Remarks on the energy equality for weak solutions to Navier Stokes equations*

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The Navier-Stokes equations

We consider the Navier–Stokes equations in a domain $\Omega \subset \mathbb{R}^3$ with smooth boundary $\partial\Omega$, unit viscosity and zero external force

$$\begin{cases} \partial_t u - \Delta u + (u \cdot \nabla)u + \nabla p = 0 & \text{in } \Omega \times (0,T) \\ \nabla \cdot u = 0 & \text{in } \Omega \times (0,T) \\ u = 0 & \text{on } \partial\Omega \times (0,T) \\ u(\cdot,0) = u_0 & \text{in } \Omega\,, \end{cases} \tag{1}$$

- $u(t,x)$: velocity field
- $p(t,x)$: pressure field

Leray–Hopf solutions

In 3D Leray-Hopf weak solutions are characterized by

$$1) \quad \int_0^\infty (u, \partial_t \phi) - (\nabla u, \nabla \phi) - (u \cdot \nabla u, \phi)\, dt = -(u_0, \phi(0))$$

for all $\phi \in C_0^\infty([0, T[\times \Omega)$ with $\nabla \cdot \phi = 0$;

2) being in $L^\infty(H) \cap L^2(V)$;

3) En. Inequality (EI)

$$\frac{1}{2}\|u(T)\|^2 + \int_0^T \|\nabla u(s)\|^2\, ds \leq \frac{1}{2}\|u_0\|^2 \qquad \forall\, T \geq 0 \qquad \text{(EI)}$$

4)

$$\|u(t) - u_0\| \to 0 \qquad t \to 0^+$$

Classical question

Which regularity of weak solutions for the validity of the energy equality?

$$\frac{1}{2}\|u(T)\|^2 + \int_0^T \|\nabla u(s)\|^2 \, ds = \frac{1}{2}\|u_0\|^2 \qquad \text{(EE)}$$

Leray–Hopf solutions: remarks

- (EI) comes by limit as $\epsilon \to 0$ of approximate solutions

$$\frac{1}{2}\|u_\epsilon(T)\|^2 + \int_0^T \|\nabla u_\epsilon(s)\|^2\, ds = \frac{1}{2}\|u_0\|^2 \qquad \text{(EE)}$$

constructed e.g. by Galerkin or Leray method (regularize Eq. by convolution $u_\epsilon \cdot \nabla$)

- (EE) for u would follow if we could use u itself as test function, not allowed.

Leray–Hopf solutions: scaling invariant regularity

- Observe that by interpolation weak solution have scaling invariant regularity

$$u \in L^p(0, T; L^q) \qquad \frac{2}{p} + \frac{3}{q} = \frac{3}{2}, \quad 2 \leq q \leq 6.$$

Strong solutions

Strong solutions are

Leray-Hopf and also $u \in L^\infty(V)$

for them

we have local existence, uniqueness, regularity for $t > 0$, and (EE)

Large classes of weak solutions which are strong are those of Leray-Prodi-Serrin-Ladyzheskaya.....

$$u \in L^p(0, T; L^q) \qquad \frac{2}{p} + \frac{3}{q} = 1 \qquad q > 3 \qquad (q \geq 3)$$

(EE) and regularity of weak solutions

One main problem when dealing with weak sol. is that only FORMALLY

$$\int_0^T (u\cdot\nabla u, u)\, dt = \frac{1}{2}\int_0^T (u, \nabla|u|^2)\, dt = -\frac{1}{2}\int_0^T (\nabla\cdot u, |u|^2)\, dt = 0,$$

in fact

$L^{4/3}(V') \ni u\cdot\nabla u$ $\qquad u \in L^2(V)$ $\qquad$ duality pairing not well-defined

The main point is to give meaning to space-time integral of $u \cdot \nabla u \cdot u$

Which regularity for (EE)? (I)

Performing this calculation and proving eventually (EE) could be done with less than critical scale invariant solution. This suggested by a pioneering result of J.J. Lions (Padova 1960) and G. Prodi (Ann. Mat. Pura Appl. 1959, Padova 1960):

$$u \in L^4(L^4) \qquad \text{implies (EE)}$$

in fact

$$\Big|\int_0^T (u\cdot\nabla u, u)\,dt\Big| \le \int_0^T \|u\|_{L^4}^2 \|\nabla u\|\,dt \le \Big(\int_0^T \|u\|_{L^4}^4\,dt\Big)^{1/2}\Big(\int_0^T \|\nabla u\|^2\,dt\Big)^{1/2} < \infty$$

hence $\int_0^T (u\cdot\nabla u, u)\,dt = 0$ and calculations leading to (EE) can be justified by approximation by $u_h = \rho_h *_t u$ (time-mollification with even kernel ρ)

$$\int_0^T (\nabla u, \nabla u_h) \to \int_0^T \|\nabla u\|^2, \quad \int_0^T (u\cdot\nabla u, \nabla u_h) \to 0, \quad (u(t), u_h(t)) \to \frac{\|u(t)\|^2}{2}$$

Remarks

In terms of scaling

$$1 < \frac{2}{4} + \frac{3}{4} = \frac{5}{4} < \frac{3}{2}$$

hence Lions' result is intermediate between

1 that is regularity and $\frac{3}{2}$ that is existence

Long-standing conjecture of Prodi is

$$\text{(EE)} \quad \Longrightarrow \quad \text{uniqueness???}$$

At present the only result coming from (EE) is the local energy inequality for solutions constructed by the Fourier-Galerkin method in the torus (Craig et al (2007))

Which regularity for (EE)? (II)

With minor changes Shinbrot (SIMA 1974) extended to

$$u \in L^p(L^q) \qquad \frac{2}{p} + \frac{2}{q} = 1 \qquad q \geq 4$$

in terms of scaling

$$1 < \frac{2}{p} + \frac{3}{q} = \frac{2}{p} + \frac{2}{q} + \frac{1}{q} = 1 + \frac{1}{q} \leq \frac{5}{4} < \frac{3}{2}$$

Observe that There is a GAP and gap decreases as $q \to +\infty$ (limiting case is the same as Serrin)

END OF CLASSICAL STORY

Pressure regularity and (EE)

Recent results Kukavica (JDDE 2006) (weaker but dimensionally equivalent to Lions's result)

$$\pi \in L^2(L^2) \qquad \text{implies (EE)}$$

observe that $\pi \in L^2(L^2) \sim u \in L^4(L^4)$ since $\Delta\pi = \nabla\nabla(uu)$. The pressure in fact scales "as u squared" in terms of regularity Berselli-Galdi (Proc AMS 2002) and Kang-Lee (IRMN notices 2006)

$$\pi \in L^p(L^q) \qquad \frac{2}{p} + \frac{3}{q} = 2 \qquad \text{gives strong solution,}$$

and Kucavica result is in the same spirit.

"Relaxed" Prodi–Lions

Recent result: Maremonti (J. Math. Fluid Mech. 2018) proves a relaxed Prodi–Serrin condition for regularity as well as a relaxed Prodi–Lions condition for (EE). In particular

$$L^4(\varepsilon, T; L^4) \qquad \text{implies (EE)}$$

for all $\varepsilon > 0$.

$\rightarrow$ No compatibility condition on the initial data

Point of view

We work with levels of regularity for ∇u and then we argue by embedding.
In this respect there is a result by Cheskidov-Friedlander-Shvydkoy (Adv. Math. Fluid Mech. 2010)

$$u \in L^3(0, T; D(A^{5/12})) \sim L^3(0, T; H^{5/6}) \subset L^3(0, T; L^{9/2})$$

considering a fractional derivative. In terms of scaling

$$1 < \frac{2}{3} + \frac{2}{9/2} = \frac{10}{9}$$

weaker than Shinbrot.

Main theorem (Berselli, C., 2018)

Our result is the following: if $\nabla u \in L^p(0,T;L^q)$ for the following ranges of p, q

(i) $\frac{3}{2} < q < \frac{9}{5}$ and $p = \frac{q}{2q-3}$ or $L^{\frac{q}{2q-3}}(0,T;L^q)$

(ii) $\frac{9}{5} \leq q < 3$ and $p = \frac{5q}{5q-6}$ or $L^{\frac{5q}{5q-6}}(0,T;L^q)$

(iii) $q \geq 3$ and $p = 1 + \frac{2}{q}$ or $L^{1+\frac{2}{q}}(0,T;L^q)$,

then u satisfies (EE).

Comments (I)

In terms of scaling invariant regularity for ∇u it is known that

$$\nabla u \in L^p(L^q) \qquad \frac{2}{p}+\frac{3}{q}=2 \quad \text{gives strong solutions}$$

Beirão da Veiga (Chinese Ann. Math. 1995, $\mathbf{R}^3$) and Berselli (DIE 2002, Ω).

For weak solutions ∇u is (x,t)-square-integrable and

$$\frac{2}{2}+\frac{3}{2}=\frac{5}{2}$$

Comments (II)

In our ranges

(i) $2 < \frac{2}{p} + \frac{3}{q} = 4 - \frac{3}{q} < \frac{5}{2}$, for any $\frac{3}{2} < q < \frac{9}{5}$,

(ii) $2 < \frac{2}{p} + \frac{3}{q} = 2 + \frac{3}{5q} < \frac{5}{2}$, for any $\frac{9}{5} \leq q < 3$,

(iii) $2 < \frac{2}{p} + \frac{3}{q} = \frac{2q}{q+2} + \frac{3}{q} < \frac{5}{2}$, for any $3 \leq q < 6$,

Comments (III)

Observe that the "Best exponent" is $q = 9/5$, that is by embedding $q^* = 9/2$ which gives

$$u \in L^3(0, T; W^{1,9/5}) \subset L^3(0, T; L^{9/2})$$

at the same level of CFS (2010)

Comments (IV)

Recalling that $q^* = \frac{3q}{3-q}$, we have

(i) $1 < \frac{2}{p} + \frac{2}{q^*} = \frac{2(5q-6)}{3q}$ for any $\frac{12}{7} < q < \frac{9}{5}$,

(ii) $1 < \frac{2}{p} + \frac{2}{q^*} = \frac{2(10q-3)}{15q}$ for any $\frac{9}{5} \leq q < 3$,

(iii) $1 < \frac{2}{p} + \frac{2}{q^*} = \frac{2(2q^2+q+6)}{3q(q+2)}$ for any $q \geq 3$,

thus showing that our range of exponents improves Shinbrot.

We recall that Shinbrot condition for the space integrability ≥ 4 corresponds to $q \geq \frac{12}{7}$ (i.e. $q^* \geq 4$) in our classification.

Comments (V)

In the range $\frac{12}{7} < q \le \frac{9}{5}$ (by embedding $3 \le p < q^*, q^* > 4$) our result improves also the ranges obtained by Leslie-Shvydkoy (SIMA 2018). They prove (EE) for

$$u \in L^p(0, T; L^r) \qquad \frac{2}{p} + \frac{2}{r} \le 1 \qquad 3 \le p \le r$$

However, Leslie and Shvydkoy studied also the case $r < 3$ corresponding in our case to $q < 3/2$ which is not covered by our ranges.
Our results are not based on Paley-Littlewood decomposition and come from a different approach.

Onsager conjecture

The validity of (EE) has also connections with Onsager conjecture. In terms of Hölder-Besov spaces we recall Cheskidov-Constantin-Friedlander-Shvydkoy (Nonlinearity 2008). This has been recently extended by Cheskidov-Luo (ArXiv 2018) to

$$u \in L^{\beta}_{w}(0,T; B_{p,\infty}^{\frac{2}{\beta}+\frac{2}{p}-1}), \quad 1 \leq \beta < p \leq \infty, \ \frac{2}{p}+\frac{1}{\beta} < 1 \qquad \text{implies (EE)}$$

Onsager conjecture and our results

Our results show by embedding the condition $u \in L^{1+\frac{2}{q}}(0, T; C^{0,1-\frac{3}{q}})$ which is in the case $q = 9/2$

$$u \in L^{\frac{13}{9}}(0, T; C^{0,1/3}),$$

which improves Cheskidov-Luo, since setting $p = \infty$, and $\beta = 3/2$ one gets

$$u \in L^{\frac{3}{2}}_w(0, T; B^{1/3}_{\infty,\infty})$$

and $1.5 = 3/2 > 13/9 = 1.\overline{4}$.

Very weak solutions

All these results are valid for Leray-Hopf weak solutions, but a recent result of Galdi (ArXiv 2017) proves that

$$u \in L^4(0; T; L^4) \quad u \text{ very weak, implies (EE)}$$

Observe also that scaling invariant very weak solutions

$$u \in L^p(0, T; L^q) \qquad \frac{2}{p} + \frac{3}{q} = 1 \quad q > 3 \qquad \text{and } C(0, T; L^3)$$

are unique [Foias (Bull. Soc. Math. France 1961)] and regular [Fabes-Jones-Riviere (ARMA 1972) and Berselli-Galdi (Nonlinear TMA 2004) for $q = 3$]

Ideas of proof

These results are based on the regularity of the adjoint (backward parabolic Oseen) equation.

$$\partial_t w(t) + u_\epsilon(T-t)\cdot\nabla w(t) - \Delta w(t) + \nabla\zeta(t) = f(T-t)$$

with vanishing initial datum and $f \in C_0^\infty$.
The exponents $p = q = 4$ play a special role in the proof since one can show that w is such that

$$w_t, \Delta w, \nabla\zeta \in L^{4/3}(0,T;L^{4/3})$$

and can be used as a test function to show, by duality if $u_0 \in L^2$, that u is also a Leray-Hopf solution, and then to apply Lions.

Extension of Galdi result (Berselli, C., 2018)

We extended this result to Shinbrot type exponents: if u is very weak and

$$u \in L^p(L^q) \qquad \frac{2}{p} + \frac{2}{q} < 1 \qquad q \geq 4$$

then (EE) holds true.

- The proof is based on a bootstrap argument to arrive to the requested space-time regularity needed to use the solution of the dual problem as test function.
- The loss < 1 is due to a certain space-time interpolation result, which fails in the limiting case.

Thank you for your attention!