to understand the reason for this uneven situation and also provide a motivation for the approach presented here, it is appropriate to briefly describe what constitutes a rigorous treatment of the phenomenon of time-periodic bifurcation. Suppose, as will be in fact show later on, that the relevant time-dependent problem can be formally written in the form

$$u_t + L(u) = N(u, \mu)\,, \tag{86}$$

where L is a linear differential operator (with appropriate *homogeneous* boundary conditions), and N is a nonlinear operator depending on the parameter $\mu \in \mathbb{R}$, such that $N(0, \mu) = 0$ for all admissible values of μ. Then, roughly speaking, time-periodic bifurcation for (86) amounts to show the existence a family of non-trivial time-periodic solutions $u = u(\mu; t)$ of (unknown) period $T = T(\mu)$ (*T-periodic* solutions) in a neighborhood of $\mu = 0$, and such that $u(\mu; \cdot) \to 0$ as $\mu \to 0$. Setting $\tau := 2\pi\, t/T \equiv \omega\, t$, (86) becomes

$$\omega\, u_\tau + L(u) = N(u,\mu) \tag{87}$$

and the problem reduces to find a family of 2π-periodic solutions to (87) with the above properties. If one now writes $u = \overline{u} + (u-\overline{u}) := v + w$, one gets that (87) is formally equivalent to the following two equations

$$\begin{aligned} &L(v) = \overline{N(v+w,\mu)} := N_1(v,w,\mu)\,,\\ &\omega\, w_\tau + L(w) = N(v+w,\mu) - \overline{N(v+w,\mu)} := N_2(v,w,\mu)\,. \end{aligned} \tag{88}$$

At this point, the crucial issue to realize is that while in the case of a *bounded* flow both "steady-state" component, v, and "oscillatory" component, w, may be taken in the *same* (Hilbert) function space [68; 70; 67], in the case of an *exterior* flow, v belongs to a space with quite less "regularity" (in the sense of behavior at large spatial distances) than w does; see also [53]. For this basic reason, as emphasized for the first time only recently in [51; 52], in the case of an *exterior* flow it is not appropriate (or even "natural") to investigate the bifurcation problem for (87) in just *one* functional setting, as done, for example, in [101]; it is instead much more spontaneous to study the two equations in (88) in two *different* function classes. As a consequence, even though *formally* being the same as differential operators, the operator L in $(88)_1$ acts on and ranges into spaces different than those the operator L in $(88)_2$ does. With this in mind, (88) becomes

$$L_1(v) = N_1(v,w,\mu)\,; \quad \omega\, w_\tau + L_2(w) = N_2(v,w,\mu)\,.$$

The above ideas will be next applied to provide sufficient conditions for time-periodic bifurcation in a viscous flow past a body. It *will be assumed throughout* $\mathcal{T}=0$, leaving the case $\mathcal{T}\neq 0$ as an open question. Set

$$\mathscr{L}_1 : \boldsymbol{v} \in X(\Omega) \mapsto \mathscr{L}(\lambda_0,\boldsymbol{v}) \in \mathcal{D}_0^{-1,2}(\Omega) \tag{89}$$

with $\mathscr{L}(\lambda_0,\boldsymbol{v})$ defined in (76). From Lemma 10 and Remark 6 it follows that under the assumption

$$\mathsf{N}\,(\mathscr{L}_1) = \{\mathbf{0}\} \tag{H1}$$

there exists a unique weak-solution analytic branch $\boldsymbol{v}_s(\lambda) := \boldsymbol{u}(\lambda) + \boldsymbol{U}$ to (8)–(7) in a neighborhood $U(\lambda_0)$, with $\boldsymbol{v}_s(\lambda_0) = \boldsymbol{u}_0 + \boldsymbol{U}$. Thus, writing $\boldsymbol{v} = \mathsf{v}(x,t;\lambda) + \boldsymbol{v}_s(x;\lambda)$, from (5) one finds that v formally satisfies the (nondimensional) problem

$$\left.\begin{aligned} \mathsf{v}_t + \lambda[(\mathsf{v}-\boldsymbol{e}_1)\cdot\nabla\mathsf{v} + \boldsymbol{v}_s(\lambda)\cdot\nabla\mathsf{v} + \mathsf{v}\cdot\nabla\boldsymbol{v}_s(\lambda)] = \Delta\mathsf{v} - \nabla\mathsf{p}\\ \operatorname{div}\mathsf{v} = 0 \end{aligned}\right\}\ \text{in } \Omega\times\mathbb{R}$$
$$\mathsf{v} = \mathbf{0}\ \text{ at } \partial\Omega\times\mathbb{R}\,, \quad \lim_{|x|\to\infty}\mathsf{v}(x,t) = \mathbf{0}\,, \quad t\in\mathbb{R}\,. \tag{89}$$

The bifurcation problem consists then in finding sufficient conditions for the existence of a non-trivial family of suitably defined time-periodic weak solutions to (89), $\mathsf{v}(t;\lambda)$, $\lambda\in U(\lambda_0)$, of period $T = T(\lambda)$ (unknown as well), such that $\mathsf{v}(t;\lambda)\to\mathbf{0}$ as $\lambda\to\lambda_0$.

Following the general approach mentioned before, one thus introduces the scaled time $\tau := \omega\, t$, split v and as the sum of its time average, $\overline{\mathsf{v}}$, over the time interval $[-\pi,\pi]$, and its "purely periodic" component $\boldsymbol{w} := \mathsf{v} - \overline{\mathsf{v}}$, and set $\mu := \lambda - \lambda_0$. In this way, problem (89) can be equivalently rewritten as the following coupled nonlinear elliptic-parabolic problem

$$\left.\begin{aligned} \Delta\bar{\mathsf{v}} + \lambda_0(\,\partial_1\bar{\mathsf{v}} - \boldsymbol{v}_0\cdot\nabla\bar{\mathsf{v}} - \boldsymbol{v}_0\cdot\nabla\bar{\mathsf{v}}) = \nabla\bar{\mathsf{p}} + \boldsymbol{N}_1(\bar{\mathsf{v}},\boldsymbol{w},\mu) \\ \operatorname{div}\bar{\mathsf{v}} = 0 \end{aligned}\right\}\ \text{in } \Omega$$
$$\bar{\mathsf{v}} = \mathbf{0} \ \text{ at } \partial\Omega\,, \quad \lim_{|x|\to\infty}\bar{\mathsf{v}}(x) = \mathbf{0} \tag{90}$$

and

$$\left.\begin{aligned} \omega\,\boldsymbol{w}_\tau - \Delta\boldsymbol{w} - \lambda_0\,(\partial_1\boldsymbol{w} - \boldsymbol{v}_0\cdot\nabla\boldsymbol{w} - \boldsymbol{w}\cdot\nabla\boldsymbol{v}_0) \\ = \nabla\varphi + \boldsymbol{N}_2(\bar{\mathsf{v}},\boldsymbol{w},\mu) \\ \operatorname{div}\boldsymbol{w} = 0 \end{aligned}\right\}\ \text{in } \Omega_{2\pi}$$
$$\boldsymbol{w} = \mathbf{0} \ \text{ at } \partial\Omega\times(-\pi,\pi)\,, \quad \lim_{|x|\to\infty}\boldsymbol{w}(x,t) = \mathbf{0}\,, \tag{91}$$

where

$$\begin{aligned} \boldsymbol{N}_1 &:= -\mu\,[\partial_1\bar{\mathsf{v}} - \boldsymbol{v}_s(\mu+\lambda_0)\cdot\nabla\bar{\mathsf{v}} - \bar{\mathsf{v}}\cdot\nabla\boldsymbol{v}_s(\mu+\lambda_0)] \\ &\quad + \lambda_0\,[(\boldsymbol{v}_s(\mu+\lambda_0) - \boldsymbol{v}_0)\cdot\nabla\bar{\mathsf{v}} + \bar{\mathsf{v}}\cdot\nabla(\boldsymbol{v}_s(\mu+\lambda_0) - \boldsymbol{v}_0)] \\ &\quad + (\mu+\lambda_0)\Big[\bar{\mathsf{v}}\cdot\nabla\bar{\mathsf{v}} + \overline{\boldsymbol{w}\cdot\nabla\boldsymbol{w}}\Big]\,, \end{aligned} \tag{92}$$

and

$$\begin{aligned} \boldsymbol{N}_2 &:= \mu\,[\partial_1\boldsymbol{w} - \boldsymbol{v}_s(\mu+\lambda_0)\cdot\nabla\boldsymbol{w} - \boldsymbol{w}\cdot\nabla\boldsymbol{v}_s(\mu+\lambda_0)] \\ &\quad - \lambda_0\Big[(\boldsymbol{v}_s(\mu+\lambda_0) - \boldsymbol{v}_0)\cdot\nabla\boldsymbol{w} + \boldsymbol{w}\cdot\nabla(\boldsymbol{v}_s(\mu+\lambda_0) - \boldsymbol{v}_0)\Big] \\ &\quad + (\mu+\lambda_0)\Big[\boldsymbol{w}\cdot\nabla\bar{\mathsf{v}} + \bar{\mathsf{v}}\cdot\nabla\boldsymbol{w} + \boldsymbol{w}\cdot\nabla\boldsymbol{w} - \overline{\boldsymbol{w}\cdot\nabla\boldsymbol{w}}\Big]\,, \end{aligned} \tag{93}$$

with $\boldsymbol{v}_0 \equiv \boldsymbol{v}_s(\lambda_0)$.

The next step is to rewrite (90)–(93) in the proper functional setting and to reformulate the bifurcation problem accordingly. To this end, one begins to introduce the operator

$$\begin{aligned} \mathscr{L}_2 : \boldsymbol{w} \in \mathsf{D}(\mathscr{L}_2) \subset H(\Omega) \mapsto -\mathrm{P}\,[\Delta\boldsymbol{w} + \lambda_0(\partial_1\boldsymbol{w} - \boldsymbol{v}_0\cdot\nabla\boldsymbol{w} - \boldsymbol{w}\cdot\nabla\boldsymbol{v}_0)] \in H(\Omega)\,, \\ \mathsf{D}(\mathscr{L}_2) := W^{2,2}(\Omega)\cap\mathcal{D}_0^{1,2}(\Omega)\,. \end{aligned} \tag{94}$$

The following result can be proved by the same arguments (slightly modified in the detail) employed in [52, Proposition 4.2].

Lemma 12 *Let* $\boldsymbol{u}_0 := \boldsymbol{v}_0 - \boldsymbol{U} \in X(\Omega)$. *Then* $\mathrm{Sp}(\mathscr{L}_{2\mathbb{C}}) \cap \{\mathrm{i}\,\mathbb{R} - \{0\}\}$ *consists, at most, of a finite or countable number of eigenvalues, each of which is isolated and of finite (algebraic) multiplicity, that can only accumulate at 0.*

Consider, next, the time-dependent operator

$$\mathscr{Q} : \boldsymbol{w} \in \mathscr{W}^2_{2\pi,0}(\Omega) \mapsto \omega_0\,\boldsymbol{w}_t + \mathscr{L}_2(\boldsymbol{w}) \in \mathscr{H}_{2\pi,0}(\Omega)\,. \tag{95}$$

Again, by a slight modification of the argument used in the proof of [52, Proposition 4.3] one can show the following.

Lemma 13 *Let* $\boldsymbol{v}_0$ *be as in* Lemma 12. *Then, the operator* $\mathscr{Q}$ *is Fredholm of index 0, for any* $\omega_0 > 0$.

Finally, one needs the functional properties of the quantities $\boldsymbol{N}_i$, $i=1,2$, defined in (92)–(93), reported in the following lemma. The proof is, one more time, a slight modification of that given in [52, Lemma 4.5, and the paragraph after it] and will be omitted.

Lemma 14 *There is a neighborhood $\mathcal{V}(0,\mathbf{0},\mathbf{0})\subset\mathbb{R}\times X(\Omega)\times\mathscr{W}^2_{2\pi,0}(\Omega)$ such that maps*

$$\mathscr{N}_1:(\mu,\boldsymbol{v}_1,\boldsymbol{v}_2)\in\mathcal{V}(0,\mathbf{0},\mathbf{0})\mapsto \mathrm{P}\,\boldsymbol{N}_1(\mu,\boldsymbol{v}_1,\boldsymbol{v}_2)\in\mathcal{D}_0^{-1,2}(\Omega)$$
$$\mathscr{N}_2:(\mu,\boldsymbol{v}_1,\boldsymbol{v}_2)\in\mathcal{V}(0,\mathbf{0},\mathbf{0})\mapsto \mathrm{P}\,\boldsymbol{N}_2(\mu,\boldsymbol{v}_1,\boldsymbol{v}_2)\in\mathscr{H}_{2\pi,0}(\Omega)$$

are analytic.

Also in view of Lemmas 12–14, one then deduces that (90)–(93) can be put in the following abstract form

$$\mathscr{L}_1(\overline{\mathsf{v}})=\mathscr{N}_1(\mu,\overline{\mathsf{v}},\boldsymbol{w})\ \text{ in } \mathcal{D}_0^{-1,2}(\Omega)\,;\quad \omega\,\boldsymbol{w}_\tau+\mathscr{L}_2(\boldsymbol{w})=\mathscr{N}_2(\mu,\overline{\mathsf{v}},\boldsymbol{w})\ \text{ in } \mathscr{H}_{2\pi,0}\,. \tag{96}$$

Notice that the spatial asymptotic conditions on $\overline{\mathsf{v}}$ in $(90)_4$, is interpreted in the sense of Remark 2, while the one in $(90)_4$ for $\boldsymbol{w}$ holds uniformly pointwise for a.a. $t\in\mathbb{R}$; see [52, Remark 3.2].

One is now in a position to give a precise definition of a time-periodic bifurcation point.

Definition 3. The triple $(\mu=0,\overline{\mathsf{v}}=\mathbf{0},\boldsymbol{w}=\mathbf{0})$ is called *time-periodic bifurcation point* for (96) if there is a sequence $\{(\mu_k,\omega_k,\overline{\mathsf{v}}_k,\boldsymbol{w}_k)\}\subset\mathbb{R}\times\mathbb{R}_+\times\mathcal{D}_0^{-1,2}(\Omega)\times\mathscr{W}^2_{2\pi,0}$ with the following properties

(i) $\{(\mu_k,\omega_k,\overline{\mathsf{v}}_k,\boldsymbol{w}_k)\}$ solves (96) for all $k\in\mathbb{N}$;

(ii) $\{(\mu_k,\overline{\mathsf{v}}_k,\boldsymbol{w}_k)\}\to(0,\mathbf{0},\mathbf{0})$ as $k\to\infty$;

(iii) $\boldsymbol{w}_k\not\equiv\mathbf{0}$, for all $k\in\mathbb{N}$.

Moreover, the bifurcation is called *supercritical* [resp. *subcritical*] if the above sequence of solutions exists only for $\mu_k>0$ [resp. $\mu_k<0$].

The goal is to give sufficient conditions for the occurrence of time-periodic bifurcation in the sense specified above. This will be achieved by means of the general result proved in [52, Theorem 4.1]. With this in mind, one has to show that the assumptions of that theorem are indeed satisfied. In this regard, supported by Lemma 12 one supposes

$$\begin{aligned}&\nu_0:=\mathrm{i}\,\omega_0\ \text{ is an eigenvalue of multiplicity 1 of } \mathscr{L}_{2\mathbb{C}}\,,\\ &k\,\nu_0\,,\,k\in\mathbb{N}-\{0,1\}\ \text{ is not an eigenvalue of } \mathscr{L}_{2\mathbb{C}}\,.\end{aligned} \tag{H2}$$

Next, consider the operator

$$\mathscr{L}_2(\mu):=\mathscr{L}_2-\mu\,\mathscr{S}\,,$$

with

$$\mathscr{S}:\boldsymbol{w}\in Z^{2,2}(\Omega)\mapsto \mathrm{P}\,[\partial_1\boldsymbol{w}-\boldsymbol{v}_0\cdot\nabla\boldsymbol{w}-\boldsymbol{w}\cdot\nabla\boldsymbol{v}_0-\lambda_0\,(\dot{\boldsymbol{v}}_s(\lambda_0)\cdot\nabla\boldsymbol{w}+\boldsymbol{w}\cdot\nabla\dot{\boldsymbol{v}}_s(\lambda_0))]\in H(\Omega)\,,$$

and where, as before, "$\cdot$" means differentiation with respect to λ. By [110, Proposition 79.15 and Corollary 79.16] one knows that for μ in a neighborhood of 0 there is a smooth map $\mu \mapsto \nu(\mu)$, with $\nu(\mu)$ simple eigenvalue of $\mathscr{L}_{2\mathbb{C}}(\mu)$ and such that $\nu_0 = \nu(0)$. The following condition will be further assumed:

$$\Re[\dot{\nu}(0)] \neq 0\,, \tag{H3}$$

which basically means that the eigenvalue $\nu(\mu)$ must cross the imaginary axis with "non-zero speed" when $\lambda \to \lambda_0$.

The general result proved in [52, Theorem 3.1] can be now applied to show the following time-periodic bifurcation result.

Theorem 12 *Suppose* (H1)–(H3) *hold. Then, the following properties are valid.*

(a) Existence. *There are analytic families*

$$(\overline{\mathsf{v}}(\varepsilon), \boldsymbol{w}(\varepsilon), \omega(\varepsilon), \mu(\varepsilon)) \in X(\Omega) \times \mathscr{W}^2_{2\pi,0}(\Omega) \times \mathbb{R}_+ \times \mathbb{R} \tag{97}$$

satisfying (96), *for all* ε *in a neighborhood* $\mathcal{I}(0)$ *and such that*

$$(\overline{\mathsf{v}}(\varepsilon), \boldsymbol{w}(\varepsilon) - \varepsilon\, \boldsymbol{v}_1, \omega(\varepsilon), \mu(\varepsilon)) \to (0, 0, \omega_0, 0) \quad \textit{as } \varepsilon \to 0\,.$$

(a) Uniqueness. *There is a neighborhood*

$$U(0, 0, \omega_0, 0) \subset X(\Omega) \times \mathscr{W}^2_{2\pi,0}(\Omega) \times \mathbb{R}_+ \times \mathbb{R}$$

such that every (nontrivial) 2π*-periodic solution to* (96), $(\boldsymbol{z}, \boldsymbol{s})$, *lying in* U *must coincide, up to a phase shift, with a member of the family* (97).

(a) Parity. *The functions* $\omega(\varepsilon)$ *and* $\mu(\varepsilon)$ *are even:*

$$\omega(\varepsilon) = \omega(-\varepsilon)\,, \quad \mu(\varepsilon) = \mu(-\varepsilon)\,, \quad \textit{for all } \varepsilon \in \mathcal{I}(0)\,.$$

Consequently, the bifurcation due to these solutions is either subcritical or supercritical, a two-sided bifurcation being excluded (unless $\mu \equiv 0$*).*

Open Problem 6 *Sufficient conditions for the occurrence of time-periodic bifurcation in the case when the body is also spinning (*$\mathcal{T} \neq 0$*) are not known.*

10 Stability and Long Time Behavior of Unsteady Perturbations

In this section, $\boldsymbol{v}_0$ will denote the velocity field of a steady-state solution to (8). As usual, λ is assumed to be positive. However, since the theories that will be described in this section have often been equally developed for both cases $\lambda = 0$ and $\lambda \neq 0$, a number of cited results also concern the case when $\lambda = 0$. The function $\boldsymbol{v}_0$ is supposed to satisfy

$$\boldsymbol{v}_0 \in L^3(\Omega), \qquad \partial_j \boldsymbol{v}_0 \in L^3(\Omega) \cap L^{3/2}(\Omega) \quad (\text{for } j = 1, 2, 3). \tag{98}$$

It follows from Lemma 9 that, if $\lambda \neq 0$, such a $\boldsymbol{v}_0$ exists for a large class of body forces $\boldsymbol{f}$ and boundary data. An associated pressure field is denoted by p_0.

One is interested in the behavior of unsteady perturbations, $(\boldsymbol{v}', p')$ to the solution $(\boldsymbol{v}_0, p_0)$. Thus, writing $\boldsymbol{v} = \boldsymbol{v}_0 + \boldsymbol{v}'$, $p = p_0 + p'$, it follows from (5) that the functions $\boldsymbol{v}'$, p' satisfy the equations

$$\left.\begin{aligned} &-\boldsymbol{v}'_t + \Delta \boldsymbol{v}' + \lambda\, \partial_1 \boldsymbol{v}' + \mathcal{T}(\boldsymbol{e}_1 \times \boldsymbol{x} \cdot \nabla \boldsymbol{v}' - \boldsymbol{e}_1 \times \boldsymbol{v}') \\ &\qquad\qquad = \lambda \boldsymbol{v}_0 \cdot \nabla \boldsymbol{v}' + \lambda \boldsymbol{v}' \cdot \nabla \boldsymbol{v}_0 + \lambda \boldsymbol{v}' \cdot \nabla \boldsymbol{v}' + \nabla p' \\ &\operatorname{div} \boldsymbol{v}' = 0 \end{aligned}\right\} \text{ in } \Omega \times (0,\infty) \tag{99}$$

and the conditions

$$\boldsymbol{v}' = \boldsymbol{0} \quad \text{at } \partial\Omega \times (0,\infty)\,; \qquad \lim_{|\boldsymbol{x}|\to\infty} \boldsymbol{v}'(\boldsymbol{x},t) = \boldsymbol{0}\,, \text{ all } t \in (0,\infty). \tag{100}$$

For simplicity, from now on the primes are omitted in the notation above. Thus, the formal application of the Helmholtz-Weyl projection P to the first equation in (99), as formulated in $L^q(\Omega)$ $(1 < q < \infty)$, yields the operator equation

$$\frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t} = \mathscr{L}\boldsymbol{v} + \mathscr{N}\boldsymbol{v} \tag{101}$$

in the space $H_q(\Omega)$. By suitably defining the domains of the operators $\mathscr{L}$ and $\mathscr{N}$, it can be easily seen that (101) is, in fact, equivalent to (99), (100). To this end, let

$$\begin{aligned} A\boldsymbol{v} &:= \mathrm{P}\, \Delta \boldsymbol{v}, \\ B_1 \boldsymbol{v} &:= \mathrm{P}\, \partial_1 \boldsymbol{v} \end{aligned}$$

for $\boldsymbol{v} \in \mathsf{D}(A) := W^{2,q}(\Omega) \cap \mathcal{D}_0^{1,q}(\Omega)$,

$$\begin{aligned} B_2 \boldsymbol{v} &:= \mathrm{P}\,(\boldsymbol{e}_1 \times \boldsymbol{x} \cdot \nabla \boldsymbol{v} - \boldsymbol{e}_1 \times \boldsymbol{v}), \\ A_{\lambda,\mathcal{T}} \boldsymbol{v} &:= A\boldsymbol{v} + \lambda B_1 \boldsymbol{v} + \mathcal{T} B_2 \boldsymbol{v} \end{aligned}$$

for

$$\boldsymbol{v} \in \mathsf{D}(A_{\lambda,\mathcal{T}}) := \begin{cases} W^{2,q}(\Omega) \cap \mathcal{D}_0^{1,q}(\Omega) & \text{if } \mathcal{T} = 0, \\ \{\boldsymbol{v} \in W^{2,q}(\Omega) \cap \mathcal{D}_0^{1,q}(\Omega);\ \boldsymbol{e}_1 \times \boldsymbol{x} \cdot \nabla \boldsymbol{v} \in L^q(\Omega)\} & \text{if } \mathcal{T} \neq 0. \end{cases}$$

Note that $A \equiv A_{0,0}$ and $A_{\lambda,0} \equiv A + \lambda B_1$ are the classical Stokes and Oseen operators, respectively. Furthermore, let

$$\begin{aligned} B_3 \boldsymbol{v} &:= \mathrm{P}\,(\boldsymbol{v}_0 \cdot \nabla \boldsymbol{v} + \boldsymbol{v} \cdot \nabla \boldsymbol{v}_0), \\ \mathscr{L}\boldsymbol{v} &:= A_{\lambda,\mathcal{T}} \boldsymbol{v} + \lambda B_3 \boldsymbol{v}, \\ \mathscr{N}\boldsymbol{v} &:= -\lambda\, \mathrm{P}\,(\boldsymbol{v} \cdot \nabla \boldsymbol{v}) \end{aligned}$$

for $\boldsymbol{v} \in \mathsf{D}(\mathscr{L}) := \mathsf{D}(A_{\lambda,\mathcal{T}})$.

Obviously, the study of the stability of the solution $(\boldsymbol{v}_0, p_0)$ is equivalent to that of the zero solution of problem (99)–(100) or equation (101). The properties of the linear operator $\mathscr{L}$ and, especially, those of its "leading part" $A_{\lambda,\mathcal{T}}$ play a fundamental role. Thus, the next two subsections will be concerned with a detailed analysis of these properties.

10.1 Spectrum of Operator $A_{\lambda,\mathcal{T}}$

The following notions and definitions from the spectral theory of linear operators will be relevant later on.

Let X be a Banach space with norm $\|\,.\,\|$, X^* be its dual, and T be a closed linear operator in X with a domain $\mathsf{D}(T)$ dense in X. (This guarantees that the adjoint operator T^* exists.)

- The symbols $\operatorname{nul}(T)$ and $\operatorname{def}(T)$ denote the *nullity* and the *deficiency* of T, respectively. If $\mathsf{R}(T)$ is closed then $\operatorname{nul}(T) = \operatorname{def}(T^*)$ and $\operatorname{def}(T) = \operatorname{nul}(T^*)$ (see e.g. Kato [71, p. 234]).
- The *approximate nullity* of T, denoted by $\operatorname{nul}'(T)$, is the maximum integer m ($m = \infty$ being permitted) with the property that to each $\epsilon > 0$ there exists an m-dimensional linear manifold M_ϵ in $\mathsf{D}(T)$ such that $\|T\boldsymbol{v}\| < \epsilon$ for all $\boldsymbol{v} \in M_\epsilon$, $\|\boldsymbol{v}\| = 1$. The *approximate deficiency* of T is denoted by $\operatorname{def}'(T)$ and defined as $\operatorname{def}'(T) := \operatorname{nul}'(T^*)$. Note that $\operatorname{nul}(T) \leq \operatorname{nul}'(T)$ and $\operatorname{def}(T) \leq \operatorname{def}'(T)$, the equalities holding if the range $\mathsf{R}(T)$ is closed. On the other hand, if $\mathsf{R}(T)$ is not closed then $\operatorname{nul}'(T) = \operatorname{def}'(T) = \infty$. The identity $\operatorname{nul}'(T) = \infty$ is equivalent to the existence of a non-compact sequence $\{\boldsymbol{u}_n\}$ on the unit sphere in X such that $T\boldsymbol{u}_n \to \boldsymbol{0}$ for $n \to \infty$ (see [71, p. 233]).
- T is called a *Fredholm operator* if both the numbers $\operatorname{nul}(T)$ and $\operatorname{def}(T)$ are finite. This implies, in particular, that $\mathsf{R}(T)$ is closed in X [110, Proposition 8.14(ii)]. Operator T is *semi-Fredholm* if the range $\mathsf{R}(T)$ is closed in X and at least one of the numbers $\operatorname{nul}(T)$ and $\operatorname{def}(T)$ is finite. Consequently, T is semi-Fredholm if and only if at least one of the numbers $\operatorname{nul}'(T)$ and $\operatorname{def}'(T)$ is finite.
- The *resolvent set* $\operatorname{Res}(T)$ is the set of all $\zeta \in \mathbb{C}$ such that $\mathsf{R}(T - \zeta I) = X$ and the operator $T - \zeta I$ has a bounded inverse in X. Consequently, $\operatorname{nul}(T - \zeta I) = \operatorname{nul}'(T - \zeta I) = \operatorname{def}(T - \zeta I) = \operatorname{def}'(T - \zeta I) = 0$ for $\zeta \in \operatorname{Res}(T)$. Note that $\operatorname{Res}(T)$ is an open subset of $\mathbb{C}$.
- The *point spectrum* $\operatorname{Sp_p}(T)$ is the set of all $\zeta \in \mathbb{C}$ such that $\operatorname{nul}(T - \zeta I) > 0$.
- The *continuous spectrum* $\operatorname{Sp_c}(T)$ is the set of all $\zeta \in \mathbb{C}$ such that $\operatorname{nul}(T - \zeta I) = 0$, $\mathsf{R}(T - \zeta I)$ is dense in X, but $\mathsf{R}(T - \zeta I) \neq X$. (In this case, $\mathsf{R}(T - \zeta I)$ is not closed in X, which implies that $\operatorname{def}(T - \zeta I) = \operatorname{def}'(T - \zeta I) = \operatorname{nul}'(T - \zeta I) = \infty$.)
- The *residual spectrum* $\operatorname{Sp_r}(T)$ is the set of all $\zeta \in \mathbb{C}$ such that $\operatorname{nul}(T - \zeta I) = 0$ and the range $\mathsf{R}(T - \zeta I)$ is not dense in X. The sets $\operatorname{Sp_p}(T)$, $\operatorname{Sp_c}(T)$ and $\operatorname{Sp_r}(T)$ are mutually disjoint and $\operatorname{Sp_p}(T) \cup \operatorname{Sp_c}(T) \cup \operatorname{Sp_r}(T) = \operatorname{Sp}(T) = \mathbb{C} \smallsetminus \operatorname{Res}(T)$ (the *spectrum* of T).
- The *essential spectrum* $\operatorname{Sp_{ess}}(T)$ is the set of all $\zeta \in \mathbb{C}$ such that $T - \zeta I$ is not semi-Fredholm. Both $\operatorname{Sp}(T)$ and $\operatorname{Sp_{ess}}(T)$ are closed in $\mathbb{C}$ and $\operatorname{Sp_{ess}}(T) \subset \operatorname{Sp}(T)$. Obviously, $\operatorname{Sp_c}(T) \subset \operatorname{Sp_{ess}}(T)$. Any point on the boundary of $\operatorname{Sp}(T)$ belongs to $\operatorname{Sp_{ess}}(T)$ unless it is an isolated point of $\operatorname{Sp}(T)$ (see [71, p. 244]).

From [72] (if $\mathcal{T} = 0$) and [104] (if $\mathcal{T} \neq 0$), it follows that the operator $A_{\lambda,\mathcal{T}}$ is closed in $H_q(\Omega)$ $(1 < q < \infty)$, and all $\zeta \in \mathbb{C}$ with a sufficiently large real part belong to

$\mathrm{Res}(A_{\lambda,\mathcal{T}})$. The effective shapes and types of spectra of the operator $A_{\lambda,\mathcal{T}}$, for various values of λ and $\mathcal{T}$, are described in [20] (in $H(\Omega)$, the case $\lambda = 0$), [21] (in $H(\Omega)$, the general case $\lambda \in \mathbb{R}$), [24] (in $H_q(\Omega)$, $\lambda = 0$) and in [22] (in $H_q(\Omega)$, $\lambda \in \mathbb{R}$). The spectrum of $A_{\lambda,0}$, as an operator in $H(\Omega)$, was studied by K.I. Babenko [4]. Babenko's result says that $\mathrm{Sp}(A_{\lambda,0}) = \mathrm{Sp_c}(A_{\lambda,0}) = \Lambda_{\lambda,0}$, where

$$\Lambda_{\lambda,0} = \{\zeta = \alpha + \mathrm{i}\beta \in \mathbb{C};\ \alpha, \beta \in \mathbb{R},\ \alpha \leq -\beta^2/\lambda^2\} \tag{102}$$

for $\lambda \neq 0$. The set $\Lambda_{\lambda,0}$ represents a parabolic region in $\mathbb{C}$, symmetric about the real axis, which shrinks to the non-negative part of the real axis if $\lambda \to 0$. In fact, $\mathrm{Sp}(A)$ ($\equiv \mathrm{Sp}(A_{0,0})$) coincides with $\mathrm{Sp_c}(A)$, and coincides with the interval $(-\infty, 0]$ in $\mathbb{R}$, as mentioned e.g. by O.A. Ladyzhenskaya in [82].

The spectrum of $A_{\lambda,\mathcal{T}}$ for general $\mathcal{T}$ is studied in [22]. Notice that the case $\mathcal{T} \neq 0$ is qualitatively different from the case $\mathcal{T} = 0$, because the magnitude of the coefficient of the "new" term $\mathcal{T}\boldsymbol{e}_1 \times \boldsymbol{x} \cdot \nabla \boldsymbol{v}$ becomes unbounded as $|\boldsymbol{x}| \to \infty$. Consequently, the operator $\mathcal{T}\boldsymbol{e}_1 \times \boldsymbol{x} \cdot \nabla$ *cannot* be treated as a lower order perturbation of Stokes or Oseen operator. The main results in [22] read as follows.

Theorem 13. *Let $1 < q < \infty$, $\lambda \neq 0$ and $\Omega = \mathbb{R}^3$. Then the spectrum of $A_{\lambda,\mathcal{T}}$, as an operator in $H_q(\mathbb{R}^3)$, satisfies the identities* $\mathrm{Sp}(A_{\lambda,\mathcal{T}}) = \mathrm{Sp_c}(A_{\lambda,\mathcal{T}}) = \mathrm{Sp_{ess}}(A_{\lambda,\mathcal{T}}) = \Lambda_{\lambda,\mathcal{T}}$, *where*

$$\Lambda_{\lambda,\mathcal{T}} := \{\zeta = \alpha + \mathrm{i}\beta + \mathrm{i}k\mathcal{T} \in \mathbb{C};\ \alpha, \beta \in \mathbb{R},\ k \in \mathbb{Z},\ \alpha \leq -\beta^2/\lambda^2\}.$$

Note that $\Lambda_{\lambda,\mathcal{T}}$ is a union of a family of overlapping solid parabolas, whose axes form an equidistant system of half-lines $\{\zeta \in \mathbb{C};\ \zeta = \alpha + k\mathcal{T}\mathrm{i},\ \alpha \leq 0,\ k \in \mathbb{Z}\}$. All the parabolas lie in the half-plane $\mathrm{Re}\,\zeta \leq 0$ and their vertices are on the imaginary axis.

Theorem 14. *Let $1 < q < \infty$, $\lambda \neq 0$ and $\Omega \subset \mathbb{R}^3$ be an exterior domain with the boundary of class $C^{1,1}$. Then the spectrum of $A_{\lambda,\mathcal{T}}$ lies in the left complex half plane $\{\zeta \in \mathbb{C}; \mathrm{Re}\,\zeta \leq 0\}$ and consists of the essential spectrum $\mathrm{Sp_{ess}}(A_{\lambda,\mathcal{T}}) = \Lambda_{\lambda,\mathcal{T}}$ and possibly a set Γ of isolated eigenvalues $\zeta \in \mathbb{C} \smallsetminus \Lambda_{\lambda,\mathcal{T}}$ with $\mathrm{Re}\,\zeta < 0$ and finite algebraic multiplicity, which can cluster only at points of $\mathrm{Sp_{ess}}(A_{\lambda,\mathcal{T}})$. The set Γ of such isolated eigenvalues is independent of $q \in (1, \infty)$.*

Sketch of the proof of Theorem 13; *see* [22] *for the details.* The proof develops along the following steps (a)–(f).

(a) Using the definition of the adjoint operator, it can be verified that the adjoint operator $A^*_{\lambda,\mathcal{T}}$ to $A_{\lambda,\mathcal{T}}$ coincides with the operator $A_{-\lambda,-\mathcal{T}}$ in $H_{q'}(\Omega)$, where $1/q + 1/q' = 1$.

(b) From [19, Theorem 1.1] one can deduce that there exist constants $C_4 > 0$ and $C_5 > 0$ such that if $\boldsymbol{u} \in \mathsf{D}(A_{\lambda,\mathcal{T}})$ and $\boldsymbol{f} \in H_q(\Omega)$ satisfy the equation $A_{\lambda,\mathcal{T}}\boldsymbol{u} = \boldsymbol{f}$ then

$$\|\boldsymbol{u}\|_{2,q} + \|(\boldsymbol{\omega} \times \boldsymbol{x}) \cdot \nabla \boldsymbol{u}\|_q \ \leq\ C_4\, \|\boldsymbol{f}\|_q + C_5\, \|\boldsymbol{u}\|_q. \tag{103}$$

(c) If $\zeta \in \mathbb{C} \smallsetminus \Lambda_{\lambda,\mathcal{T}}$, then each solution of the resolvent equation $(A_{\lambda,\mathcal{T}} - \zeta I)\boldsymbol{u} = \boldsymbol{f}$, for $\boldsymbol{f} \in H_q(\Omega)$, satisfies the estimate

$$\|\boldsymbol{u}\|_q \ \leq\ C_6\, \|\boldsymbol{f}\|_q, \tag{104}$$

where $C_6 = C_6(\zeta, q)$. This estimate is derived by means of the Fourier transform, and a subtle and rather technical application of the Michlin-Lizorkin multiplier theorem. Using inequalities (103) and (104), one can prove that $\mathsf{R}(A_{\lambda,\mathcal{T}} - \zeta I)$ is closed and the operator $A_{\lambda,\mathcal{T}} - \zeta I$ is injective. The same statement also holds on the adjoint operator $A^*_{\lambda,\mathcal{T}}$ in $H_{q'}(\Omega)$. As $A^*_{\lambda,\mathcal{T}} - \overline{\zeta} I$ is injective, the range $\mathsf{R}(A_{\lambda,\mathcal{T}} - \zeta I)$ is the whole space $H_q(\Omega)$. Consequently, $\zeta \in \mathrm{Res}(A_{\lambda,\mathcal{T}})$. This shows that $\mathbb{C} \setminus \Lambda_{\lambda,\mathcal{T}} \subset \mathrm{Res}(A_{\lambda,\mathcal{T}})$.

(d) We show that $\mathrm{Sp_p}(A_{\lambda,\mathcal{T}}) = \emptyset$. Assume that $\zeta \in \Lambda_{\lambda,\mathcal{T}}$ and $\boldsymbol{u} \in \mathsf{D}(A_{\lambda,\mathcal{T}})$ satisfies the equation $(A_{\lambda,\mathcal{T}} - \zeta I)\boldsymbol{u} = \boldsymbol{0}$. Applying the Fourier transform $\mathcal{F}$, this equation yields

$$\mathcal{T}(\boldsymbol{e}_1 \times \boldsymbol{\xi} \cdot \nabla \widehat{\boldsymbol{u}}) - (\zeta - \mathrm{i}\lambda\xi_1 + |\boldsymbol{\xi}|^2)\widehat{\boldsymbol{u}} - \mathcal{T}\boldsymbol{e}_1 \times \widehat{\boldsymbol{v}} \;=\; 0, \tag{105}$$

where $\widehat{\boldsymbol{u}} = \mathcal{F}(\boldsymbol{u})$ and $\boldsymbol{\xi} = (\xi_1, \xi_2, \xi_3)$ denotes the Fourier variable. The case $1 < q \le 2$ is simpler, because $\widehat{\boldsymbol{u}}$ is a function from $L^{q'}(\Omega)$: if ξ_1, r and ϕ denote the cylindrical coordinates in the space of Fourier variables, then one can calculate that $\boldsymbol{e}_1 \times \boldsymbol{\xi} \cdot \nabla \widehat{\boldsymbol{u}} = \partial_\phi \widehat{\boldsymbol{u}}$. Substituting this to (105), one obtains the equation

$$\mathcal{T}\,\partial_\phi \widehat{\boldsymbol{u}} - (\zeta - \mathrm{i}\lambda\xi_1 + |\boldsymbol{\xi}|^2)\widehat{\boldsymbol{u}} - \mathcal{T}\boldsymbol{e}_1 \times \widehat{\boldsymbol{v}} \;=\; 0.$$

If $O(\phi)$ denotes the matrix of rotation about the ξ_1–axis by angle ϕ and $\widehat{\boldsymbol{w}}(\xi_1, r, \phi) := O(\phi)\,\widehat{\boldsymbol{u}}(\xi_1, r, \phi)$ then one arrives at the ordinary differential equation

$$\mathcal{T}\,\partial_\phi \widehat{\boldsymbol{w}} - (\zeta - \mathrm{i}\lambda\xi_1 + r^2 + \xi_1^2)\,\widehat{\boldsymbol{w}} \;=\; \boldsymbol{0}.$$

This equation can be solved explicitly. The solution satisfies: $\widehat{\boldsymbol{w}}(\xi_1, r, \phi + 2\pi) = \widehat{\boldsymbol{w}}(\xi_1, r, \phi)\,\mathrm{e}^{2\pi(\zeta - \mathrm{i}\lambda\xi_1 + r^2 + \xi_1^2)}$. As $\widehat{\boldsymbol{w}}$ is 2π–periodic in variable ϕ, and $\mathrm{Re}\,\zeta + r^2 + \xi_1^2 = \mathrm{Re}\,\zeta + |\boldsymbol{\xi}|^2 \neq 0$ for a.a. $\boldsymbol{\xi} \in \mathbb{R}^3$, $\widehat{\boldsymbol{w}}$ is equal to zero a.e. in $\mathbb{R}^3$. It means that $\boldsymbol{u}$ is the zero element of $H_q(\Omega)$, which implies that it cannot be an eigenfunction and ζ therefore cannot be an eigenvalue. The case $2 < q < \infty$ is rather more complicated because $\widehat{\boldsymbol{u}}$ is only a tempered distribution. Nevertheless, one can also arrive at the same conclusion, i.e. that any $\zeta \in \Lambda_{\lambda,\mathcal{T}}$ cannot be an eigenvalue of $A_{\lambda,\mathcal{T}}$.

(e) The identity $\mathrm{Sp_r}(A_{\lambda,\mathcal{T}}) = \emptyset$ can be proven by means of the duality argument: $\zeta \in \mathrm{Sp_r}(A_{\lambda,\mathcal{T}})$ would imply that $\overline{\zeta} \in \mathrm{Sp_p}(A^*_{\lambda,\mathcal{T}})$. However, the same considerations as in step (d), applied to the adjoint operator $A^*_{\lambda,\mathcal{T}}$, show that $\mathrm{Sp_p}(A^*_{\lambda,\mathcal{T}}) = \emptyset$.

(f) The identities $\mathrm{Sp}(A_{\lambda,\mathcal{T}}) = \mathrm{Sp_c}(A_{\lambda,\mathcal{T}}) = \mathrm{Sp_{ess}}(A_{\lambda,\mathcal{T}})$ follow from the facts that $\mathrm{Sp_p}(A_{\lambda,\mathcal{T}})$ and $\mathrm{Sp_r}(A_{\lambda,\mathcal{T}})$ are empty and $\mathrm{Sp_c}(A_{\lambda,\mathcal{T}}) \subset \mathrm{Sp_{ess}}(A_{\lambda,\mathcal{T}})$. The inclusion $\mathrm{Sp}(A_{\lambda,\mathcal{T}}) \subset \Lambda_{\lambda,\mathcal{T}}$ follows from item c). The inclusion $\Lambda_{\lambda,\mathcal{T}} \subset \mathrm{Sp_{ess}}(A_{\lambda,\mathcal{T}})$ is proven in [22] so that ζ is assumed to be in $\Lambda^\circ_{\lambda,\mathcal{T}}$ (the interior of $\Lambda_{\lambda,\mathcal{T}}$), and a concrete sequence $\{\boldsymbol{u}_n\}$, such that $\|(A_{\lambda,\mathcal{T}} - \zeta I)\boldsymbol{u}_n\|_q \to 0$ for $n \to \infty$, is constructed on the unit sphere in $H_q(\Omega)$. The construction is quite technical, so the readers are referred to [22] for the details. Thus, $\zeta \in \mathrm{Sp_{ess}}(A_{\lambda,\mathcal{T}})$, which implies that $\Lambda^\circ_{\lambda,\mathcal{T}} \subset \mathrm{Sp_{ess}}(A_{\lambda,\mathcal{T}})$. The inclusion $\Lambda_{\lambda,\mathcal{T}} \subset \mathrm{Sp_{ess}}(A_{\lambda,\mathcal{T}})$ now follows from the fact that $\mathrm{Sp_{ess}}(A_{\lambda,\mathcal{T}})$ is closed. □

Sketch of the proof of Theorem 14; *see* [22] *for the details.* The proof is a consequence of the following steps (g)–(j).

(g) One can show the same inequality as (103), applying the cut-off function technique and splitting the equation $A_{\lambda,\mathcal{T}}\boldsymbol{u} = \boldsymbol{f}$ into an equation for the unknown $\boldsymbol{u}_1$ in a bounded domain Ω_ρ (where $\rho > 0$ is sufficiently large) and an equation for the unknown $\boldsymbol{u}_2$ in the whole space $\mathbb{R}^3$. Due to [19, Theorem 1.1], the function $\boldsymbol{u}_2$ satisfies (103), while $\boldsymbol{u}_1$ satisfies (103) because $\lambda B_1\boldsymbol{u}$ and $\mathcal{T}B_2\boldsymbol{u}$ can be brought into the right-hand side and then one can apply the estimates of solutions of the Stokes problem in a bounded domain. The L^q–norms of $\lambda B_1\boldsymbol{u}$ and $\mathcal{T}B_2\boldsymbol{u}$ over Ω_ρ can be interpolated between $\|\boldsymbol{u}\|_q$ and $\|\boldsymbol{u}_1\|_{2,q}$, and the norm $\|\boldsymbol{u}_1\|_{2,q}$ can be absorbed by the left-hand side. Finally, the sum of the estimates of $\boldsymbol{u}_1$ (over Ω_ρ) and $\boldsymbol{u}_2$ (over $\mathbb{R}^3$) leads to (103).

(h) The inclusion $\Lambda_{\lambda,\mathcal{T}} \subset \mathrm{Sp}_{\mathrm{ess}}(A_{\lambda,\mathcal{T}})$: assume that $\zeta \in \Lambda^\circ_{\lambda,\mathcal{T}}$. By analogy with f), one can construct a sequence $\{\boldsymbol{u}_n\}$ on the unit sphere in $H_q(\Omega)$, such that $\|(A_{\lambda,\mathcal{T}}-\zeta I)\boldsymbol{u}_n\|_q \to 0$ for $n \to \infty$. Since $\mathrm{Sp}_{\mathrm{p}}(A_{\lambda,\mathcal{T}})$ is not known to be empty, and it is necessary to show that $\zeta \in \mathrm{Sp}_{\mathrm{ess}}(A_{\lambda,\mathcal{T}})$, it is important that the sequence $\{\boldsymbol{u}_n\}$ is non-compact in $H_q(\Omega)$. The details can be found in [22], where the functions $\boldsymbol{u}_n$ are defined so that they have compact supports S_n in Ω, and the intersection $\cap_{n=1}^{\infty} S_{k_n}$ (where $\{S_{k_n}\}$ is any subsequence of $\{S_n\}$) is empty.

(i) The opposite inclusion $\mathrm{Sp}_{\mathrm{ess}}(A_{\lambda,\mathcal{T}}) \subset \Lambda_{\lambda,\mathcal{T}}$: if $\zeta \in \mathrm{Sp}_{\mathrm{ess}}(A_{\lambda,\mathcal{T}})$ then, by definition, $\mathrm{nul}'(A_{\lambda,\mathcal{T}}-\zeta I)=\infty$ or $\mathrm{def}'(A_{\lambda,\mathcal{T}}-\zeta I)=\infty$. The latter means that $\mathrm{nul}'(A^*_{\lambda,\mathcal{T}}-\overline{\zeta}I)=\infty$. Thus, that $\mathrm{nul}'(A_{\lambda,\mathcal{T}}-\zeta I)$ may be assumed to be infinity, otherwise one can deal with the operator $A^*_{\lambda,\mathcal{T}}$ instead of $A_{\lambda,\mathcal{T}}$. The identity $\mathrm{nul}'(A_{\lambda,\mathcal{T}}-\zeta I)=\infty$ enables one to construct, by mathematical induction, a sequence $\{\boldsymbol{u}_n\}$ in $\mathsf{D}(A_{\lambda,\mathcal{T}})$ satisfying $\|\boldsymbol{u}^n\|_q=1$, $\|(A_{\lambda,\mathcal{T}}-\zeta I)\boldsymbol{u}_n\|_q \to 0$ as $n\to\infty$ and $\mathrm{dist}(\boldsymbol{u}^n;\,\mathcal{L}_{n-1})=1$ for all $n\in\mathbb{N}$, where $\mathcal{L}_{n-1}$ denotes the linear hull of the functions $\boldsymbol{u}^1, \dots, \boldsymbol{u}^{n-1}$. Using a cut–off function technique, the functions $\boldsymbol{u}_n$ can be modified so that they are all supported for $|\boldsymbol{x}|>\rho$ (for sufficiently large ρ), and the modified functions (let us denote them $\widetilde{\boldsymbol{u}}_n$) are on the unit sphere in $H_q(\Omega)$ and satisfy $\|(A_{\lambda,\mathcal{T}}-\zeta I)\widetilde{\boldsymbol{u}}_n\|_q \to 0$ for $n\to\infty$ as well. However, as $\mathrm{supp}\,\widetilde{\boldsymbol{u}}_n \subset \Omega$, $\widetilde{\boldsymbol{u}}_n$ can be considered to be a function from $\mathsf{D}((A_{\lambda,\mathcal{T}})_{\mathbb{R}^3})$, where $(A_{\lambda,\mathcal{T}})_{\mathbb{R}^3}$ denotes the operator $A_{\lambda,\mathcal{T}}$ in $H_q(\mathbb{R}^3)$. This yields the equality $\mathrm{nul}'((A_{\lambda,\mathcal{T}})_{\mathbb{R}^3}-\zeta I)=\infty$, which implies, due to item c), that $\zeta\in\Lambda_{\lambda,\mathcal{T}}$.

(j) The domain $\zeta \in \mathbb{C}\smallsetminus\Lambda_{\lambda,\mathcal{T}}$ consists of points in $\mathrm{Res}(A_{\lambda,\mathcal{T}})$ and possibly also of isolated eigenvalues of $A_{\lambda,\mathcal{T}}$ with finite algebraic multiplicities, which may possibly cluster only at points of $\partial\Lambda_{\lambda,\mathcal{T}}$. (See [71, pp. 243, 244].) Assume that $\zeta\in\mathbb{C}\smallsetminus\Lambda_{\lambda,\mathcal{T}}$ is an eigenvalue of $A_{\lambda,\mathcal{T}}$ with an eigenfunction $\boldsymbol{u}$. Applying again an appropriate cut–off function technique, and treating the equation $(A_{\lambda,\mathcal{T}}-\zeta I)\boldsymbol{u}=\boldsymbol{0}$ separately in a bounded domain Ω_ρ (for a sufficiently large ρ) and in the whole space $\mathbb{R}^3$, one can show that $\boldsymbol{u}$ is in $W^{2,s}(\Omega)$ for any $1<s<\infty$. (This follows from estimates valid in a bounded domain and the result from item c), implying that ζ in in the resolvent set of $(A_{\lambda,\mathcal{T}})_{\mathbb{R}^3}$.) Finally, multiplying the equation $(A_{\lambda,\mathcal{T}}-\zeta I)\boldsymbol{u}=\boldsymbol{0}$ by $\overline{\boldsymbol{u}}$ and integrating in Ω, one can show that $\mathrm{Re}\,\zeta<0$. □

When $q=2$, in [21] it is shown that if $\mathscr{B}$ is axially symmetric about the x_1–axis then $\mathrm{Sp}(A_{\lambda,\mathcal{T}})=\Lambda_{\lambda,\mathcal{T}}$. It means that the set of eigenvalues of $A_{\lambda,\mathcal{T}}$, lying outside $\Lambda_{\lambda,\mathcal{T}}$, is empty. The same statement for operator $A_{\lambda,\mathcal{T}}$ in $H_q(\Omega)$ for general $q\in(1,\infty)$ follows from Theorem 14. The proof in [21] comes from the fact that an eigenfunction $\boldsymbol{u}$, corresponding to a hypothetic eigenvalue ζ, is 2π–periodic in the cylindrical variable, which is the angle φ measured about the x_1–axis. Then the proof uses the Fourier expansion of $\boldsymbol{u}$ in φ and splitting of the equation $(A_{\lambda,\mathcal{T}} - \zeta I)\boldsymbol{u} = \mathbf{0}$ to individual Fourier modes.

Open Problem 7 *In the general case when body $\mathscr{B}$ is not axially symmetric, it is not known whether the set of eigenvalues of $A_{\lambda,\mathcal{T}}$ in $\mathbb{C} \setminus \Lambda_{\lambda,\mathcal{T}}$ is empty.*

Finally, note that $\mathrm{Sp}(A_{0,\mathcal{T}})$ can be formally obtained, by letting $\lambda \to 0$ in $\Lambda_{\lambda,\mathcal{T}}$. Then $\Lambda_{\lambda,\mathcal{T}}$ shrinks to a system of infinitely many equidistant half-lines. The spectrum of operator $A_{0,\mathcal{T}}$ is studied in detail in [24].

10.2 A Semigroup, Generated by the Operator $A_{\lambda,\mathcal{T}}$

10.2.1 The Case $\mathcal{T} = 0$

It is well known that the Stokes operator A generates a bounded analytic semigroup, e^{At}, in $H_q(\Omega)$ [57]. The fact that the Oseen operator $A_{\lambda,0} \equiv A + \lambda B_1$ also generates an analytic semigroup in $H_q(\Omega)$ was proved by T. Miyakawa [91]. The main tool is the inequality

$$\|B_1\boldsymbol{u}\|_q \;\leq\; \epsilon\,\|A\boldsymbol{u}\|_q + C(\epsilon)\,\|\boldsymbol{u}\|_q \tag{106}$$

for all $\boldsymbol{u} \in \mathsf{D}(A)$ and $\epsilon > 0$, which implies that B_1 is relatively bounded with respect to A with the relative bound equal to zero. Then the existence and analyticity of the semigroup $\mathrm{e}^{(A+B_1)t}$ follows e.g. from [71, Theorem IX.2.4].

The so called "L^r–L^q estimates" of the semigroup $\mathrm{e}^{A_{\lambda,0}t}$ play an important role in the analysis of stability of steady flow. They were first derived by T. Kobayashi and Y. Shibata in [72], whose main result is given next.

Theorem 15. *If $1 < r \leq q < \infty$ then there exists $C = C(\lambda, q, r) > 0$ such that*

$$\|\mathrm{e}^{A_{\lambda,0}t}\boldsymbol{a}\|_q \;\leq\; C\,t^{-\frac{3}{2}\left(\frac{1}{r}-\frac{1}{q}\right)}\,\|\boldsymbol{a}\|_r \tag{107}$$

for all $\boldsymbol{a} \in H_r(\Omega)$ and $t > 0$. Moreover, if $1 < r \leq q \leq 3$ then

$$|\mathrm{e}^{A_{\lambda,0}t}\boldsymbol{a}|_{1,q} \;\leq\; C\,t^{-\frac{3}{2}\left(\frac{1}{r}-\frac{1}{q}\right)-\frac{1}{2}}\,\|\boldsymbol{a}\|_r \tag{108}$$

for all $\boldsymbol{a} \in H_r(\Omega)$ and $t > 0$.

Sketch of the proof; *see* [72] *for the details.* Following [72], the starting point is the following representation formula of the semigroup

$$\mathrm{e}^{A_{\lambda,0}t}\boldsymbol{a} \;=\; \frac{1}{2\pi\mathrm{i}t}\int_{\omega-\mathrm{i}\infty}^{\omega+\mathrm{i}\infty} \mathrm{e}^{\zeta t}\,\frac{\partial}{\partial\zeta}(A_{\lambda,0} - \zeta I)^{-1}\boldsymbol{a}\,d\zeta\,, \quad \omega > 0\,.$$

In order to estimate $(A_{\lambda,0}-\zeta I)^{-1}\boldsymbol{a}$, the Oseen resolvent problem $(A_{\lambda,0}-\zeta)\boldsymbol{a} = \boldsymbol{f}$ is split into the problem in the bounded domain Ω_ρ (for sufficiently large ρ), and in the whole space $\mathbb{R}^3$. The estimates in Ω_ρ follow from the fact that the Oseen operator in Ω_ρ has a compact resolvent and the spectrum (which coincides with the point spectrum) is in the left half-plane in $\mathbb{C}$, with a positive distance from the imaginary axis. The estimates in $\mathbb{R}^3$ are obtained by means of the Fourier transform and the Michlin–Lizorkin multiplier theorem. The next step is the construction of a parametrix, which enables the authors to combine the estimates in Ω_ρ and in $\mathbb{R}^3$, and obtain the estimates of $|(A_{\lambda,0}-\zeta I)^{-1}\boldsymbol{a}|_{2,r}$ and $|\zeta|\,\|(A_{\lambda,0}-\zeta I)^{-1}\boldsymbol{a}\|_r$ in terms of $C\,\|\boldsymbol{a}\|_r$ in the exterior domain Ω. Then the limit procedure for $\omega \to 0^+$ is considered. However, due to subtle technical reasons, the limit procedure works only in a norm over a bounded domain and one only gets the inequality

$$\|\partial_t^m \nabla^2 \mathrm{e}^{A_{\lambda,0}t}\boldsymbol{a}\|_{r;\,\Omega_\rho} \le C\,t^{-3/2}\,\|\boldsymbol{a}\|_r \tag{109}$$

for $t \geq 1$n and $\boldsymbol{a} \in H_r(\Omega)$ with the support in Ω_ρ, where $C = C(m,r,\lambda,\rho)$. On the other hand, using the formula

$$\boldsymbol{u}(\boldsymbol{x},t) = \Big(\frac{1}{4\pi t}\Big)^{3/2} \int_{\mathbb{R}^3} \mathrm{e}^{-|\boldsymbol{x}-t\lambda-\boldsymbol{y}|^2/4t}\,\boldsymbol{a}(\boldsymbol{y})\,d\boldsymbol{y}.$$

for solution of the unsteady Oseen equations

$$\left.\begin{aligned} -\boldsymbol{u}_t + \Delta\boldsymbol{u} + \lambda\,\partial_1\boldsymbol{u} &= 0 \\ \operatorname{div}\boldsymbol{u} &= 0 \end{aligned}\right\} \quad \text{in } \mathbb{R}^3\times(0,\infty) \tag{110}$$

with the initial condition $\boldsymbol{u}(\boldsymbol{x},0) = \boldsymbol{a}(\boldsymbol{x})$ (for $\boldsymbol{x} \in \mathbb{R}^3$), one can derive the estimate

$$\|\partial_t^j \nabla^k \boldsymbol{u}(t)\|_{q;\mathbb{R}^3} \;\le\; C\,t^{-\frac{3}{2}\left(\frac{1}{r}-\frac{1}{q}\right)-\frac{k}{2}}\,\|\boldsymbol{a}\|_r \tag{111}$$

for all $\boldsymbol{a} \in H_r(\mathbb{R}^3)$ and $t \geq 1$. The constant C on the right-hand side depends only on j, k, q, r and λ. Finally, combining appropriately (109) with (111), one can obtain (107) and (108). □

10.2.2 The Case $\mathcal{T} \neq 0$

The operator $A_{\lambda,\mathcal{T}} \equiv A + \lambda B_1 + \mathcal{T} B_2$ is the Oseen operator with the effect of rotation. T. Hishida [64] considered the case $\lambda = 0$ and proved that $A_{0,\mathcal{T}} \equiv A + \mathcal{T} B_2$ generates a C_0–semigroup $\mathrm{e}^{A_{0,\mathcal{T}}t}$ in $H(\Omega)$. M. Geissert, H. Heck and M. Hieber [55] also considered $\lambda = 0$ and proved that $A_{0,\mathcal{T}}$ generates a C_0–semigroup $\mathrm{e}^{A_{0,\mathcal{T}}t}$ in $H_q(\Omega)$ for $1 < q < \infty$. The case $\lambda \neq 0$ was studied by Y. Shibata in [104], whose main finding is given next.

Theorem 16. *Let $1 < r \le q < \infty$. The operator $A_{\lambda,\mathcal{T}}$ generates a C_0 semigroup $\mathrm{e}^{A_{\lambda,\mathcal{T}}t}$ in $H_q(\Omega)$. Moreover, it satisfies the same inequalities (107) and (108) as the semigroup $\mathrm{e}^{A_{\lambda,0}t}$.*

Sketch of the proof; *see* [104] *for the details.* One begins to study the linear Cauchy problem, defined by the equations

$$\left.\begin{aligned} -\boldsymbol{u}_t + \Delta\boldsymbol{u} + \lambda\,\partial_1\boldsymbol{u} + \mathcal{T}(\boldsymbol{e}_1\times\boldsymbol{x}\cdot\nabla\boldsymbol{u} - \boldsymbol{e}_1\times\boldsymbol{u}) \;&=\; \nabla p \\ \operatorname{div}\boldsymbol{u} &= 0 \end{aligned}\right\} \quad \text{in } \mathbb{R}^3\times(0,\infty) \tag{112}$$

and the initial condition $\boldsymbol{u}(\boldsymbol{x},0)=\boldsymbol{a}(\boldsymbol{x})$. The solution $\boldsymbol{u}(\boldsymbol{x},t)$ is expressed by means of the Fourier transform and the solution of the corresponding resolvent problem with the resolvent parameter ζ (denoted by $\mathcal{A}_{\mathbb{R}^3,\mathcal{T},\lambda}(\zeta)$) is expressed by the combined Laplace-Fourier transform. Then the estimates of $\|\mathcal{A}_{\mathbb{R}^3,\mathcal{T},\lambda}(\zeta)\|_{q;\mathbb{R}^3}$ and $|\mathcal{A}_{\mathbb{R}^3,\mathcal{T},\lambda}(\zeta)|_{m,q;\mathbb{R}^3}$ in terms of powers of $|\zeta|$ and $\|\boldsymbol{a}\|_q$ are derived. The solution $\mathcal{A}_{\mathbb{R}^3,\mathcal{T},\lambda}(\zeta)$ is then split into the part $\mathcal{A}_1(\zeta)$, which "neglects" the term $\mathcal{T}(\boldsymbol{e}_1\times\boldsymbol{x}\cdot\nabla\boldsymbol{u}-\boldsymbol{e}_1\times\boldsymbol{u})$ in (112), and $\mathcal{A}_2(\zeta)$, which is a correction due to this term. While the estimates of $\mathcal{A}_1(\zeta)$ are shown in a similar way and for the same values of ζ as in the proof of Theorem 15, the estimates of $\mathcal{A}_2(\zeta)$ impose sharper restrictions on ζ and hold only for $\zeta\in\mathbb{C}_+:=\{\zeta\in\mathbb{C};\ \mathrm{Re}\,\zeta>0\}$. However, remarkably enough, in [104], subtle estimates of $\mathcal{A}_2(\gamma+\mathrm{i}s)$ (for $\gamma>0$ and $s\in\mathbb{R}$) are derived independent of γ (for $0<\gamma<\gamma_0$), provided $\boldsymbol{a}$ has a support in $B_R(\mathbf{0})$ for $R>0$. The essential role in the expression of the solution of (112) is played by the integral of $\mathcal{A}_2(\zeta)$ on the line $\{\zeta=\gamma+\mathrm{i}s;\ s\in\mathbb{R}\}$, parallel to the imaginary axis. The estimates independent of γ enable one to pass to the limit for $\gamma\to 0$. Then the appropriate cut-off function procedure, and the limit process for $R\to\infty$, lead to an expression, that confirms that $\boldsymbol{u}(\,.\,,t)$ depends on the initial datum $\boldsymbol{a}$ through a C_0–semigroup.

One can immediately observe from the shape of the spectrum of the operator $A_{\lambda,\mathcal{T}}$ (see Theorems 13 and 14) that $A_{\lambda,\mathcal{T}}$ is not a sectorial operator in $H_q(\Omega)$. Thus, unlike the case $\mathcal{T}=0$, the semigroup generated by $A_{\lambda,\mathcal{T}}$ is only a C_0–semigroup and not an analytic semigroup.

The idea used to derive estimates analogous to (107) and (108) is similar to that employed in the proof of Theorem 15. However, in contrast to the case $\mathcal{T}=0$ (when, expressing the solution by the line integral on a line parallel to the imaginary axis, one especially needs to control the behavior of the resolvent for the values of the resolvent parameter ζ near $\mathbf{0}$), the case $\mathcal{T}\neq 0$ requires the control of the resolvent "uniformly" on the whole line. This is caused by the fact that as the line approaches the imaginary axis in the considered limit procedure, it approaches the spectrum of operator $A_{\lambda,\mathcal{T}}$ not only in the neighborhood of $\mathbf{0}$, but in the neighborhood of the infinitely many points $\mathrm{i}k\mathcal{T}$, $k\in\mathbb{Z}$. □

10.3 Existence and Uniqueness of Solutions of the Initial–Boundary Value Problem

This subsection presents a brief survey of results on the existence and uniqueness of weak and strong solutions to the initial–boundary value problem, consisting of equation (5) and the initial condition

$$\boldsymbol{v}(\boldsymbol{x},0)\ =\ \boldsymbol{a}(\boldsymbol{x})\qquad \text{for } \boldsymbol{x}\in\Omega. \tag{113}$$

Referring to other chapters in this Handbook for a detailed analysis, here only those results are recalled that are relevant to our study. The definition of the weak solution, in the unsteady case, is analogous to that provided for the steady-sate problem (8)–(7). More precisely, $\boldsymbol{v}$ is called a *weak solution* to problem (5)–(113) if

(i) $\boldsymbol{v} \in L^\infty(0,T;H(\Omega)) \cap L^2(0,T;D^{1,2}(\Omega))$, for all $T>0$;

(ii) $\lim_{t\to 0^+} \|\boldsymbol{v}(t)-\boldsymbol{a}\|_2 = 0$;

(iii) $\boldsymbol{v}$ satisfies (5) in the sense of distributions.

10.3.1 The Case $\mathcal{T}=0$

In absence of rotation, existence results can be found in many works. They mostly concern the Navier–Stokes equations, but their extension to the more general problem (5), (113) with $\mathcal{T}=0$ is rather straightforward.

The first results on the global in time existence of weak solutions, $\boldsymbol{v}$, assuming the initial velocity $\boldsymbol{a} \in H(\Omega)$, are due to J. Leray [84] (for $\Omega=\mathbb{R}^3$) and E. Hopf [66] (for arbitrary open set $\Omega \subset \mathbb{R}^3$). A more recent and detailed presentation of these classical results can be found, e.g., in the book [106] or in the survey paper [39]. In particular, one shows the existence of a weak solution for any $\boldsymbol{a} \in H(\Omega)$ and $\boldsymbol{f} \in L^2(0,T;\, \mathcal{D}_0^{-1,2}(\Omega))$ (and $\boldsymbol{v} \equiv \boldsymbol{0}$ at $\partial\Omega$). However, the uniqueness of such solutions in the same class of existence remains an open problem. The weak solution is known to be unique if, in addition, it is in $L^r(0,T;\, L^s(\Omega))$, where $2 \le r \le \infty$, $3 \le s \le \infty$, $2/r + 3/s \le 1$. More precisely, if $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$ are two weak solutions, with $\boldsymbol{v}_1$ in the class $L^r(0,T;\, L^s(\Omega))$ above, and $\boldsymbol{v}_2$ satisfying the so called energy inequality (see (118) with $\boldsymbol{v}_0 \equiv \boldsymbol{0}$ and $s=0$), then $\boldsymbol{v}_1 = \boldsymbol{v}_2$. As the solutions in the class $L^r(0,T;\, L^s(\Omega))$ satisfy the energy inequality automatically, one can speak of "uniqueness in the class $L^r(0,T;\, L^s(\Omega))$".

Following [106], a weak solution $\boldsymbol{v}$ in the class $L^r(0,T;\, L^s(\Omega))$ with r, and s as above, is called a *strong solution.* In addition to be unique, strong solutions are also known to be "smooth" (= regular), provided that the body force $\boldsymbol{f}$ is either a potential vector field (and can be therefore absorbed by the pressure term) or "sufficiently smooth". (See e.g. [39] for more details.) In particular, if $\partial\Omega$ is of class C^2 and $\boldsymbol{f} \in L^2(0,T;\, L^2(\Omega))$ then the strong solution $\boldsymbol{v}$ belongs to $C((\epsilon,T);H(\Omega)) \cap L^2(\epsilon,T;\, W^{2,2}(\Omega))$ for any $\epsilon \in (0,T)$. (It depends on the regularity of the initial velocity $\boldsymbol{a}$ whether $\epsilon=0$ can also be considered.) For initial velocity $\boldsymbol{a}$ and body force $\boldsymbol{f}$ in appropriate function spaces and of "arbitrary size", strong solutions are known to exist in some time interval $(0,T_0)$, but it is not known whether one can take $T_0=\infty$, in general. If, however, the size of the data is sufficiently restricted, then one can show $T_0=\infty$. There exists a vast literature on the subject dealing with various types of domains and different choices of functions spaces for $\boldsymbol{a}$ and $\boldsymbol{f}$, starting from the pioneering and fundamental papers of A. A. Kiselev and O. A. Ladyzhenskaya, G. Prodi, and H. Fujita and T. Kato in the early 60's, and continuing with J.G. Heywood (1980), T. Miyakawa (1982), Y. Giga (1986), H. Amann (2000) and R. Farwig, H. Sohr and W. Varnhorn (2009). Among these papers, especially [62] (by Heywood) [91] (by Miyakawa), and [1] (by Amann) deal with the Navier-Stokes problems in exterior domains. An important result concerning the length of the time interval $(0, T_0)$ where a strong solution exists without restriction on the "size" of the data, states that (e.g. [1] or [61]), that if $\boldsymbol{f}$ is e.g. in $L^2(0, \infty;\, L^2(\Omega))$ then either $T_0 = \infty$, or else $|\boldsymbol{v}(\,.\,, t)|_{1,2} \to \infty$ for $t \to T_0{}^-$.

10.3.2 The Case $\mathcal{T} \neq 0$

The existence of a weak solution to the problem (5), (113) with $\mathcal{T} \neq 0$ has been proven by W. Borchers [7]. It also follows from a more general result proven in [96] on the existence of weak solutions in domains with moving boundaries. Recall that the weak solution is a function in the same class as for the case $\mathcal{T} = 0$, i.e. belongs to $L^\infty(0, T;\, H(\Omega)) \cap L^2(0, T;\, W_0^{1,2}(\Omega))$.

Regarding the local in time existence of a strong solution to the problem (5), (113) with $\mathcal{T} \neq 0$, only a few results are available. Below the contributions of T. Hishida [64], G. P. Galdi and A. L. Silvestre [42] and P. Cumsille and M. Tucsnak [11] are explained. They all concern the case when the motion of body $\mathscr{B}$ in the fluid reduces to the rotation and the translational velocity is zero. It means that the term $\lambda\, \partial_1 \boldsymbol{v}$ in the momentum equation $(5)_1$ vanishes.

T. Hishida [64] assumes that the body force $\boldsymbol{f}$ is zero and the initial velocity is in $\mathsf{D}(A^{1/4})$ and proves the existence of a solution in the class $C([0, T_0];\, \mathsf{D}(A^{1/4})) \cap C((0, T_0];\, \mathsf{D}(A))$ for certain $T_0 > 0$. Recall that A denotes the Stokes operator. Hishida's proof is based on a non-trivial generalization of the semigroup method, formerly used by Fujita and Kato [32].

G. P. Galdi and A. L. Silvestre [42] also deal with the case of the zero body force $\boldsymbol{f}$. They assume that the initial velocity $\boldsymbol{a}$ is in $W^{2,2}(\Omega)$, satisfies $\operatorname{div} \boldsymbol{a} = 0$ and $\boldsymbol{e}_1 \times \boldsymbol{x} \cdot \nabla \boldsymbol{a} \in L^2(\Omega)$, and they obtain a solution in $C([0, T_0];\, W^{1,2}(\Omega)) \cap C((0, T_0);\, W^{2,2}(\Omega))$ for some $T_0 > 0$. The proof is based on the construction of classical Faedo-Galerkin approximations in Ω_R, getting a solution in Ω_R, and letting $R \to \infty$. The procedure is, however, not standard because of the "troublesome" term $\mathcal{T} \boldsymbol{e}_1 \times \boldsymbol{x} \cdot \nabla \boldsymbol{v}$ whose influence has to be controlled.

P. Cumsille and M. Tucsnak [11] consider the equations of motion of the viscous incompressible fluid around body $\mathscr{B}$ in a frame in which the velocity of the fluid vanishes in infinity and the body is rotating with a constant angular velocity about one of the coordinate axes. Thus, the domain filled in by the fluid is time dependent and it is denoted by $\Omega(t)$. The authors consider a body force $\boldsymbol{f}$ locally square integrable from $(0, \infty)$ to $W^{1,\infty}(\mathbb{R}^3)$ and the no-slip boundary condition for the velocity on $\partial\Omega(t)$. The main theorem from [11] says that if the initial velocity $\boldsymbol{a}$ is in $W_0^{1,2}(\Omega(0))$ and it is divergence-free then there exists $T_0 > 0$ and a unique strong solution $\boldsymbol{u} \in L^2(0, T_0;\, W^{2,2}(\Omega(t))) \cap C([0, T_0];\, W^{1,2}(\Omega(t)))$ such that $\boldsymbol{u}_t \in L^2(0, T_0;\, L^2(\Omega(t)))$. Moreover, either T_0 can be extended up to infinity or the norm of $\boldsymbol{u}$ in $W^{1,2}(\Omega(t))$ tends to infinity for $t \to T_0-$. In order to obtain a problem in a fixed exterior domain, the authors use a change of variables which coincides with the rotation in the neighborhood of body $\mathscr{B}$, but it equals the identity far from the body. Then they solve the problem in the fixed exterior domain Ω. Using the relations between the solutions of the equations in the frame considered in [11] on one hand, and the body fixed frame on the other hand, the result of Cumsille and Tucsnak from [11] can be reformulated in terms of solution to the problem (5), (113) as follows: given $\boldsymbol{a} \in H(\Omega) \cap W_0^{1,2}(\Omega)$, there exists $T_0 > 0$ and a unique solution $\boldsymbol{v}$ of the problem (5), (113) such that

$$\left.\begin{aligned} \boldsymbol{v} \ &\in \ L^2\left(0, T_0;\, W^{2,2}(\Omega)\right) \cap C\left([0, T_0];\, W^{1,2}(\Omega)\right), \\ \boldsymbol{v}_t - \mathcal{T}(\boldsymbol{e}_1 \times \boldsymbol{x} \cdot \nabla \boldsymbol{v} - \boldsymbol{e}_1 \times \boldsymbol{v}) \ &\in \ L^2\left(0, T_0;\, L^2(\Omega)\right). \end{aligned}\right\} \tag{114}$$

Cumsille & Tucsnak's result is applied in subsection 10.5. Since it is also used in the case $\lambda \neq 0$, we note that following the proof in [11] and using the fact that the translation-related term $\lambda\,\partial_1 \boldsymbol{v}$ in the first equation in (5) can be considered to be a subordinate perturbation of $\Delta \boldsymbol{v}$, the above formulated result can be extended to the case when it also includes the translation of $\mathscr{B}$ in the direction parallel to the axis of rotation. Consequently, the result also holds for the equations in (5) with the term $\lambda\,\partial_1 \boldsymbol{v}$ and the second inclusion in (114) can be modified:

$$\boldsymbol{v}_t - \lambda\,\partial_1 \boldsymbol{v} - \mathcal{T}(\boldsymbol{e}_1 \times \boldsymbol{x} \cdot \nabla \boldsymbol{v} - \boldsymbol{e}_1 \times \boldsymbol{v}) \ \in \ L^2\left((0, T_0;\, L^2(\Omega)\right). \tag{115}$$

10.4 Attractivity and Asymptotic Stability with Smallness Assumptions on v_0

Recall that $\boldsymbol{v}_0$ denotes (the velocity field of) a solution to problem (8) (i.e. a steady-state solution to problem (5)), and that its associated "perturbation" $\boldsymbol{v}$ satisfies (99) and (100). Since in studying long-time behavior the dependence of $\boldsymbol{v}$ on time is more relevant than the one on spatial variables, in the following considerations, $\boldsymbol{v}(\boldsymbol{x}, t)$ is often abbreviated to $\boldsymbol{v}(t)$. Thus, for example, the initial condition (113) may be written in the form

$$\boldsymbol{v}(0) \ = \ \boldsymbol{a}. \tag{116}$$

10.4.1 The Case $\mathcal{T} = 0$

A number of results concern the long-time behavior of the unsteady perturbations $\boldsymbol{v}$ in the class of weak solutions. The first relevant contribution in this direction is due to K. Masuda [90], who assumes that $\boldsymbol{v}_0$ is continuously differentiable, $\nabla \boldsymbol{v}_0 \in L^3(\Omega)$ along with the smallness condition which, according to our notation, yields

$$\sup_{\boldsymbol{x} \in \Omega} \ \lambda\,|\boldsymbol{x}|\,|\boldsymbol{v}_0(\boldsymbol{x})| \ < \ \frac{1}{2}. \tag{117}$$

The perturbed unsteady solution is supposed to satisfy the momentum equation in (5) with a perturbed body force. Thus, the corresponding perturbation $\boldsymbol{v}$ satisfies (99), (100), with an additional right-hand side $\boldsymbol{f}'$ in (99), representing the perturbation to the steady body force $\boldsymbol{f}$. The Helmholtz-Weyl projected function $P\boldsymbol{f}'$ is assumed to be in $C^1\left([0, \infty);\, H(\Omega)\right) \cap L^1\left(0, \infty;\, H(\Omega)\right)$ and such that $\sup_{t>0} \int_t^{t+1} \|(d/ds)P\boldsymbol{f}'(s)\|_2^2\, ds +$

$\int_0^\infty s^{1/2}\,\|(d/ds)P\boldsymbol{f}'(s)\|_2\,ds < \infty$. As for the class of perturbations, the author assumes that $\boldsymbol{v}$ is a weak solution to the problem (99) (with a nonzero $\boldsymbol{f}'$ on its right-hand side), (100), and (116), with initial data $\boldsymbol{a} \in H(\Omega)$, and satisfies the so called "strong energy inequality", namely,

$$\tfrac{1}{2}\|\boldsymbol{v}(t)\|_2^2 \le \tfrac{1}{2}\|\boldsymbol{v}(s)\|_2^2 - \int_s^t \left[\lambda(\boldsymbol{v}(\tau)\cdot\nabla\boldsymbol{v}_0, \boldsymbol{v}(\tau)) + |\boldsymbol{v}(\tau)|_{1,2}^2 + (\boldsymbol{f}'(\tau), \boldsymbol{v}(\tau))\right]\,d\tau\,, \quad (118)$$

for a.a. $s > 0$ (including $s = 0$) and all $t \in [s, T]$, arbitrary $T > 0$. Notice that the latter is formally obtained by multiplying equation (99) by $\boldsymbol{v}$ and integrating over $\Omega \times (s, t)$, and relaxing the equality sign to the inequality one. Under the above conditions, Masuda shows that there exists $T_* > 0$ such that $\boldsymbol{v}(t)$ becomes regular for $t > T_*$, and decays at the following rate

$$|\boldsymbol{v}(t)|_{1,2} \le Ct^{-1/4}\,,\quad \|\boldsymbol{v}(t)\|_\infty \le Ct^{-1/8}\,,\quad \text{for all } t > T_*. \quad (119)$$

The proof uses (118) and the assumptions on the integrability of $\boldsymbol{f}'$ to deduce, first, that $\boldsymbol{v}(t)$ is "small" for large t. Then, combining this with the estimates of $A^{1/2}\boldsymbol{v}$ one shows that $\boldsymbol{v}(t)$ is regular and tends to zero for $t \to \infty$ in the norm $|\,.\,|_{1,2}$. The rate of decay is calculated from the energy-type inequality, satisfied by $\boldsymbol{v}_t$. The author also generalizes these results to the case when the unperturbed solution $\boldsymbol{v}_0$ is time-dependent. It should be noted that in [90] no assumption on the size of the initial perturbation $\boldsymbol{a}$ is used: it can be arbitrarily large. However, nothing can be said about the behavior of $\boldsymbol{v}(t)$ for $t \in (0, T_*)$.

If $\boldsymbol{v}_0 \equiv \boldsymbol{0}$, the decay rates (119) are sharpened by J.G. Heywood in [62].

The above results have been further elaborated on by P. Maremonti in [88]. Maremonti studied the attractivity of steady as well as unsteady solutions $\boldsymbol{v}_0$ to problem (5) in the same class of weak solutions considered by Masuda with $\boldsymbol{f}' \equiv \boldsymbol{0}$. In particular, for the case $\boldsymbol{v}_0$ steady, he shows the following decay rates

$$\|\boldsymbol{v}_t(t)\|_2 \le C\,t^{-1}\,,\quad |\boldsymbol{v}(t)|_{1,2} \le C\,t^{-1/2}\,,\quad \|\boldsymbol{v}(t)\|_\infty \le C\,t^{-1/2}\,,$$

thus improving and extending the analogous finding of [90] and [62]. Instead of condition (117), the author assumes that the maximum of certain variational problem involving $\boldsymbol{v}_0$ is not "too large". The latter condition is certainly satisfied if $\boldsymbol{v}_0$ is sufficiently regular and obeys (117).

The somehow more complicated question of *asymptotic* stability of $\boldsymbol{v}$ in the L^2-norm was first addressed by P. Maremonti in [89]. In particular, he shows that all $\boldsymbol{v}$ in the class of weak solutions, with $\boldsymbol{a} \in H(\Omega)$ and satisfying the strong energy inequality (118) with $\boldsymbol{f}' \equiv \boldsymbol{0}$, must decay to 0 in the L^2-norm, provided that the magnitude of $\boldsymbol{v}_0$ is restricted in the same way as specified in [88] and discussed earlier on.

An important contribution to the studies of the asymptotic stability of the steady solution $\boldsymbol{v}_0$ was also made by T. Miyakawa and H. Sohr in [92]. The authors show that if the basic steady solution $\boldsymbol{v}_0$ of (5) is such that $\boldsymbol{v}_0 \in L^\infty(\Omega)$, $\nabla\boldsymbol{v}_0 \in L^3(\Omega)$ and the smallness condition (117) is satisfied, and if, in addition, the perturbation $\boldsymbol{f}'$ to the body force $\boldsymbol{f}$ is in $L^2([0,T); H(\Omega))$ for all $T > 0$ and in $L^1([0,\infty); H(\Omega))$, then the L^2-norm of each weak solution $\boldsymbol{v}$ to problem (99), (100) satisfying the energy inequality (118) tends to 0 for $t \to \infty$. In [92] it is also shown that the class of such weak solutions is not empty, thus solving a problem left open in [89] and partially solved in [35]. Further results concerning the L^2–decay of the perturbation $\boldsymbol{v}(t)$ (as a weak solution to (99), (100)) for $t \to \infty$ are provided in the paper [8] by W. Borchers and T. Miyakawa: the authors assume that the steady solution $\boldsymbol{v}_0$ of (5) is in $L^3(\Omega)$, $\nabla \boldsymbol{v}_0 \in L^3(\Omega)$, the smallness condition (117) is satisfied and the perturbation $\boldsymbol{f}'$ to the body force $\boldsymbol{f}$ is in $L^2_{loc}([0,\infty);\, H(\Omega)) \cap L^1(0,\infty;\, H(\Omega)) \cap L^1\big(0,\infty;\, \mathcal{D}_0^{-1,2}(\Omega)\big)$. They show that then the L^2-norm of each weak solution $\boldsymbol{v}$ to problem (99), (100) obeying (118) tends to 0 for $t \to \infty$. Moreover, if $\|\mathrm{e}^{\mathscr{L}t}\boldsymbol{a}\|_2 = O(t^{-\alpha})$ for some $\alpha > 0$ then $\|\boldsymbol{v}(t)\|_2 = O\big((\ln t)^{\epsilon-1/2}\big)$ for any $\epsilon > 0$. (Here, $\mathrm{e}^{\mathscr{L}t}$ denotes the semigroup generated by operator $\mathscr{L}$; see Section 10.5.1.) The results of [8] are generalized by the same authors to the case of n–space dimensions ($n \geq 3$) in [9].

Even sharper rates of decay of the norms $\|\boldsymbol{v}(t)\|_r$ ($2 \leq r \leq \infty$) and $\|\nabla \boldsymbol{v}(t)\|_r$ ($2 \leq r \leq 3$) were obtained by H. Kozono in [77], provided the perturbation $\boldsymbol{f}'$ to the body force is in $L^1(0,\infty;\, L^2(\Omega)) \cap C((0,\infty);\, L^2(\Omega))$ and decays like t^{-1} for $t \to \infty$. Kozono does not use any condition of smallness of the basic flow $\boldsymbol{v}_0$ or its initial perturbation $\boldsymbol{a}$, but needs $\boldsymbol{v}_0$ in Serrin's class $L^r(0,\infty;\, L^s(\Omega))$ ($2/r + 3/s = 1$, $3 < s \leq \infty$). This implies that $\boldsymbol{v}_0$ is in fact a strong solution and it is in a suitable sense small for large t. Obviously, the only time-independent solution in the considered Serrin class is $\boldsymbol{v}_0 = \boldsymbol{0}$.

There exists a series of results on stability of solution $\boldsymbol{v}_0$ in the class of strong unsteady perturbations, which, unlike the cited papers [90], [62], [88], [8], [9] and [77], provide an information on the size of the perturbations at *all times* $t > 0$, and not just for "large" t. However, on the other hand, the initial value of the perturbation is always required to be "small" as well as $\boldsymbol{v}_0$ is also supposed to be "sufficiently small" in appropriate norms. The first results of this kind come from the early seventies of the 20th century and new results on this topic still appear.

The next paragraphs contain the sketch of the main steps to obtain a result of the above type. Assume that $\boldsymbol{v}$ is a strong solution to problem (99), (100) in the time interval $(0, T_0)$, for some $T_0 > 0$. Multiplying the first equation in (99) (where $\boldsymbol{v}' = \boldsymbol{v}$) by $\boldsymbol{v}$ and integrating by parts over Ω, one obtains

$$\begin{aligned}\frac{1}{2}\,\frac{d}{dt}\,\|\boldsymbol{v}\|_2^2 + |\boldsymbol{v}|_{1,2}^2 \;&= \lambda\,(\boldsymbol{v}\cdot\nabla\boldsymbol{v}_0, \boldsymbol{v}) \;\leq\; \lambda\,|\boldsymbol{v}_0|_{1,3/2}\,\|\boldsymbol{v}\|_6^2 \\ &\leq C_7^2\lambda\,|\boldsymbol{v}_0|_{1,3/2}\,|\boldsymbol{v}|_{1,2}^2. \end{aligned}\tag{120}$$

(The norm $\|\boldsymbol{v}\|_6$ has been estimated by Sobolev's inequality: $\|\boldsymbol{v}\|_6 \leq C_7\,|\boldsymbol{v}|_{1,2}$, see e.g. [47, p. 54].) Multiplying the first equation in (99) by $A\boldsymbol{v} \equiv P_2\Delta\boldsymbol{v}$ and integrating over Ω, one obtains

$$\begin{aligned}\frac{1}{2}\,\frac{d}{dt}\,|\boldsymbol{v}|_{1,2}^2 + \|A\boldsymbol{v}\|_2^2 \;&=\; \lambda\,\big((-\partial_1\boldsymbol{v} + \boldsymbol{v}_0\cdot\nabla\boldsymbol{v} + \boldsymbol{v}\cdot\nabla\boldsymbol{v}_0 + \boldsymbol{v}\cdot\nabla\boldsymbol{v}),\, A\boldsymbol{v}\big)\; d\boldsymbol{x} \\ &\leq \frac{1}{4}\,\|A\boldsymbol{v}\|_2^2 + 4\lambda^2\left(\|\partial_1\boldsymbol{v}\|_2^2 + \|\boldsymbol{v}_0\cdot\nabla\boldsymbol{v}\|_0^2 + \|\boldsymbol{v}\cdot\nabla\boldsymbol{v}_0\|_2^2 + \|\boldsymbol{v}\cdot\nabla\boldsymbol{v}\|_2^2\right)\,. \end{aligned}\tag{121}$$

The first term on the right-hand side can be absorbed by the left hand side. The other terms on the right-hand side can be estimated by means of the inequalities

$$|\boldsymbol{v}|_{1,6} \;\leq\; C\,\|\nabla \boldsymbol{v}\|_{1,2} \;=\; C\,\left(|\boldsymbol{v}|_{1,2}^2 + |\boldsymbol{v}|_{2,2}^2\right)^{1/2} \;\leq\; C\,\left(\|A\boldsymbol{v}\|_2^2 + |\boldsymbol{v}|_{1,2}^2\right)^{1/2},$$

where the first one follows from the continuous imbedding $W^{1,2}(\Omega) \hookrightarrow L^6(\Omega)$ and the second one follows e.g. from [47, pp. 322–323]. Thus, if $\mathcal{Y}[\boldsymbol{v}]$ is defined by the formula

$$\mathcal{Y}[\boldsymbol{v}] \;:=\; \|A\boldsymbol{v}\|_2^2 + |\boldsymbol{v}|_{1,2}^2,$$

then

$$\|\partial_1 \boldsymbol{v}\|_2^2 \leq 2\lambda^2\,|\boldsymbol{v}|_{1,2}^2,$$

$$\begin{aligned}\|\boldsymbol{v}_0\cdot\nabla\boldsymbol{v}\|_2^2 &\leq \|\boldsymbol{v}_0\|_6^2\;|\boldsymbol{v}|_{1,2}\;|\boldsymbol{v}|_{1,6} \;\leq\; C\,|\boldsymbol{v}_0|_{1,2}^2\;|\boldsymbol{v}|_{1,2}\;\mathcal{Y}[\boldsymbol{v}]^{1/2}\\ &\leq \epsilon\,\|A\boldsymbol{v}\|_2^2 + C(\epsilon)\;\left(|\boldsymbol{v}_0|_{1,2}^2 + |\boldsymbol{v}_0|_{1,2}^4\right)\;|\boldsymbol{v}|_{1,2}^2,\end{aligned}$$

$$\|\boldsymbol{v}\cdot\nabla\boldsymbol{v}_0\|_2^2 \leq |\boldsymbol{v}_0|_{1,3}^2\;\|\boldsymbol{v}\|_6^2 \;\leq\; C_7^2\,|\boldsymbol{v}_0|_{1,3}^2\;|\boldsymbol{v}|_{1,2}^2,$$

$$\|\boldsymbol{v}\cdot\nabla\boldsymbol{v}\|_2^2 \leq \|\boldsymbol{v}\|_6^2\;|\boldsymbol{v}|_{1,3}^2 \;\leq\; C_7^2\,|\boldsymbol{v}|_{1,2}^3\;|\boldsymbol{v}|_{1,6} \;\leq\; C\,|\boldsymbol{v}|_{1,2}^3\;\mathcal{Y}[\boldsymbol{v}]^{1/2} \;\leq\; C\,|\boldsymbol{v}|_{1,2}^2\;\mathcal{Y}[\boldsymbol{v}].$$

Employing these inequalities into (121), and choosing e.g. $\epsilon = \frac{1}{4}$, one gets

$$\frac{d}{dt}\,|\boldsymbol{v}|_{1,2}^2 + \|A\boldsymbol{v}\|_2^2 \;\leq\; C_8\lambda^2\,\left(1 + |\boldsymbol{v}_0|_{1,2}^2 + |\boldsymbol{v}_0|_{1,2}^4\right)\,|\boldsymbol{v}|_{1,2}^2 + C_9\lambda^2\,|\boldsymbol{v}|_{1,2}^2\;\mathcal{Y}[\boldsymbol{v}]. \tag{122}$$

Adding the inequalities (120) (multiplied by 2) and (122) (multiplied by $\alpha > 0$), and passing everything to the left-hand side, one obtains

$$\begin{aligned}&\frac{d}{dt}\,\left(\|\boldsymbol{v}\|_2^2 + \alpha\,|\boldsymbol{v}|_{1,2}^2\right)\\ &\quad + |\boldsymbol{v}|_{1,2}^2\;\left[2 - 2C_7^2\lambda\,|\boldsymbol{v}_0|_{1,3/2} - C_8\lambda^2\alpha\,\left(1 + |\boldsymbol{v}_0|_{1,2}^2 + |\boldsymbol{v}_0|_{1,2}^4\right) - C_9\lambda^2\alpha\,|\boldsymbol{v}|_{1,2}^2\right]\\ &\quad + \|A\boldsymbol{v}\|_2^2\;\left[\alpha - C_9\lambda^2\alpha\,|\boldsymbol{v}|_{1,2}^2\right] \;\leq\; 0.\end{aligned}$$

This implies that

$$\begin{aligned}&\frac{d}{dt}\,\left(\|\boldsymbol{v}\|_2^2 + \alpha\,|\boldsymbol{v}|_{1,2}^2\right) + |\boldsymbol{v}|_{1,2}^2\;\left[2 - 2C_7^2\lambda\,|\boldsymbol{v}_0|_{1,3/2} - C_8\lambda^2\alpha\,\left(1 + |\boldsymbol{v}_0|_{1,2}^2 + |\boldsymbol{v}_0|_{1,2}^4\right)\right.\\ &\quad \left. - C_9\lambda^2\,\left(\|\boldsymbol{v}\|_2^2 + \alpha\,|\boldsymbol{v}|_{1,2}^2\right)\right] + \|A\boldsymbol{v}\|_2^2\;\left[\alpha - C_9\lambda^2\,\left(\|\boldsymbol{v}\|_2^2 + \alpha\,|\boldsymbol{v}|_{1,2}^2\right)\right] \;\leq\; 0.\end{aligned} \tag{123}$$

This inequality shows that if

$$2C_7^2\lambda\,|\boldsymbol{v}_0|_{1,3/2} + C_8\lambda^2\alpha\,\left(1 + |\boldsymbol{v}_0|_{1,2}^2 + |\boldsymbol{v}_0|_{1,2}^4\right) \;<\; 2 \tag{124}$$

and $\|\boldsymbol{v}\|_2^2 + \alpha\,|\boldsymbol{v}|_{1,2}^2$ is initially so small that

$$\begin{aligned}&C_9\lambda^2\,\left(\|\boldsymbol{a}\|_2^2 + \alpha\,|\boldsymbol{a}|_{1,2}^2\right)\\ &\quad <\; \min\left\{2 - 2C_7^2\lambda\,|\boldsymbol{v}_0|_{1,3/2} - C_8\lambda^2\alpha\,\left(1 + |\boldsymbol{v}_0|_{1,2}^2 + |\boldsymbol{v}_0|_{1,2}^4\right);\;\alpha\right\}\end{aligned} \tag{125}$$

(recall that $\boldsymbol{v}(0) = \boldsymbol{a}$) then $\|\boldsymbol{v}(t)\|_2^2 + \alpha\,|\boldsymbol{v}(t)|_{1,2}^2$ is non-decreasing for t in some right neighborhood of 0. This consideration can be simply extended, by the bootstrapping argument, to the whole interval of existence of the strong solution $\boldsymbol{v}$ (let it be $(0, T_0)$) so that one obtains: $\|\boldsymbol{v}(t)\|_2^2 + \alpha\,|\boldsymbol{v}(t)|_{1,2}^2 < \|\boldsymbol{a}\|_2^2 + \alpha\,|\boldsymbol{a}|_{1,2}^2$ for all $t \in (0, T_0)$. This shows, among other things, that the norm $\|\boldsymbol{v}(t)\|_{1,2}$ cannot blow up when $t \to T_0-$. Consequently, $T_0 = \infty$ and the inequality $\|\boldsymbol{v}(t)\|_2^2 + \alpha\,|\boldsymbol{v}(t)|_{1,2}^2 < \|\boldsymbol{a}\|_2^2 + \alpha\,|\boldsymbol{a}|_{1,2}^2$ holds for all $t \in (0, \infty)$. Note that if

$$C_7^2\lambda\,|\boldsymbol{v}_0|_{1,3/2} \;<\; 1 \tag{126}$$

then one can choose $\alpha > 0$ so small that (124) holds. (α is further supposed to be chosen in this way.)

Integrating inequality (123) with respect to t, one can also derive an information on the integrability of $\mathcal{Y}[\boldsymbol{v}(t)]$ and on the asymptotic decay of $|\boldsymbol{v}(t)|_{1,2}$. Thus, also including the information on the uniqueness of strong solutions, and using the result of [89], one obtains the following Lyapunov-type asymptotic stability of $\boldsymbol{v}_0$ in the $W^{1,2}$-norm.

Theorem 17. *Suppose the steady solution $\boldsymbol{v}_0$ to the problem (8) satisfies conditions (98) and (126), and $\alpha > 0$ is chosen so that (124) holds. Then, if $\boldsymbol{a} \in H(\Omega) \cap W_0^{1,2}(\Omega)$ satisfies (125), problem (99), (100) with the initial condition $\boldsymbol{v}(0) = \boldsymbol{a}$ has a unique strong solution $\boldsymbol{v}$ on the time interval $(0, \infty)$. Furthermore, there exists $C_{10} > 0$ such that this solution satisfies*

$$\|\boldsymbol{v}(t)\|_2^2 + \alpha\,|\boldsymbol{v}(t)|_{1,2}^2 + C_{10} \int_0^t \left(|\boldsymbol{v}(s)|_{1,2}^2 + \alpha\,\|A\boldsymbol{v}(s)\|_2^2\right)\,ds \;\leq\; \|\boldsymbol{a}\|_2^2 + \alpha\,|\boldsymbol{a}|_{1,2}^2 \qquad (127)$$

for all $t > 0$ and

$$\lim_{t\to\infty}\, \|\boldsymbol{v}(t)\|_{1,2} \;=\; 0. \qquad (128)$$

The noteworthy assumption in the above theorem is condition (126) of "sufficient smallness" of the solution $\boldsymbol{v}_0$.

The ideas of proof described previously and similar energy-type considerations have been applied to many other studies of stability or instability of steady-state solutions to Navier–Stokes and related equations. Concerning flows in exterior domains, the readers are referred e.g. to [60; 61; 33; 34; 36].

A different approach, based on a representation of a solution by means of semigroups generated by the operators $A_{\lambda,0}$ or $\mathscr{L}$ and on estimates of the semigroups, has been employed by H. Kozono and T. Ogawa [75], H. Kozono and M. Yamazaki [76] and Y. Shibata [102]. In particular, Kozono and Yamazaki [76] study the flow in an exterior "smooth" domain Ω in $\mathbb{R}^n$ ($n \geq 3$), under the assumption that the translational velocity of the moving body is zero, which in our notation means $\lambda = 0$ in the first equation in (99). The steady-state solution $\boldsymbol{v}_0$ is supposed to belong to $L^{n,\infty}_\sigma(\Omega) \cap L^\infty(\Omega)$ and its gradient is supposed to be in $L^{r_*}(\Omega)$ for some $r_* \in (n, \infty)$. (The Lorentz–type space $L^{r,q}_\sigma(\Omega)$ for $1 < r < \infty$ and $1 \leq q \leq \infty$ is defined by means of the real interpolation to be $(H_{r_0}(\Omega), H_{r_1}(\Omega))_{\theta,q}$, where $1 < r_0 < r < r_1 < \infty$ and $1 < \theta < 1$ satisfy $1/r = (1-\theta)/r_0 + \theta/r_1$, see [76]. It is shown in [9] that $L^{r,q}_\sigma(\Omega)$ coincides with the space of all $\boldsymbol{u} \in L^{r,q}(\Omega)$ such that $\operatorname{div} \boldsymbol{u} = 0$ in Ω in the sense of distributions and $\boldsymbol{u} \cdot \boldsymbol{n} = 0$ on $\partial\Omega$ in the sense of traces.) Equations (99) are treated in the equivalent form (101). The operators $\mathscr{L}$ and $\mathscr{N}$ are defined in the introductory part of this section for $n = 3$, but the definition in the general case $n \in \mathbb{N}$, $n \geq 3$ is analogous. In the considered case, $\mathscr{L}$ has the concrete form $\mathscr{L} = A + \lambda B_3$. The operator $\mathscr{L}$ generates a quasi-bounded analytic semigroup in $H_q(\Omega)$ – this is shown in [76] by means of appropriate resolvent estimates which imply that operator $\mathscr{L}$ is sectorial. The strong solution is identified with the mild solution, which satisfies the integral equation

$$\boldsymbol{v}(t) \;=\; \mathrm{e}^{\mathscr{L}t}\boldsymbol{a} + \int_0^\infty \mathrm{e}^{\mathscr{L}(t-s)}\mathscr{N}\boldsymbol{v}(s)\,ds. \tag{129}$$

The solution of this equation is constructed as a limit of a sequence of approximations, which are defined by the equations $\boldsymbol{v}^0 := \mathrm{e}^{-\mathscr{L}t}\boldsymbol{a}$ and

$$\boldsymbol{v}^j(t) \;:=\; \boldsymbol{v}^0(t) + \int_0^t \mathrm{e}^{\mathscr{L}(t-s)}\mathscr{N}\boldsymbol{v}^{j-1}(s)\,ds \qquad (j = 1, 2, 3, \dots).$$

The authors define $K_j := \sup_{0<t<\infty}\; t^{\frac{n}{2}\left(\frac{1}{n}-\frac{1}{r_*}\right)}\;\|\boldsymbol{v}^j(t)\|_{r_*}$ (for $j = 0, 1, 2, \dots$) and show that the sequence $\{K_j\}_{j=0}^\infty$ is bounded if K_0 is sufficiently small. Here, the important role play the estimates of $\int_0^t \mathrm{e}^{\mathscr{L}(t-s)}\mathscr{N}\boldsymbol{v}^j(s)\,ds$ in $L^{r_*}(\Omega)$, which are based on the duality argument and L^p–$L^{r'_*}$ estimates of $\nabla\mathrm{e}^{\mathscr{L}^*(t-s)}$, where $\mathrm{e}^{\mathscr{L}^*(t-s)}$ is the adjoint semigroup to $\mathrm{e}^{\mathscr{L}(t-s)}$ in the dual space $H_{r'_*}(\Omega)$. Similar arguments enable one to derive the inequality

$$\sup_{0<t<\infty}\; t^{\frac{n}{2}\left(\frac{1}{n}-\frac{1}{r_*}\right)}\;\|\boldsymbol{v}^{j+1}(t) - \boldsymbol{v}^j(t)\|_{r_*} \;\le\; C(K_0, n, r_*)^j \qquad (j = 0, 1, 2, \dots),$$

where $C(K_0, n, r_*)$ is less than one for K_0 "small enough". From this, the authors deduce that there exists $\boldsymbol{v}$ such that $t^{\frac{n}{2}\left(\frac{1}{n}-\frac{1}{r_*}\right)}\,\boldsymbol{v}(t) \in BC((0,\infty);\, H_{r_*}(\Omega))$ and

$$\sup_{0<t<\infty}\; t^{\frac{n}{2}\left(\frac{1}{n}-\frac{1}{r_*}\right)}\;\|\boldsymbol{v}^j(t) - \boldsymbol{v}(t)\|_{r_*} \;\longrightarrow\; 0 \qquad \text{as } j \to \infty.$$

The following theorem is the main result proved in [76]:

Theorem 18. *Let $\boldsymbol{v}_0 \in L^{n,\infty}_\sigma(\Omega) \cap L^\infty(\Omega)$, $\nabla\boldsymbol{v}_0 \in L^{r_*}(\Omega)$ for some $r_* \in (n,\infty)$. There exists $\kappa = \kappa(n, r_*) > 0$ such that if $\|\boldsymbol{v}_0\|_{L^{n,\infty}(\Omega)} \le \kappa$ and $\|\boldsymbol{a}\|_{L^{n,\infty}(\Omega)} \le \kappa$ then there exists a strong solution $\boldsymbol{v}$ of the problem (101), (116) which, among other things, is in $BC\left((0,\infty);\, L^{n,\infty}_\sigma(\Omega)\right) \cap C\left((0,\infty);\, \mathsf{D}(A)\right) \cap C^1\left((0,\infty);\, H_{r_*}(\Omega)\right)$ and satisfies*

$$\|\boldsymbol{v}(t)\|_r \;\le\; C\,t^{-\frac{n}{2}\left(\frac{1}{n}-\frac{1}{r}\right)}, \qquad n < r \le r_*$$

for all $t > 0$ with a constant C depending only on n, r and r_*.

In [76] it is also shown that the better is the information on the spatial decay of the initial velocity $\boldsymbol{a}$, the sharper is the asymptotic behavior of $\boldsymbol{v}(t)$ for $t \to \infty$.

The case of a non-zero translational velocity of the body (i.e. $\lambda \ne 0$, in the first equation in (99)) is dealt with by Y. Shibata [102]. Shibata's approach strongly uses the L^q–L^r estimates of the semigroup $\mathrm{e}^{A_{\lambda,0}t}$ generated by the Oseen operator $A_{\lambda,0}$, provided by Theorem 15. Shibata considers the case $\boldsymbol{f} \in L^\infty(\Omega)$ and assuming that λ and

$$\langle\!\langle\, \boldsymbol{f}\,\rangle\!\rangle_{2\delta} \;:=\; \sup_{\boldsymbol{x}\in\Omega}\,(1+|\boldsymbol{x}|)^{5/2}\;\left(1+|\boldsymbol{x}|+x_1\right)^{1/2+2\delta}\;|\boldsymbol{f}(\boldsymbol{x})|$$

(for some $0 < \delta < \frac{1}{4}$) are "small", he at first proves the existence of a steady solution $\boldsymbol{v}_0$ to problem (8) (with $\mathcal{T} = 0$), satisfying the condition $\boldsymbol{v}_0(\boldsymbol{x}) \to \mathbf{0}$ for $|\boldsymbol{x}| \to \infty$, which is, among other things, "small" in the norm of $W^{2,q}(\Omega)$ (where $3 < q < \infty$). The author considers general Dirichlet's boundary condition $\boldsymbol{v}_0 = \boldsymbol{g}$ on $\partial\Omega$, where $\boldsymbol{g} - \boldsymbol{e}_1$ is supposed to be "small enough" in an appropriate norm. The next theorem follows from [102, Theorem 1.4] if $\boldsymbol{g} = \mathbf{0}$.

Theorem 19. *Let $3 < q < \infty$ and β, δ be any numbers such that $0 < \delta < \frac{1}{4}$ and $\delta < \beta < 1 - \delta$. Let $\boldsymbol{f} \in L^\infty(\Omega)$ and $\boldsymbol{a} \in H_3(\Omega)$. Then there exists $\epsilon = \epsilon(q, \beta, \delta) \in (0, 1]$ such that if $0 < \lambda < \epsilon$, $\langle\!\langle\, \boldsymbol{f} \,\rangle\!\rangle_{2\delta} \le \lambda^{\beta+\delta}$ and $\|\boldsymbol{a}\|_3 < \epsilon$ then the problem (99), (100), (116) has a unique solution $\boldsymbol{v} \in BC\,(0, \infty;\, H_3(\Omega))$ such that*

$$[\boldsymbol{v}]_{3,0,t} + [\boldsymbol{v}]_{q,(1-3/q)/2,t} + [\nabla \boldsymbol{v}]_{3,1/2,t} \;\le\; \sqrt{\epsilon},$$

where

$$[\boldsymbol{v}]_{q,\rho,t} \;:=\; \sup_{0<s<t} \; s^\rho \; \|\boldsymbol{v}(s)\|_q. \tag{130}$$

Moreover, the inequalities

$$[\boldsymbol{v}]_{r,(1-3/r)/2,t} \;\le\; C(r) \,\left(\epsilon + \epsilon^{1/2+\beta}\right),$$

$$\|\boldsymbol{v}(t)\|_\infty \;\le\; C(s) \,\left(\epsilon + \epsilon^{1/2+\beta}\right) \left(t^{-1/2} + t^{-(1-3/2s)}\right)$$

hold for any $t > 0$, where $3 < r < \infty$ and $3 < s < q$.

The proof is based on solution of the integral equation

$$\boldsymbol{v}(t) \;=\; \mathrm{e}^{A_{\lambda,0} t} \boldsymbol{a} + \int_0^\infty \mathrm{e}^{A_{\lambda,0}(t-s)} \left[\lambda B_3 \boldsymbol{v}(s) + \mathscr{N} \boldsymbol{v}(s)\right] \, ds. \tag{131}$$

The author derives a series of subtle estimates of the right-hand side, which finally enable him to solve the equation (131) by means of the contraction mapping principle.

10.4.2 The Case $\mathcal{T} \neq 0$

To the best of our knowledge, if the body is allowed to rotate there are no results analogous to [90], [62], [88], [8], [9] and [77] regarding the behavior of unsteady perturbations to the steady solution $\boldsymbol{v}_0$ in the class of weak solutions.

The asymptotic stability of a steady-state solution $\boldsymbol{v}_0$ when $\lambda = 0$ (body rotates without translating in absence of external forces) was first proved by G. P. Galdi and A. L. Silvestre in [42]. Their approach is based on the combined use of the classical Galerkin method (suitably adapted to the situation at hand) and the spatial asymptotic properties of $\boldsymbol{v}_0$ determined in [41]. More precisely, they show that there exists $C_{11} > 0$ such that if $\|\boldsymbol{a}\|_{1,2} + |\mathcal{T}| < C_{11}$, and if $\boldsymbol{e}_1 \times \boldsymbol{x} \cdot \nabla \boldsymbol{a} \in L^2(\Omega)$, then the initial–value problem (99), (100), (116) has a strong solution $\boldsymbol{v}$ on the time interval $(0, \infty)$, which is (together with its first order and second order spatial derivatives) in $C\left([0, \infty);\, L^2(\Omega)\right)$, and satisfies, among others, the following asymptotic property

$$\lim_{t \to \infty} \; |\boldsymbol{v}(t)|_{1,2} \;=\; 0. \tag{132}$$

The idea of the proof is as follows: due to the local in time existential theorems, the strong solution $\boldsymbol{v}$ exists on a "short" time interval $(0, T_0)$. Considering at first the equations in a bounded domain Ω_R, multiplying equation (99) by $\boldsymbol{v}$, applying the limit procedure for $R \to \infty$, and assuming that $\mathcal{T}$ is "sufficiently small", the authors show that $\|\boldsymbol{v}(t)\|_2$ is bounded in $(0, T_0)$ and $|\boldsymbol{v}(t)|_{1,2}^2$ is integrable over $(0, T_0)$. Similarly, multiplying the momentum equation by $A\boldsymbol{v}$ (respectively differentiating with respect to t and multiplying by $\boldsymbol{v}_t$) and using the smallness of $\mathcal{T}$, one obtains an inequality which shows that $|\boldsymbol{v}(t)|_{1,2}$ is uniformly bounded in $(0,T_0)$ and $\|A\boldsymbol{v}(t)\|_2^2$ (respectively $\|\nabla\boldsymbol{v}_t(t)\|_2^2$) is integrable over $(0,T_0)$. Since $\|\boldsymbol{v}(t)\|_{1,2}$ cannot blow up as $t \to T_0-$, the interval $(0,T_0)$ can be extended to $(0,\infty)$. Since the integrals $\int_0^\infty |\boldsymbol{v}|_{1,2}^2 \, dt$ and $\int_0^\infty |\boldsymbol{v}_t(t)|_{1,2}^2 \, dt$ are finite, one can deduce that (132) holds.

In [104] and [103], Y. Shibata generalized his previous results from [102] (see Section 10.4.1 on the existence of a "small" steady solution and its stability) to the case when $\mathcal{T} \neq 0$. Applying the same arguments as in [102] and using the fact that the operator $A_{\lambda,\mathcal{T}}$ with the rotational effect generates a C_0–semigroup $\mathrm{e}^{A_{\lambda,\mathcal{T}}t}$ which satisfies the same L^q–L^r estimates as the semigroup $\mathrm{e}^{A_{\lambda,0}t}$ (see Theorem 16), he proved the following.

Theorem 20. *Let $3 < q < \infty$ and σ be a small positive number. Then, there exists $\epsilon = \epsilon(q,\sigma) > 0$ such that if $\boldsymbol{a} \in H_3(\Omega)$ and*

$$\|\boldsymbol{v}_0\|_{3-\sigma} + \|\boldsymbol{v}_0\|_{3+\sigma} + |\boldsymbol{v}_0|_{1,\frac{3}{2}-\sigma} + |\boldsymbol{v}_0|_{1,\frac{3}{2}+\sigma} + \|\boldsymbol{a}\|_3 \;\leq\; \epsilon\,,$$

the problem (99), (100), (116) has a unique solution

$$\boldsymbol{v} \in C\left([0,\infty);\, H_3(\Omega)\right) \cap C^0\left((0,\infty);\, L^q(\Omega) \cap W_0^{1,3}(\Omega)\right),$$

which satisfies (see (130))

$$[\boldsymbol{v}]_{3,0,t} + [\boldsymbol{v}]_{q,(1-3/q)/2,t} + [\nabla\boldsymbol{v}]_{3,1/2,t} \;\leq\; \sqrt{\epsilon} \qquad \textit{for any } t > 0.$$

Note that similar results have also been obtained by T. Hishida and Y. Shibata [65] in the case when operator $A_{\lambda,\mathcal{T}}$ reduces to $A_{0,\mathcal{T}}$.

10.5 Spectral Stability and Related Results

The previous section presented, among other things, a number of results concerning the attractivity/asymptotic stability under "smallness" assumptions on $\boldsymbol{v}_0$. Objective of this section is to formulate analogous result, but with more general hypotheses that involve the spectral properties of the relevant linearization around $\boldsymbol{v}_0$. Recall that the perturbation $\boldsymbol{v}$ satisfies the operator equation (101), i.e.

$$\frac{d\boldsymbol{v}}{dt} \;=\; \mathscr{L}\boldsymbol{v} + \mathscr{N}\boldsymbol{v}.$$

Now, assume, temporarily, Ω *bounded*. It is then well known that $\mathrm{Sp}(\mathscr{L}) \equiv \mathrm{Sp_p}(\mathscr{L})$ [98]. In a series of fundamental papers going back to the pioneering works of G. Prodi [98] and D.H. Sattinger [99], it is shown that the zero solution of the above equation is stable, in fact, even exponentially stable, if

$$\exists\,\delta > 0 \;\;:\;\; \mathrm{Re}\,\zeta \leq -\delta, \quad \forall \zeta \in \mathrm{Sp}(\mathscr{L})\,. \tag{133}$$

It can be easily shown that if $\boldsymbol{v}_0$ is "small" enough (i.e., the operator B_3 is a "small" perturbation of $A_{\lambda,\mathcal{T}}$), then (133) holds, whereas the converse is not necessarily true. However, if as in our case, Ω is an *exterior domain, condition (133) cannot be satisfied.*

The reason is that the essential spectrum of $\mathscr{L}$, located in the half-plane $\{\zeta \in \mathbb{C};\ \mathrm{Re}\,\zeta \leq 0\}$, *touches the imaginary axis at the point* 0 *if* $\mathcal{T} = 0$*, or at infinitely many points* $\mathrm{i}k\mathcal{T}$ *(*$k \in \mathbb{Z}$*) if* $\mathcal{T} \neq 0$; see (102) if $\mathcal{T} = 0$, and Theorem 13 if $\mathcal{T} \neq 0$. Now, as shown in [95] if $\mathcal{T} = 0$ and in [54] if $\mathcal{T} \neq 0$, the operator B_3 is relatively compact with respect to $A_{\lambda,\mathcal{T}}$. Consequently, the operator $\mathscr{L} \equiv A_{\lambda,\mathcal{T}} + \lambda B_3$ has the same essential spectrum as $A_{\lambda,\mathcal{T}}$ and the spectra of $\mathscr{L}$ and $A_{\lambda,\mathcal{T}}$ differ at most by a countable number of isolated eigenvalues which may possibly cluster only on the boundary of $\mathrm{Sp}_{\mathrm{ess}}(A_{\lambda,\mathcal{T}})$, see [71, pp. 243–244]. However, as shown in [15] (by P. Deuring and J. Neustupa) and [97] (by J. Neustupa) the essential spectrum of $\mathscr{L}$ does not play a decisive role in the stability issue. In fact, if condition (133) is replaced by a certain assumption on the eigenvalues of $\mathscr{L}$ then one can show the stability of the zero solution of equation (101), regardless of the properties of $\mathrm{Sp}_{\mathrm{ess}}(\mathscr{L})$. The papers [15] and [97] both concern the case $\mathcal{T} = 0$: in the whole space [15], and in exterior domain [97]. Furthermore, they use results from the previous research [95] by J. Neustupa, where the author also treats the case $\mathcal{T} = 0$ and formulates a sufficient condition for stability, under the assumption that the semigroup $\mathrm{e}^{\mathscr{L}t}$, applied to a *finite* family of certain functions, is L^1– and L^2–integrable on $(0, \infty)$ in an appropriate norm defined only over a *bounded* sub–region of Ω. Similar ideas have also been employed in papers [93] and [94] which concern a general parabolic equation in a Hilbert space or a parabolic system in an exterior domain, and in paper [54], which brings a generalization of the results from [95] to the case $\mathcal{T} \neq 0$.

The next sections present in some details the results of [95], [15], [97] and [54]. To this end, the cases $\mathcal{T} = 0$ [95; 15; 97] and $\mathcal{T} \neq 0$ [54] are considered separately.

10.5.1 The Case $\mathcal{T} = 0$

The paper [95] uses the following important facts and steps.

- As the operator $A_{\lambda,0}$ generates an analytic semigroup in $H(\Omega)$ and operator B_3 is relatively compact with respect to $A_{\lambda,0}$, the operator $\mathscr{L} \equiv A_{\lambda,0} + \lambda B_3 = A + \lambda B_1 + \lambda B_3$ generates an analytic semigroup in $H(\Omega)$ as well. (See [71, p. 498].) We denote this semigroup by $\mathrm{e}^{\mathscr{L}t}$.
- The operator B_1 is skew–symmetric in $H(\Omega)$. Set
$$B_{3s}\boldsymbol{v} := \mathrm{P}\ (\boldsymbol{v} \cdot (\nabla \boldsymbol{v}_0)_s)\,,$$
$$B_{3a}\boldsymbol{v} := \mathrm{P}\ (\boldsymbol{v}_0 \cdot \nabla \boldsymbol{v} + \boldsymbol{v} \cdot (\nabla \boldsymbol{v}_0)_a)\,.$$
The subscript s (respectively a) denotes the symmetric (respectively skew–symmetric = anti–symmetric) part of operator B_3 or of the tensor $\nabla \boldsymbol{v}_0$.
- Let $\kappa > 0$ be fixed. The operator $A + (1+\kappa)\lambda B_{3s}$ is selfadjoint in $H(\Omega)$. The spectrum of $A + (1+\kappa)\lambda B_{3s}$ consists of $\mathrm{Sp}_{\mathrm{ess}}\,(A + (1+\kappa)\lambda B_{3s}) = (-\infty, 0]$ and at most a finite set of positive eigenvalues, each of whose has a finite multiplicity. Let the positive eigenvalues be $\zeta_1 \leq \zeta_2 \leq \ldots \leq \zeta_N$, each of them being counted as many times as is its multiplicity. Let $\boldsymbol{\phi}_1, \ldots, \boldsymbol{\phi}_N$ be the associated eigenfunctions. They can be chosen in a way that they constitute an orthonormal system in $H(\Omega)$.

– Denote by $H(\Omega)'$ the linear hull of $\boldsymbol{\phi}_1, \dots, \boldsymbol{\phi}_N$ and by P' the orthogonal projection of $H(\Omega)$ onto $H(\Omega)'$. Furthermore, denote by $H(\Omega)''$ the orthogonal complement to $H(\Omega)'$ in $H(\Omega)$ and by P'' the orthogonal projection of $H(\Omega)$ onto $H(\Omega)''$. Then $H(\Omega)$ admits the orthogonal decomposition $H(\Omega) = H(\Omega)' \oplus H(\Omega)''$ and the operator $A + (1+\kappa)\lambda B_{3s}$ is reduced on each of the subspaces $H(\Omega)'$ and $H(\Omega)''$. Moreover, it is positive on $H(\Omega)'$ and non–positive on $H(\Omega)''$.
– Since $(A\boldsymbol{\phi} + (1+\kappa)\lambda B_{3s}\boldsymbol{\phi}, \boldsymbol{\phi}) \leq 0$ for all $\boldsymbol{\phi} \in H(\Omega)'' \cap \mathsf{D}(A)$, operator $\mathscr{L}$ satisfies

$$(\mathscr{L}\boldsymbol{\phi}, \boldsymbol{\phi}) \;=\; ((A + \lambda B_{3s})\boldsymbol{\phi}, \boldsymbol{\phi})_2 \;=\; \frac{\kappa}{1+\kappa}\,(A\boldsymbol{\phi}, \boldsymbol{\phi})$$
$$+\,\frac{1}{1+\kappa}\,((A + \lambda B_{3s} + \kappa\lambda B_{3s})\boldsymbol{\phi}, \boldsymbol{\phi}) \;\leq\; \frac{\kappa}{1+\kappa}\,(A\boldsymbol{\phi}, \boldsymbol{\phi}) \;=\; -C_{12}\,|\boldsymbol{\phi}|_{1,2}^2,$$

for all $\boldsymbol{\phi} \in H(\Omega)'' \cap \mathsf{D}(A)$, where $C_{12} = \kappa/(1+\kappa)$. This inequality expresses the so called "essential dissipativity" of $\mathscr{L}$ in space $H(\Omega)''$.
– All functions $\boldsymbol{\phi}_1, \dots, \boldsymbol{\phi}_N$ belong to $\mathcal{D}_0^{-1,2}(\Omega)$ (the dual to $\mathcal{D}_0^{1,2}(\Omega)$).

The main result of the paper [95] is the following.

Theorem 21. *Suppose that the steady solution $\boldsymbol{v}_0$ to the problem* (8) *satisfies conditions (98), and let $\rho^* > 0$ be so large that $|\boldsymbol{v}_0|_{1,3/2,\Omega^{\rho^*}} \leq \frac{1}{8}$. Moreover, assume*

(A) there exists a function $\varphi \in L^1(0,\infty) \cap L^2(0,\infty)$ such that $\|\mathrm{e}^{\mathscr{L}t}\boldsymbol{\phi}_i\|_{2;\,\Omega_{\rho^*}} \leq \varphi(t)$ for all $i = 1, \dots, N$ and $t > 0$.

Then there are positive constants δ, C_{13}, C_{14} such that if $\boldsymbol{a} \in H(\Omega) \cap W_0^{1,2}(\Omega)$ and $\|\boldsymbol{a}\|_{1,2} \leq \delta$, the equation (101) with the initial condition $\boldsymbol{v}(0) = \boldsymbol{a}$ has a unique solution $\boldsymbol{v}$ on the time interval $(0,\infty)$. The solution satisfies

$$\|\boldsymbol{v}(t)\|_{1,2}^2 + C_{13} \int_0^t \left(|\boldsymbol{v}(s)|_{1,2}^2 + \|A\boldsymbol{v}(s)\|_2^2 \right) ds \;\leq\; C_{14}\,\|\boldsymbol{a}\|_{1,2}^2 \tag{134}$$

(for all $t > 0$) and

$$\lim_{t\to\infty} |\boldsymbol{v}(t)|_{1,2} \;=\; 0. \tag{135}$$

The proof is based on splitting the equation (101) into an equation in $H(\Omega)''$, where $\mathscr{L}$ is essentially dissipative, and a complementary equation, where one uses the decay of the semigroup $\mathrm{e}^{\mathscr{L}t}$ following from assumption (A).

Theorem 21 tells us that the question of stability of the steady solution $\boldsymbol{v}_0$ reduces to the L^1– and L^2–integrability of a finite family of certain functions in the interval $(0,\infty)$, i.e., condition (A). In the paper [15], the authors consider the case $\Omega = \mathbb{R}^3$, and show that condition (A) is indeed satisfied under some assumptions on the spectrum of $\mathscr{L}$. The latter amounts to assume that all eigenvalues of $\mathscr{L}$ have negative real parts, without any request on the essential spectrum of $\mathscr{L}$. The important tool used in [15] is the fundamental solution of the Oseen equation in $\mathbb{R}^3$ and the estimates for the corresponding resolvent problem.

Sufficient conditions for the stability of the null solution to (99), in terms of eigenvalues of $\mathscr{L}$ and when $\Omega \not\equiv \mathbb{R}^3$, have been recently formulated in the paper [97]. Here, the author shows at first that condition (A) in Theorem 21 can be replaced by the following one

(B) Given $\xi > 0$ there exist functions $\varphi \in L^1(0,\infty) \cap L^2(0,\infty)$ and $\boldsymbol{\psi}_1, \dots, \boldsymbol{\psi}_N \in H(\Omega) \cap \mathcal{D}_0^{-1,2}(\Omega)$ such that

$$\|\boldsymbol{\phi}_j - \boldsymbol{\psi}_j\|_{-1,2} \le \xi \qquad \text{for } j = 1, \dots, N, \tag{136}$$

$$\big|(\mathrm{e}^{\mathscr{L}t}\boldsymbol{\phi}_i, \boldsymbol{\psi}_j)\big| \le \varphi(t) \qquad \text{for } t > 0 \text{ and } i, j = 1, \dots, N. \tag{137}$$

(Condition (B) reduces to the requirement that $\big|(\mathrm{e}^{\mathscr{L}t}\boldsymbol{\phi}_i, \boldsymbol{\phi}_j)\big| \le \varphi(t)$ if one chooses $\boldsymbol{\psi}_j = \boldsymbol{\phi}_j$ for $j = 1, \dots, N$.) In [97] it is further assumed that the steady solution $\boldsymbol{v}_0$ is in $L^r(\Omega)$ for all $r \in (2, \infty]$ and $\partial_j \boldsymbol{v}_0 \in \boldsymbol{L}^s(\Omega)$, for $j = 1, 2, 3$ and for all $s \in (\frac{4}{3}, \infty]$. This assumption is fulfilled if the acting body force $\boldsymbol{f}$ is in $L^q(\Omega)$ for all $q \in (1, q_0]$, where $q_0 > 3$; see Lemma 9. Moreover, if $\boldsymbol{f}$ has a compact support then $\boldsymbol{v}_0 = \mathbb{E}(\boldsymbol{x}) \cdot \boldsymbol{m} + \boldsymbol{v}_0'(\boldsymbol{x})$, where $\boldsymbol{m}$ is a certain constant vector, $\mathbb{E}$ is the Oseen fundamental tensor and $\boldsymbol{v}_0'(\boldsymbol{x})$ is a perturbation which decays faster than $\mathbb{E}(\boldsymbol{x})$ for $|\boldsymbol{x}| \to \infty$; see Theorem 6. This form of $\boldsymbol{v}_0$ is used in [97], where the main result states the following.

Theorem 22. *Let the conditions*

(C_1) there exist $\delta > 0$ and $a_0 > 0$ such that all eigenvalues ζ of operator $\mathscr{L}$ satisfy $\operatorname{Re}\zeta < \max\{-\delta; -a_0 (\operatorname{Im}\zeta)^2\}$,

(C_2) 0 is not an eigenvalue of the operator $\mathscr{L}_{\mathrm{ext}}$

be fulfilled. Then the conclusions of Theorem 21 hold.

Here, $\mathscr{L}_{\mathrm{ext}}$ denotes the operator $\mathscr{L}$ with the domain extended to $D^{2,2}(\Omega) \cap \mathcal{D}_0^{1,2}(\Omega)$. Condition ($\mathrm{C}_1$) implies that $\mathscr{L}$ has no eigenvalues with non–negative real parts.

Sketch of the proof of Theorem 22; *see* [97] *for the details.* The proof is based on showing that $[(\mathrm{C}_1) \wedge (\mathrm{C}_2)] \Longrightarrow (\mathrm{B})$. The function $\mathrm{e}^{\mathscr{L}t}\boldsymbol{\phi}_i$ is expressed by the formula

$$\mathrm{e}^{\mathscr{L}t}\boldsymbol{\phi}_i \; = \; (2\pi \mathrm{i})^{-1} \int_{\Gamma^\epsilon} \mathrm{e}^{\zeta t} \, (\zeta I - \mathscr{L})^{-1} \boldsymbol{\phi}_i \, d\zeta, \tag{138}$$

where Γ^ϵ is a curve in $\mathbb{C} \smallsetminus \mathrm{Sp}(\mathscr{L})$, which depends on a parameter $\epsilon > 0$. Recall that $\mathrm{Sp}(\mathscr{L})$ consists of the essential spectrum in the half-plane $\{\zeta \in \mathbb{C};\ \operatorname{Re}\zeta \le 0\}$, see (102), and at most a countable number of isolated eigenvalues. Curve Γ^ϵ has three parts Γ_1, Γ_2^ϵ and Γ_3, where Γ_1 and Γ_3 coincide with the half-lines $\arg(\zeta) = \pi \pm \alpha$, respectively, for some fixed $\alpha \in (0, \pi/2)$ and large $|\zeta|$. Both the curves Γ_1 and Γ_3 lie in the half-plane $\{\zeta \in \mathbb{C};\ \operatorname{Re}\zeta < 0\}$. Since $\mathrm{Sp}_{\mathrm{ess}}(\mathscr{L})$ touches the imaginary axis at point 0, the curve $\Gamma_2^\epsilon := \{\lambda \in \mathbb{C};\ \lambda = -as^2 + \epsilon(s_0^2 - s^2) + \mathrm{i}s \text{ for } -s_0 \le s \le s_0\}$ (where $a > 0$ and $s_0 > 0$ are appropriate fixed positive numbers) extends into the half-plane $\{\zeta \in \mathbb{C};\ \operatorname{Re}\zeta > 0\}$. If $\epsilon \to 0+$ then Γ_2^ϵ approaches $\Gamma_2^0 := \{\lambda \in \mathbb{C};\ \lambda = -as^2 + \mathrm{i}s \text{ for } -s_0 \le s \le s_0\}$, and consequently, Γ^ϵ approaches $\Gamma^0 = \Gamma_1 \cup \Gamma_2^0 \cup \Gamma_3$. (Number $a > 0$ is chosen so that $\mathrm{Sp}(\mathscr{L})$ lies on the left from Γ^0, with the exception of point 0.) In order to verify inequality (137) in condition (B), one has to estimate $\big(\mathrm{e}^{\mathscr{L}t}\boldsymbol{\phi}_i, \boldsymbol{\psi}_j\big)$. Since one can prove that the range of $\mathscr{L}^*|_{\mathcal{D}(\Omega)}$ (the adjoint operator to $\mathscr{L}$ reduced to $\mathcal{D}(\Omega)$) is dense in $\mathcal{D}_0^{-1,2}(\Omega)$, in order to satisfy (136) one can choose functions $\boldsymbol{\psi}_j$ in the form $\boldsymbol{\psi}_j := \mathscr{L}^* \boldsymbol{\psi}_j'$, where $\boldsymbol{\psi}_j' \in \mathcal{D}(\Omega)$ (for $j = 1, \dots, N$). Then

$$\big(\mathrm{e}^{\mathscr{L}t}\boldsymbol{\phi}_i, \boldsymbol{\psi}_j\big) \; = \; \frac{1}{2\pi \mathrm{i}} \int_{\Gamma^\epsilon} \mathrm{e}^{\zeta t} \, \big((\zeta I - \mathscr{L})^{-1} \boldsymbol{\phi}_i, \boldsymbol{\psi}_j\big) \; d\zeta$$

$$= \frac{1}{2\pi\mathrm{i}} \int_{\Gamma^\epsilon} \mathrm{e}^{\zeta t} \, \big((\zeta I - \mathscr{L})^{-1} \boldsymbol{\phi}_i, (\mathscr{L}^* - \overline{\zeta} I + \overline{\zeta} I) \boldsymbol{\psi}'_j \big) \; d\zeta$$

$$= -\frac{1}{2\pi\mathrm{i}} \int_{\Gamma^\epsilon} \mathrm{e}^{\zeta t} \, \big(\boldsymbol{\phi}_i, \boldsymbol{\psi}'_j \big) \; d\zeta + \frac{1}{2\pi\mathrm{i}} \int_{\Gamma^\epsilon} \mathrm{e}^{\zeta t} \zeta \, \big((\zeta I - \mathscr{L})^{-1} \boldsymbol{\phi}_i, \boldsymbol{\psi}'_j \big) \; d\zeta. \tag{139}$$

As the integrand in the first integral on the right-hand side depends on ζ only through $\mathrm{e}^{\zeta t}$, one can consider the limit for $\epsilon \to 0+$ and show that the integral equals the integral on the curve Γ^0. A simple calculation yields that the integral on Γ^0, as a function of t, is in $L^1(0,\infty) \cap L^2(0,\infty)$. (Here, it is important that $\Gamma^0 \subset \{\zeta \in \mathbb{C};\ \mathrm{Re}\,\zeta \leq 0\}$ and Γ^0 touches the imaginary axis only at the point 0.) The treatment of the second integral on the right-hand side of (139) is much more complicated. It is necessary to derive a series of estimates of $\boldsymbol{u}_\zeta := (\mathscr{L} - \zeta I)^{-1} \boldsymbol{\phi}_i$, which satisfies the equation

$$(A + \lambda B_1 + \lambda B_3 - \zeta I) \boldsymbol{u}_\zeta \;=\; \boldsymbol{\phi}_i. \tag{140}$$

This equation can be treated as the perturbed Oseen resolvent equation with the resolvent parameter ζ. Especially the estimates for $\zeta \notin \mathrm{Sp}(\mathscr{L})$ in the neighborhood of 0 (hence also in the neighborhood of $\mathrm{Sp}_{\mathrm{ess}}(\mathscr{L})$) are very subtle. They finally enable one to pass to the limit for $\epsilon \to 0+$ and show that the integral of $\mathrm{e}^{\zeta t} \zeta \, \big((\zeta I - \mathscr{L})^{-1} \boldsymbol{\phi}_i, \boldsymbol{\psi}'_j \big)$ on curve Γ^0 is, as a function of t, in $L^1(0,\infty) \cap L^2(0,\infty)$. The factor ζ plays a decisive role because it allows one to control the integrand for ζ on the critical part of curve Γ^0, i.e. near $\zeta = 0$.

Note that the assumption on the non–zero translational motion of body $\mathscr{B}$ in the fluid (i.e. $\lambda \neq 0$) is important because it enables one to apply the theory of the Oseen equation and to obtain appropriate estimates of function $\boldsymbol{u}_\zeta$. □

Remark 10. A result similar to Theorem 22 was stated by L.I. Sazonov [100]. There, the main theorem on stability claims that the steady solution $\boldsymbol{v}_0$ is asymptotically stable in the L^3-norm if $\mathscr{L}$, as an operator in $H_3(\Omega)$, does not have eigenvalues in the half-plane $\mathrm{Re}\,\zeta > 0$. However, the proofs of the fundamental estimates of the Oseen semigroup as well as of the main theorem do not contain all the necessary details, which makes it difficult to assess the validity of author's arguments.

10.5.2 The Case $\mathcal{T} \neq 0$

The results of [95] are generalized to the case $\mathcal{T} \neq 0$ (i.e. $\mathcal{B}$ is allowed to spin at constant rate) involving the rotational motion of body $\mathscr{B}$, in the paper [36] by G. P. Galdi and J. Neustupa. The steady solution $\boldsymbol{v}_0$ is assumed to satisfy the properties $\boldsymbol{v}_0 \in L^3(\Omega)$, $\partial_j \boldsymbol{v}_0 \in L^3(\Omega) \cap L^{3/2}(\Omega)$ $(j = 1, 2, 3)$, and the estimate $|\nabla \boldsymbol{v}_0(\boldsymbol{x})| \leq C\, |\boldsymbol{x}|^{-1}$ for $\boldsymbol{x} \in \Omega$. The existence of such a solution is known for a large class of body forces $\boldsymbol{f}$, provided $\lambda \neq 0$; see Lemma 9 and Theorem 6. The main theorem on stability of the zero solution of equation (101) is analogous to Theorem 21, that is why it is not repeated here.

The presence of the term $\mathcal{T} B_2 \boldsymbol{v}$ in the operator $\mathscr{L}$ defined in equation (101) causes a series of new problems that one has to face and overcome. For example, unlike the case $\mathcal{T} = 0$, the time derivative of $\boldsymbol{v}$ need not be an element of $H(\Omega)$. However, one can show that $(d\boldsymbol{v}/dt) - \lambda B_1\boldsymbol{v} - \mathcal{T}B_2\boldsymbol{v} \in H(\Omega)$ and

$$\int_\Omega \Big(\frac{d\boldsymbol{v}}{dt} - \lambda B_1\boldsymbol{v} - \mathcal{T}B_2\boldsymbol{v}\Big) \cdot \boldsymbol{v}\; d\boldsymbol{x} = \frac{d}{dt}\,\frac{1}{2}\,\|\boldsymbol{v}\|_2^2,$$

$$\int_\Omega \Big(\frac{d\boldsymbol{v}}{dt} - \lambda B_1\boldsymbol{v} - \mathcal{T}B_2\boldsymbol{v}\Big) \cdot A\boldsymbol{v}\; d\boldsymbol{x} = -\frac{d}{dt}\,\frac{1}{2}\,|\boldsymbol{v}|_{1,2}^2.$$

These identities play an important role in the proof of the theorem on stability. Another important step is to show that the functions $\nabla\boldsymbol{\phi}_i$ and $\Delta\boldsymbol{\phi}_i$ $(i = 1, \dots, N)$ are square-integrable with the weight $|\boldsymbol{x}|^2$ in Ω. This enables one to estimate the norm $\|B_2\boldsymbol{v}\|_2$ by $C\,|\boldsymbol{v}|_{1,2}$, which is again a crucial property in the proof of the stability result. All details can be found in [54].

Open Problem 8 *The question whether –in analogy with the case $\mathcal{T} = 0$ and paper [95]– the stability of the zero solution of equation (101) can be determined by the location of the eigenvalues of operator $\mathscr{L}$ is open.*

The difficulties related to this problem are generated by the fact that now, being $\mathcal{T} \neq 0$, the operator $\mathscr{L}$ is no longer sectorial. Thus, even if all the eigenvalues have negative real parts, one cannot express the $\mathrm{e}^{\mathscr{L}t}\boldsymbol{\phi}_i$ by a formula similar to (138), where the curve Γ^ϵ coincides with the half-lines $\arg(\zeta) = \pi \pm \alpha$ (for some $\alpha \in (0, \pi/2)$ and large $|\zeta|$) and touches or intersects the half-plane $\mathbb{C}_+$ only in a small neighborhood of $\mathbf{0}$. On the contrary, the curve Γ^ϵ must lie at the right of infinitely many points $\mathrm{i}k\mathcal{T}$ $(k \in \mathbb{Z})$ on the imaginary axis, and even if one formally passes to the limit $\epsilon \to 0$ in order to obtain a curve Γ^0 in the half-plane $\{\zeta \in \mathbb{C};\ \mathrm{Re}\,\zeta \leq 0\}$, then Γ^0 must pass through the points $\mathrm{i}k\mathcal{T}$ $(k \in \mathbb{Z})$. Consequently, the integral on the right-hand side of (138) cannot be treated and estimated in the same way as in the case $\mathcal{T} = 0$.

11 Conclusion

The article is an updated survey of important known qualitative properties of mathematical models of viscous incompressible flows past rigid and rotating bodies. The models are based on the Navier-Stokes equations. Greatest attention is paid to steady problems, as well as problems that are quasi-steady in the sense that the transformed equations describing the motion of the fluid around a rotating body are steady in the body-fixed frame. The presented results concern the existence, regularity and uniqueness of solutions (see Sections 4–6). Section 7 deals with the spatial asymptotic properties of steady solutions, like the questions of presence of a wake behind the body and the decay of velocity and vorticity in or outside the wake in dependence on the distance from the body. Here, the case $\mathcal{T} \neq 0$ (i.e. the case when the body rotates with a non–zero constant angular velocity) is much more difficult than the case $\mathcal{T} = 0$ and the relevant results are therefore of a relatively recent date. The structure of the set of steady solutions for arbitrarily large given data is studied in Section 8 by means of tools of nonlinear analysis, like the theory of proper Fredholm operators, corresponding mod 2 degree, etc. One of the results asserts that, to a given nonzero translational velocity and angular velocity, the solution set is generically finite and has an odd number of elements. Section 9 analyzes sufficient and necessary conditions for bifurcations of steady or time-periodic solutions from steady solutions. The corresponding theorems provide a theoretical explanation of the well known phenomenon, i.e. that the properties and shape of a steady solution may considerably change if some characteristic parameters of the flow field vary. The long time behavior of unsteady perturbations of a given steady solution $\boldsymbol{v}_0$ are finally studied in Section 10. This section also brings some necessary results on the existence and uniqueness of solutions. The core of the section are 1) the results on the stability of $\boldsymbol{v}_0$ under the assumption that $\boldsymbol{v}_0$ is in some sense "sufficiently small", and 2) the results that do not need any condition of smallness of $\boldsymbol{v}_0$, and instead of it they use either an assumption on a "sufficiently fast" time-decay of a certain finite family of functions related to $\boldsymbol{v}_0$, or an assumption on the position of eigenvalues of a certain associated linear operator. (Here, one has to overcome the difficulties following from the presence of the essential spectrum, having a non-empty intersection with the imaginary axis.)

The readers find a series of references to related papers or books inside each section. The article also brings the formulation of altogether eight open problems that concern the discussed topics and represent a challenge for future research.

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