Theoretical and numerical results for a chemo-repulsion model with quadratic production

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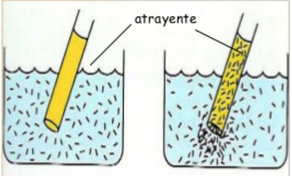
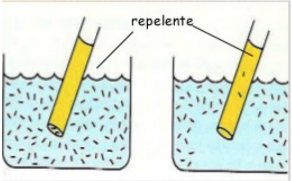
Chemotaxis is the biological process of the movement of living organisms in response to a chemical stimulus which can be given towards a higher (attractive) or lower (repulsive) concentration of a chemical substance. At the same time, the presence of living organisms can produce or consume chemical substance.





Chemotaxis system [Keller-Segel, 1970-1971]

$$\begin{cases} \partial_t u - D_u \Delta u + \chi \nabla \cdot (u \nabla v) = 0 \ \text{ in } \Omega, \ t > 0, \\ \partial_t v - D_v \Delta v + \beta v = \alpha u \text{ in } \ \Omega, \ t > 0, \\ \frac{\partial v}{\partial \mathbf{n}} = \frac{\partial u}{\partial \mathbf{n}} = 0 \ \text{ on } \partial\Omega, \ t > 0, \\ v(\mathbf{x}, 0) = v_0(\mathbf{x}) > 0, \ u(\mathbf{x}, 0) = u_0(\mathbf{x}) > 0 \ \text{ in } \Omega, \end{cases} \quad (1)$$

- v is the chemical concentration,
- u denotes the cell density,
- The term $\chi u \nabla v$ models the transport of cells towards the higher concentrations of chemical signal if $\chi > 0$, and towards the lower concentrations of chemical signal if $\chi < 0$.

Chemorepulsion production system with quadratic production term

$$\begin{cases} \partial_t u - D_u \Delta u - \nabla \cdot (u \nabla v) = 0 \quad \text{in } \Omega, \ t > 0, \\ \partial_t v - D_v \Delta v + v = u^2 \text{ in } \ \Omega, \ t > 0, \\ \frac{\partial v}{\partial \mathbf{n}} = \frac{\partial u}{\partial \mathbf{n}} = 0 \quad \text{on } \partial\Omega, \ t > 0, \\ v(\mathbf{x}, 0) = v_0(\mathbf{x}) > 0, \ u(\mathbf{x}, 0) = u_0(\mathbf{x}) > 0 \quad \text{in } \Omega, \end{cases} \tag{2}$$

- In the case of linear production, [Cieslak et all, 2008], prove that model (1) is well-posed: there exists global in time weak solution (based on an energy inequality) and, for $2D$ domains, there exists a unique global in time strong solution.
- The quadratic production term allows to control an energy in $L^2(\Omega)$-norm for u which is very useful for performing numerical analysis.

Weak solution

$$\left\{\begin{array}{rcl} u & \in & L^\infty(0,+\infty;L^2(\Omega))\cap L^2(0,T;H^1(\Omega)), \\ v & \in & L^\infty(0,+\infty;H^2(\Omega))\cap L^2(0,T;H^2(\Omega)) \\ (\partial_t u, \partial_t v) & \in & L^{q'}(0,T;H^1(\Omega)'\times L^2(\Omega)), \end{array}\right. \quad (3)$$

where $q' = 2$ in the $(2D)$ case and $q' = 4/3$ in the $(3D)$ case

It is possible to prove that model (2) is well-posed: there exists global in time weak solution (based on an energy inequality) and, for $2D$ domains, there exists a unique global in time strong solution satisfying

$$\left\{\begin{array}{rcl} u & \in & L^\infty(0,+\infty;H^1(\Omega))\cap L^2(0,T;H^2(\Omega)), \\ v & \in & L^\infty(0,+\infty;H^2(\Omega))\cap L^2(0,T;H^3(\Omega)) \\ \partial_t u & \in & L^\infty(0,+\infty;L^2(\Omega))\cap L^2(0,+\infty;H^1(\Omega)), \\ \partial_t v & \in & L^\infty(0,+\infty;H^1(\Omega))\cap L^2(0,+\infty;H^2(\Omega)). \end{array}\right. \quad (4)$$

Some Properties

- The problem is conservative in u, as we can check integrating $(2)_1$ in Ω,

$$\frac{d}{dt}\left(\int_\Omega u\right) = 0, \quad \text{i.e.} \quad \int_\Omega u(t) = \int_\Omega u_0, \quad \forall t > 0.$$

Also, integrating $(2)_2$ in Ω, we deduce the following behavior of v,

$$\frac{d}{dt}\left(\int_\Omega v\right) = \int_\Omega u^2 - \int_\Omega v.$$

- $u \geq 0$ and $v \geq 0$.
- The following energy inequality holds for a.e. t_0, t_1 with $t_1 \geq t_0 \geq 0$:

$$\mathcal{E}(u(t_1), v(t_1)) - \mathcal{E}(u(t_0), v(t_0)) + \int_{t_0}^{t_1} (\|\nabla u(s)\|_0^2 + \frac{1}{2}\|\Delta v(s)\|_0^2 + \frac{1}{2}\|\nabla v(s)\|_0^2)\, ds \leq 0, \tag{5}$$

for

$$\mathcal{E}(u(t), v(t)) = \|u(t)\|_0^2 + \frac{1}{2}\|\nabla v(t)\|_0^2$$

Chemorepulsion production system with quadratic production term

$$\left\{\begin{array}{l} \partial_t u - D_u \Delta u - \nabla \cdot (u \nabla v) = 0 \ \text{ in } \Omega,\ t > 0, \\ \partial_t v - D_v \Delta v - v = u^2 \text{ in } \ \Omega,\ t > 0, \\ \frac{\partial v}{\partial \mathbf{n}} = \frac{\partial u}{\partial \mathbf{n}} = 0 \ \text{ on } \partial\Omega,\ t > 0, \\ v(\mathbf{x}, 0) = v_0(\mathbf{x}) > 0,\ u(\mathbf{x}, 0) = u_0(\mathbf{x}) > 0 \ \text{ in } \Omega, \end{array}\right. \tag{6}$$

Chemorepulsion production system with quadratic production term

$$\begin{cases} \partial_t u - D_u \Delta u - \nabla \cdot (u \nabla v) = 0 \ \text{ in } \Omega, \ t > 0, \\ \partial_t v - D_v \Delta v - v = u^2 \ \text{in } \ \Omega, \ t > 0, \\ \frac{\partial v}{\partial \mathbf{n}} = \frac{\partial u}{\partial \mathbf{n}} = 0 \ \text{ on } \partial\Omega, \ t > 0, \\ v(\mathbf{x}, 0) = v_0(\mathbf{x}) > 0, \ u(\mathbf{x}, 0) = u_0(\mathbf{x}) > 0 \ \text{ in } \Omega, \end{cases} \tag{6}$$

An equivalent problem ($\boldsymbol{\sigma} = \nabla v$), [Zhang & Zhu, 2016]

$$\begin{cases} \partial_t u - \nabla \cdot (\nabla u) = \nabla \cdot (u \boldsymbol{\sigma}) \ \text{ in } \Omega, \ t > 0, \\ \partial_t \boldsymbol{\sigma} - \nabla(\nabla \cdot \boldsymbol{\sigma}) + \text{rot}(\text{rot } \boldsymbol{\sigma}) + \boldsymbol{\sigma} = \nabla(u^2) \ \text{in } \ \Omega, \ t > 0, \\ \frac{\partial u}{\partial \mathbf{n}} = 0 \ \text{ on } \partial\Omega, \ t > 0, \\ \boldsymbol{\sigma} \cdot \mathbf{n} = 0, \ \ [\text{rot } \boldsymbol{\sigma} \times \mathbf{n}]_{tang} = 0 \ \text{ on } \partial\Omega, \ t > 0, \\ u(x, 0) = u_0(x) \geq 0, \ \boldsymbol{\sigma}(x, 0) = \nabla v_0(x) \ \text{ in } \Omega, \end{cases} \tag{7}$$

Once solved (7), we can recover v from u^2 solving

$$\begin{cases} \partial_t v - \Delta v + v = u^2 \ \text{in } \ \Omega, \ t > 0, \\ \frac{\partial v}{\partial \mathbf{n}} = 0 \ \text{ on } \partial\Omega, \ t > 0, \\ v(x, 0) = v_0(x) \geq 0 \ \text{ in } \Omega. \end{cases} \tag{8}$$

Problems (2) and (7)-(8) are equivalents!

Some Properties of (7)

- The problem is conservative in u, as we can check integrating $(7)_1$ in Ω,

$$\frac{d}{dt}\left(\int_\Omega u\right) = 0, \quad \text{i.e.} \quad \int_\Omega u(t) = \int_\Omega u_0, \quad \forall t > 0.$$

  Also, integrating (8) in Ω, we deduce the following behavior of v,

$$\frac{d}{dt}\left(\int_\Omega v\right) = \int_\Omega u^2 - \int_\Omega v.$$

- $u \geq 0$
- The following energy inequality holds for a.e. t_0, t_1 with $t_1 \geq t_0 \geq 0$:

$$\begin{aligned}&\mathcal{E}(u(t_1), v(t_1)) - \mathcal{E}(u(t_0), v(t_0)) \\ &+ \int_{t_0}^{t_1} \left(\|\nabla u(s)\|_0^2 + \frac{1}{2}\|\nabla \cdot \boldsymbol{\sigma}\|_0^2 + \frac{1}{2}\|\text{rot}\ \boldsymbol{\sigma}\|_0^2 + \|\boldsymbol{\sigma}\|_0^2 \right) ds \leq 0,\end{aligned} \tag{9}$$

  for $\mathcal{E}(u(t), v(t)) = \|u(t)\|_0^2 + \frac{1}{2}\|\sigma(t)\|_0^2$

For the variable $\boldsymbol{\sigma}$ we will use the following space

$$\mathbf{H}^1_\sigma(\Omega) := \{\mathbf{u} \in \mathbf{H}^1(\Omega) : \mathbf{u} \cdot \mathbf{n} = 0 \text{ on } \partial\Omega\}$$

endowed with the equivalent norm in $\mathbf{H}^1(\Omega)$ ([Amrouche and Seloula, 2013]):

$$\|\boldsymbol{\sigma}\|_1^2 = \|\boldsymbol{\sigma}\|_0^2 + \|\text{rot } \boldsymbol{\sigma}\|_0^2 + \|\nabla \cdot \boldsymbol{\sigma}\|_0^2, \quad \forall \boldsymbol{\sigma} \in \mathbf{H}^1_\sigma(\Omega).$$

TIME-DISCRETE SCHEME

Scheme **US**

Time step n: Given $(u_{n-1}, \boldsymbol{\sigma}_{n-1})$, compute $(u_n, \boldsymbol{\sigma}_n)$ solving

$$\left\{ \begin{array}{l} (\delta_t u_n, \bar{u}) + (\nabla u_n, \nabla \bar{u}) + (u_n \boldsymbol{\sigma}_n, \nabla \bar{u}) = 0, \quad \forall \bar{u}, \\ (\delta_t \boldsymbol{\sigma}_n, \bar{\boldsymbol{\sigma}}) + \langle B_2 \boldsymbol{\sigma}_n, \bar{\boldsymbol{\sigma}} \rangle - 2(u_n \nabla u_n, \bar{\boldsymbol{\sigma}}) = 0, \quad \forall \bar{\boldsymbol{\sigma}}. \end{array} \right. \tag{10}$$

where $\langle B_2 \boldsymbol{\sigma}, \bar{\boldsymbol{\sigma}} \rangle = (\boldsymbol{\sigma}, \bar{\boldsymbol{\sigma}}) + (\nabla \cdot \boldsymbol{\sigma}, \nabla \cdot \bar{\boldsymbol{\sigma}}) + (\text{rot } \boldsymbol{\sigma}, \text{rot } \bar{\boldsymbol{\sigma}})$. Moreover, we recover v_n solving

$$\delta_t v_n + B_1 v_n = u_n^2, \quad \text{a.e. } \mathbf{x} \in \Omega, \tag{11}$$

Main results

1. Scheme **US** is **conservative** in u, that is, $\int_\Omega u_n = \int_\Omega u_{n-1} = \cdots = \int_\Omega u_0$.
2. $u_n \geq 0$, $v_n \geq 0$
3. It can be proved the **existence of nonnegative solution** $(u_n, \boldsymbol{\sigma}_n, v_n)$ of scheme **US**, and under a smallness condition on k, this solution is **unique**.

TIME-DISCRETE SCHEME

Scheme **US**

Time step n: Given $(u_{n-1}, \boldsymbol{\sigma}_{n-1})$, compute $(u_n, \boldsymbol{\sigma}_n)$ solving

$$\left\{ \begin{array}{l} (\delta_t u_n, \bar{u}) + (\nabla u_n, \nabla \bar{u}) + (u_n \boldsymbol{\sigma}_n, \nabla \bar{u}) = 0, \quad \forall \bar{u}, \\ (\delta_t \boldsymbol{\sigma}_n, \bar{\boldsymbol{\sigma}}) + \langle B_2 \boldsymbol{\sigma}_n, \bar{\boldsymbol{\sigma}} \rangle - 2(u_n \nabla u_n, \bar{\boldsymbol{\sigma}}) = 0, \quad \forall \bar{\boldsymbol{\sigma}}. \end{array} \right. \tag{12}$$

where $\langle B_2 \boldsymbol{\sigma}, \bar{\boldsymbol{\sigma}} \rangle = (\boldsymbol{\sigma}, \bar{\boldsymbol{\sigma}}) + (\nabla \cdot \boldsymbol{\sigma}, \nabla \cdot \bar{\boldsymbol{\sigma}}) + (\text{rot } \boldsymbol{\sigma}, \text{rot } \bar{\boldsymbol{\sigma}})$. Moreover, we recover v_n solving

$$\delta_t v_n + B_1 v_n = u_n^2, \quad \text{a.e. } \mathbf{x} \in \Omega, \tag{13}$$

Main results

4. Scheme **US** is **unconditionally energy-stable**:

$$\delta_t \mathcal{E}(u_n, \boldsymbol{\sigma}_n) + \frac{k}{2} \|\delta_t u_n\|_0^2 + \frac{k}{4} \|\delta_t \boldsymbol{\sigma}_n\|_0^2 + \|\nabla u_n\|_0^2 + \frac{1}{2} \|\boldsymbol{\sigma}_n\|_1^2 = 0, \tag{14}$$

holds, where $\mathcal{E}(u_n, \boldsymbol{\sigma}_n) = \frac{1}{2} \|u_n\|_0^2 + \frac{1}{4} \|\boldsymbol{\sigma}_n\|_0^2$.

TIME-DISCRETE SCHEME

Scheme **US**

Time step n: Given $(u_{n-1}, \boldsymbol{\sigma}_{n-1})$, compute $(u_n, \boldsymbol{\sigma}_n)$ solving

$$\left\{ \begin{array}{l} (\delta_t u_n, \bar{u}) + (\nabla u_n, \nabla \bar{u}) + (u_n \boldsymbol{\sigma}_n, \nabla \bar{u}) = 0, \quad \forall \bar{u}, \\ (\delta_t \boldsymbol{\sigma}_n, \bar{\boldsymbol{\sigma}}) + \langle B_2 \boldsymbol{\sigma}_n, \bar{\boldsymbol{\sigma}} \rangle - 2(u_n \nabla u_n, \bar{\boldsymbol{\sigma}}) = 0, \quad \forall \bar{\boldsymbol{\sigma}}. \end{array} \right. \tag{15}$$

where $\langle B_2 \boldsymbol{\sigma}, \bar{\boldsymbol{\sigma}} \rangle = (\boldsymbol{\sigma}, \bar{\boldsymbol{\sigma}}) + (\nabla \cdot \boldsymbol{\sigma}, \nabla \cdot \bar{\boldsymbol{\sigma}}) + (\text{rot } \boldsymbol{\sigma}, \text{rot } \bar{\boldsymbol{\sigma}})$. Moreover, we recover v_n solving

$$\delta_t v_n + B_1 v_n = u_n^2, \quad \text{a.e. } \mathbf{x} \in \Omega, \tag{16}$$

Main results

5. As consequence, we have estimates $(u_n, \boldsymbol{\sigma}_n) \in \updownarrow^\infty L^2 \cap l^2 H^1$.
6. Moreover, we obtain the estimate $\int_\Omega v_n \leq K_0$, and therefore $v_n \in l^\infty H^1 \cap l^2 H^2$.
7. **Convergence** towards weak solutions. [Marion-Temam]
8. Error estimates $(\mathcal{O}(k))$ for $e_u^n = u(t_n) - u_n$, $e_{\boldsymbol{\sigma}}^n = \boldsymbol{\sigma}(t_n) - \boldsymbol{\sigma}_n$, $e_v^n = v(t_n) - v_n$
9. Uniform strong estimates in 2D domains.

Lemma (Convergence)

There exists a subsequence (k'), with $k' \to 0$, and a weak solution $(u, \boldsymbol{\sigma})$ of (7) in $(0, T)$, such that the sequence $(u_{k'}, \boldsymbol{\sigma}_{k'})$ of solutions of discrete scheme **US** corresponding to k' converges to $(u, \boldsymbol{\sigma})$ weakly-* in $L^\infty(0, T; L^2(\Omega))$, weakly in $L^2(0, T; H^1(\Omega))$ and strongly in $L^2(0, T; L^2(\Omega))$.

Lemma (Error estimates in weak norms for **US**)

Let $(u_n, \boldsymbol{\sigma}_n)$ be a solution of scheme **US**, v_n solution of (26) and assume the following regularity for $(u, \boldsymbol{\sigma})$ exact solution of (7) and v exact solution of (8):

$$u \in L^\infty(0,T;H^1(\Omega)), \quad u_{tt} \in L^2(0,T;(H^1(\Omega))'), \tag{17}$$

$$\boldsymbol{\sigma} \in \mathbf{L}^\infty(0,T;\mathbf{H}^1(\Omega)), \quad \boldsymbol{\sigma}_{tt} \in \mathbf{L}^2(0,T;(\mathbf{H}^1(\Omega))'), \tag{18}$$

$$v_{tt} \in L^2(0,T;(H^1(\Omega))'). \tag{19}$$

If $k < \dfrac{1}{4C\,\|\nabla u, \nabla\cdot\boldsymbol{\sigma}\|^4_{L^\infty L^2}}$, then the a priori error estimates

$$\|(e^n_{u,}, e^n_{\boldsymbol{\sigma}})\|_{l^\infty L^2 \cap l^2 H^1} \le C(T)\,k$$

$$\|e^n_v\|_{l^\infty L^2 \cap l^2 H^1} \le C(T)\,k$$

hold, where $C(T) = K_1 exp(K_2 T)$.

A LINEAR TIME-DISCRETE SCHEME

Scheme **LC**

Time step n: Given $(u_{n-1}, \boldsymbol{\sigma}_{n-1})$, compute $(u_n, \boldsymbol{\sigma}_n)$ solving

$$\begin{cases} (\delta_t u_n, \bar{u}) + \langle A u_n, \bar{u} \rangle = -(u_{n-1} \boldsymbol{\sigma}_n, \nabla \bar{u}), & \forall \bar{u}, \\ (\delta_t \boldsymbol{\sigma}_n, \bar{\boldsymbol{\sigma}}) + \langle B_2 \boldsymbol{\sigma}_n, \bar{\boldsymbol{\sigma}} \rangle = 2(u_{n-1} \nabla u_n, \bar{\boldsymbol{\sigma}}), & \forall \bar{\boldsymbol{\sigma}} \end{cases} \quad (20)$$

Moreover, we recover v_n solving

$$\delta_t v_n + B_1 v_n = u_n^2, \quad \text{a.e. } \mathbf{x} \in \Omega, \quad (21)$$

Main results

1. The same results hold; but: It is not clear $u_n \geq 0$.
2. We do not have the relation $\boldsymbol{\sigma}_n = \nabla v_n$.

FULLY DISCRETE SCHEME

Scheme **US**

Time step n: Given $(u_{n-1}, \boldsymbol{\sigma}_{n-1})$, compute $(u_n, \boldsymbol{\sigma}_n)$ solving

$$\left\{ \begin{array}{l} (\delta_t u_n, \bar{u}) + (\nabla u_n, \nabla \bar{u}) + (u_n \boldsymbol{\sigma}_n, \nabla \bar{u}) = 0, \quad \forall \bar{u}, \\ (\delta_t \boldsymbol{\sigma}_n, \bar{\boldsymbol{\sigma}}) + \langle B_2 \boldsymbol{\sigma}_n, \bar{\boldsymbol{\sigma}} \rangle - 2(u_n \nabla u_n, \bar{\boldsymbol{\sigma}}) = 0, \quad \forall \bar{\boldsymbol{\sigma}}. \end{array} \right. \tag{22}$$

where $\langle B_2 \boldsymbol{\sigma}, \bar{\boldsymbol{\sigma}} \rangle = (\boldsymbol{\sigma}, \bar{\boldsymbol{\sigma}}) + (\nabla \cdot \boldsymbol{\sigma}, \nabla \cdot \bar{\boldsymbol{\sigma}}) + (\text{rot } \boldsymbol{\sigma}, \text{rot } \bar{\boldsymbol{\sigma}})$. Moreover, we recover v_n solving

$$\delta_t v_n + B_1 v_n = u_n^2, \quad \text{a.e. } \mathbf{x} \in \Omega, \tag{23}$$

Main results

1. Total mass conservation.
2. **Existence of solution** $(u_n, \boldsymbol{\sigma}_n, v_n)$, and under a smallness condition on k, this solution is **unique**.
3. **Unconditional energy-stability**:

$$\delta_t \mathcal{E}(u_n, \boldsymbol{\sigma}_n) + \frac{k}{2} \|\delta_t u_n\|_0^2 + \frac{k}{4} \|\delta_t \boldsymbol{\sigma}_n\|_0^2 + \|\nabla u_n\|_0^2 + \frac{1}{2} \|\boldsymbol{\sigma}_n\|_1^2 = 0. \tag{24}$$

FULLY DISCRETE SCHEME

Scheme **US**

Time step n: Given $(u_{n-1}, \boldsymbol{\sigma}_{n-1})$, compute $(u_n, \boldsymbol{\sigma}_n)$ solving

$$\left\{ \begin{array}{l} (\delta_t u_n, \bar{u}) + (\nabla u_n, \nabla \bar{u}) + (u_n \boldsymbol{\sigma}_n, \nabla \bar{u}) = 0, \quad \forall \bar{u}, \\ (\delta_t \boldsymbol{\sigma}_n, \bar{\boldsymbol{\sigma}}) + \langle B_2 \boldsymbol{\sigma}_n, \bar{\boldsymbol{\sigma}} \rangle - 2(u_n \nabla u_n, \bar{\boldsymbol{\sigma}}) = 0, \quad \forall \bar{\boldsymbol{\sigma}}. \end{array} \right. \tag{25}$$

where $\langle B_2 \boldsymbol{\sigma}, \bar{\boldsymbol{\sigma}} \rangle = (\boldsymbol{\sigma}, \bar{\boldsymbol{\sigma}}) + (\nabla \cdot \boldsymbol{\sigma}, \nabla \cdot \bar{\boldsymbol{\sigma}}) + (\mathrm{rot}\, \boldsymbol{\sigma}, \mathrm{rot}\, \bar{\boldsymbol{\sigma}})$. Moreover, we recover v_n solving

$$\delta_t v_n + B_1 v_n = u_n^2, \quad \text{a.e. } \mathbf{x} \in \Omega, \tag{26}$$

Main results

4. Weak estimates.

5. Convergence towards weak solutions.

6. Optimal error estimates for $e_u^n = u(t_n) - u_h^n$, $\mathcal{R}_h^u : H^1(\Omega) \to U_h$,
$e_u^n = (\mathcal{I} - \mathcal{R}_h^u) u(t_n) + \mathcal{R}_h^u u(t_n) - u_h^n = e_{u,i}^n + e_{u,h}^n$,

$$\|(e_{u,h}^n, e_{\boldsymbol{\sigma},h}^n)\|_{l^\infty L^2 \cap l^2 H^1}, \|e_{v,h}^n\|_{l^\infty H^1 \cap l^2 W^{1,6}} \le C(T)(k + h^{m+1}).$$

7. Uniform strong estimates in 2D domains.

ASYMPTOTIC ANALYSIS

Convergence at infinite time

Let (u, v) be any weak-strong solution of problem (2) obtained by Galerkin approximations. Then, the following estimates hold

$$\|(u(t) - m_0, \nabla v(t))\|_0^2 \leq C_0 e^{-2t}, \quad \text{a.e.} \quad t \geq 0. \tag{27}$$

$$\|v(t) - (m_0)^2\|_0^2 \leq C_0 e^{-t}, \quad \forall t \geq 0, \tag{28}$$

where $m_0 := \dfrac{1}{|\Omega|}\displaystyle\int_\Omega u_0$ and C_0 is a positive constant depending on the data (u_0, v_0), but independent of t.

Convergence in stronger norms

Let $\varepsilon > 0$. There exists a constant $C_1 > 0$ such that if $\varepsilon^2 \leq \frac{1}{2C_1}$ it holds

$$\|(u(t) - m_0, \nabla v(t))\|_1^2 \leq 2\varepsilon e^{-\frac{1}{2}(t-t_2)}, \quad \text{a.e. } t \geq t_2(\varepsilon), \tag{29}$$

with $t_2 := t_2(\varepsilon) \geq 0$ a large enough time that will be obtained in the proof.

Main results

1. Total mass conservation.
2. Unconditional existence,
3. **Unconditional energy-stability**:
4. Weak estimates.
5. Convergence towards weak solutions.

Large-time behavior of the scheme **UV**

Let (u_h^n, v_h^n) be a solution of the scheme **UV** associated to an initial data $(u_h^0, v_h^0) \in U_h \times V_h$ which is a suitable approximation of $(u_0, v_0) \in L^2(\Omega) \times H^1(\Omega)$, as $h \downarrow 0$, with $\frac{1}{|\Omega|} \int_\Omega u_h^0 = \frac{1}{|\Omega|} \int_\Omega u_0 = m_0$. Then,

$$\|(u_h^n - m_0, \nabla v_h^n)\|_0^2 \leq C_0 e^{-\frac{2}{1+2k} kn}, \quad \forall n \geq 0, \tag{30}$$

$$\|v_h^n - (m_0)^2\|_0^2 \leq C_0 e^{-\frac{1}{1+k} kn}, \quad \forall n \geq 0, \tag{31}$$

$$k \sum_{m>n} \left(\|\widetilde{u}_h^m\|_1^2 + \frac{1}{2}\|(A_h - I) v_h^n\|_0^2 + \frac{1}{2}\|\nabla v_h^n\|_0^2 \right) \leq C_0 e^{-\frac{2}{1+2k} kn}, \quad \forall n \geq 0, \tag{32}$$

where C_0 is a positive constant depending on the data (u_0, v_0), but independent of (k, h) and n.

Large-time behavior of scheme **US**

Let $(u_h^n, \boldsymbol{\sigma}_h^n)$ be a solution of the scheme **US** associated to an initial data $(u_h^0, \boldsymbol{\sigma}_h^0) \in U_h \times \boldsymbol{\Sigma}_h$ which is a suitable approximation of $(u_0, \boldsymbol{\sigma}_0) \in L^2(\Omega) \times \boldsymbol{L}^2(\Omega)$, as $h \to 0$, with $\frac{1}{|\Omega|}\int_\Omega u_h^0 = \frac{1}{|\Omega|}\int_\Omega u_0 = m_0$. Then,

$$\|(u_h^n - m_0, \boldsymbol{\sigma}_h^n)\|_0^2 \leq C_0 e^{-\frac{2}{1+2k}kn}, \quad \forall n \geq 0, \tag{33}$$

$$k \sum_{m>n} \left(\|\widetilde{u}_h^m\|_1^2 + \frac{1}{2}\|\boldsymbol{\sigma}_h^m\|_1^2 \right) \leq C_0 e^{-\frac{2}{1+2k}kn}, \quad \forall n \geq 0, \tag{34}$$

where C_0 is a positive constant depending on the data $(u_0, \boldsymbol{\sigma}_0)$, but independent of (k, h) and n.

Numerical Simulations

- We are considering a finite element discretization in space associated to the variational formulation of schemes **BE** and **LC**, where the $\mathcal{P}_1$-continuous approximation is taken for u_h, $\boldsymbol{\sigma}_h$ and v_h.
- We have chosen the domain $\Omega = [0,2]^2$ using a structured mesh.
- All the simulations are carried out using **FreeFem++** software.
- The linear iterative method used to approach the nonlinear scheme **US** is the Newton Method, and in all the cases, the iterative method stops when the relative error in L^2-norm is less than $\varepsilon = 10^{-6}$.

Positivity

- Positivity of u_h, v_h in fully discrete schemes???

Remember that

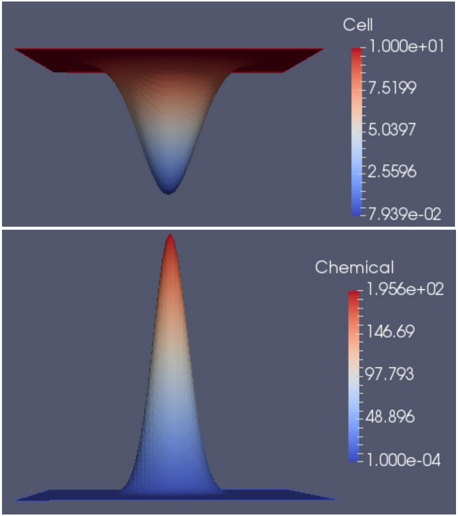
- In scheme **US**, we prove $u_n \geq 0$ and $v_n \geq 0$.
- In scheme **LC**, we prove $v_n \geq 0$, but $u_n \geq 0$??? it is not clear.

Then, we compare the positivity of the variables u_h and v_h in both schemes, taking meshes in space increasingly thinner ($h = \frac{2}{70}$, $h = \frac{2}{150}$ and $h = \frac{2}{300}$). In all the cases, we choose $k = 10^{-5}$ and the initial conditions are

$$u_0 = -10xy(2-x)(2-y)exp(-10(y-1)^2 - 10(x-1)^2) + 10.0001$$

and

$$v_0 = 200xy(2-x)(2-y)exp(-30(y-1)^2 - 30(x-1)^2) + 0.0001.$$


Cell
1.000e+01
7.5199
5.0397
2.5596
7.939e-02
Chemical
1.956e+02
146.69
97.793
48.896
1.000e-04

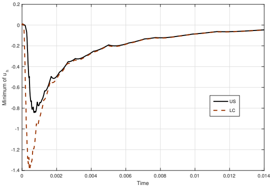


Figure : Minimum values of u_h, with $h = \frac{1}{35}$

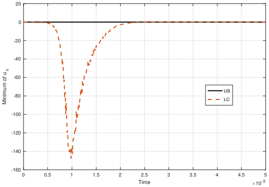


Figure : Minimum values of u_h, with $h = \frac{1}{75}$

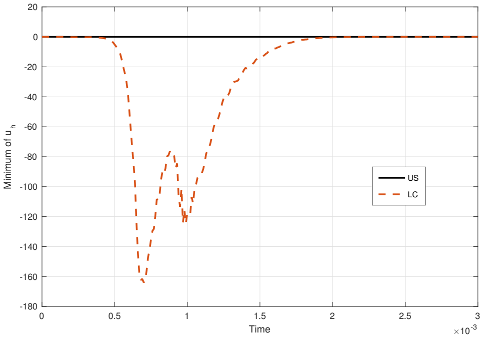
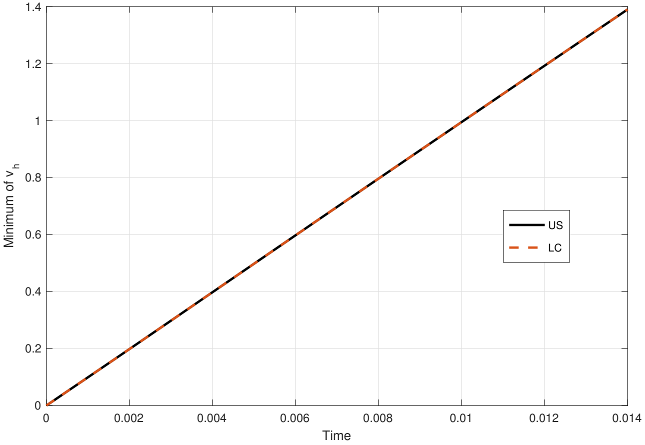


Figure : Minimum values of u_h, with $h = \frac{1}{150}$.


1.4
1.2
1
0.8
0.6
0.4
0.2
0
Minimum of v_h
0
0.002
0.004
0.006
0.008
0.01
0.012
0.014
Time
US
LC

Energy-Stability: If $(u_n, \boldsymbol{\sigma}_n)$ is any solution of the fully discrete schemes corresponding to schemes **US** (or. **LC**), the following relation holds

$$RE(u_n, \boldsymbol{\sigma}_n) := \delta_t E(u_n, \boldsymbol{\sigma}_n) + \|\nabla u_n\|_0^2 + \frac{1}{2}\|\boldsymbol{\sigma}_n\|_1^2 \leq 0, \quad \forall n, \quad (35)$$

where $E(u_n, \boldsymbol{\sigma}_n)$ was defined as:

$$E(u_n, \boldsymbol{\sigma}_n) = \frac{1}{2}\|u_n\|_0^2 + \frac{1}{4}\|\boldsymbol{\sigma}_n\|_0^2. \quad (36)$$

Taking $k = 10^{-6}$, $h = \frac{1}{25}$ and the initial conditions

$$u_0 = -10xy(2-x)(2-y)exp(-10(y-1)^2 - 10(x-1)^2) + 10.0001$$

and

$$v_0 = 20xy(2-x)(2-y)exp(-30(y-1)^2 - 30(x-1)^2) + 0.0001,$$

we obtain

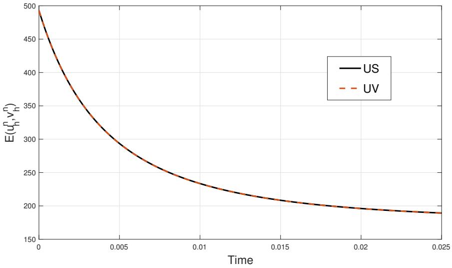


Figure : Energy $\mathcal{E}(u_h^n, v_h^n)$ of schemes **UV** and **US**

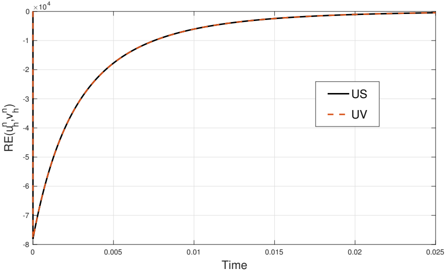


Figure : Residue $RE(u_h^n, v_h^n)$ of schemes **UV** and **US**.

Numeric error orders:

We consider an exact solution for problem (2) but in the variables (u, σ, v), $k = 10^{-5}$ and obtain the following spatial numerical error orders:

Error - h	1/40 - 1/50	1/50 - 1/60	1/60 - 1/70	1/70 - 1/80
$\|e_u^n\|_{l^2 L^2}$	1.9974	1.9986	1.9995	2.0002
$\|e_u^n\|_{l^\infty L^2}$	1.9970	1.9980	1.9985	1.9989
$\|e_u^n\|_{l^2 H^1}$	0.9978	0.9985	0.9989	0.9992
$\|e_u^n\|_{l^\infty H^1}$	0.9981	0.9987	0.9991	0.9993
$\|e_v^n\|_{l^2 L^2}$	2.9481	2.9312	2.9073	2.8763
$\|e_v^n\|_{l^\infty L^2}$	2.8875	2.8421	2.7855	2.7189
$\|e_v^n\|_{l^2 H^1}$	1.9941	1.9956	1.9966	1.9972
$\|e_v^n\|_{l^\infty H^1}$	1.9985	1.9990	1.9993	1.9995

Extensions

- Design unconditional energy stables and mass-conservative numerical schemes for the model

$$\begin{cases} \partial_t u - D_u \Delta u - \nabla \cdot (u \nabla v) = 0 \ \text{ in } \Omega, \ t > 0, \\ \partial_t v - D_v \Delta v + v = u^p \text{ in } \ \Omega, \ t > 0, \\ \frac{\partial v}{\partial \mathbf{n}} = \frac{\partial u}{\partial \mathbf{n}} = 0 \ \text{ on } \partial\Omega, \ t > 0, \\ v(\mathbf{x}, 0) = v_0(\mathbf{x}) > 0, \ u(\mathbf{x}, 0) = u_0(\mathbf{x}) > 0 \ \text{ in } \Omega, \end{cases} \tag{37}$$

  for $p = 1$ and $p \in (1, 2)$.
- The attractive case: Pattern formation.

C. Amrouche and N. E. H. Seloula, *Lp-theory for vector potentials and Sobolev?s inequalities for vector fields: application to the Stokes equations with pressure boundary conditions*. Math. Models Methods Appl. Sci. 23 (2013), no. 1, 37–92.

T. Cieslak, P. Laurençot and C. Morales-Rodrigo, *Global existence and convergence to steady states in a chemorepulsion system.* Parabolic and Navier-Stokes equations. Part 1, 105–117, Banach Center Publ., 81, Part 1, Polish Acad. Sci. Inst. Math., Warsaw, 2008.

F. Guillén-González, M. A. Roríguez-Bellido, D. A. Rueda-Gómez, *Study of a chemo-repulsion model with quadratic production. Part I: Analysis of the continuous problem and time-discrete numerical schemes.* Submitted.

F. Guillén-González, M. A. Roríguez-Bellido, D. A. Rueda-Gómez, *Study of a chemo-repulsion model with quadratic production. Part II: Analysis of an unconditional energy-stable fully discrete scheme.* Submitted.

F. Guillén-González, M. A. Roríguez-Bellido, D. A. Rueda-Gómez, *Asymptotic behaviour for a chemo-repulsion system with quadratic production: The continuous problem and two fully discrete numerical schemes.* Submitted.

M. Marion, R. Temam, *Navier-Stokes equations: theory and approximation*. Handbook of numerical analysis, Vol. VI, 503–688, Handb. Numer. Anal., VI, North-Holland, Amsterdam (1998).

J. Zhang, J. Zhu and R. Zhang, *Characteristic splitting mixed finite element analysis of Keller-Segel chemotaxis models*. Appl. Math. Comput. 278 (2016), 33–44.

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