Lecture 2: Model equations for water waves

Walter Craig

Department of Mathematics & Statistics

McMaster University

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Outline

Hamiltonian perturbation theory

Shallow water equations

Transformation theory

Boussinesq and KdV scaling regimes

Modulational scaling regimes

Hamiltonian perturbation theory

- Suppose the Hamiltonian H depends upon ε a small parameter

$$H(v;\varepsilon) = H^{(0)} + \varepsilon H^{(1)} + \ldots \varepsilon^m H^{(m)} + \varepsilon^{m+1} R(v;\varepsilon)$$

- It is natural to approximate orbits $v(t;\varepsilon)$ by the truncated problem

$$\begin{aligned} \dot{v} &= J grad_v \Big(H^{(0)} + \varepsilon H^{(1)} + \ldots \varepsilon^m H^{(m)} \Big) \\ & v(0) = v_0 \,, \qquad v(t) = v(t;\varepsilon, m) \end{aligned}$$

Proposition

Suppose $H \in C^{2,m+1}(M \times \mathcal{E})$. Then for $|t| \leq T_0$, approximate orbits $v(t;\varepsilon,m)$ are ε^m close to orbits of the full system

- NB Bounded time of validity, may be extended to $T_\varepsilon \sim \log(1/\varepsilon)$

  Averaging theory methods, including Birkhoff normal forms, allow one to extend T_ε the time interval of validity

Shallow water equations

- **Burger's equation** A simplified form of nonlinear wave equation, where $x \in \mathbb{R}^1$ and normally $u(t,x)$ has units of velocity

$$\partial_t u = u\partial_x u \ , \qquad u(0,x) = u_0(x) \tag{1}$$

Hamiltonian for Burger's flow, with symplectic structure $J = \partial_x$

$$H_{Burgers}(u) = \int \tfrac{1}{6}u^3\,dx \qquad \partial_t u = J\,\mathrm{grad}\,H_{Burgers}(u) = \partial_x\big(\tfrac{1}{2}u^2\big)$$

- **Shallow water equations** Scaling regime of water waves in which $X := \varepsilon x$ and $(\eta, \xi) = (\eta(X), \xi(X)/\varepsilon)$ are slowly varying

$$\begin{aligned} \partial_t \eta &= -\partial_x \cdot \big((h+\eta)\partial_x \xi\big) \\ \partial_t \xi &= -g\eta - \tfrac{1}{2}|\partial_x \xi|^2 \end{aligned} \tag{2}$$

The work of T. Kano & T. Nishida (1979) was first to study the rigorous justification of this scaling limit. In their case it was in the setting of $(x,y) \in \mathbb{R}^2$ and with analytic initial data

Shallow water equations

- **Green Naghdi equations** are higher order equations in the shallow water scaling regime. These are studied in detail by D. Lannes (2000).
  The Hamiltonian is (setting $\partial_x = iD$)

$$H(\eta,\xi) = \tfrac{1}{2}\int \Big(\xi D\cdot(h+\eta)D\xi + g\eta^2\Big) + \varepsilon^2\Big(-h^2\xi D^2(\frac{h}{3}+\eta)D^2\xi - 2\xi D^2 h\eta^2 D^2\Big)\,dx$$

  This gives rise to the system of equations

$$\partial_t\eta = D\cdot(h+\eta)D\xi + \varepsilon^2\big(-h^2D^2(\frac{h}{3}+\eta)D^2\xi\big) \tag{3}$$

$$\partial_t\xi = -g\eta - \tfrac{1}{2}|D\xi|^2 + \frac{\varepsilon^2}{2}\big(h^2+4h\eta\big)|D^2\xi|^2$$

calculus of transformations

Goal: to introduce small parameters into the Hamiltonian for water waves in order to systematically treat these and further model equations with Hamiltonian perturbation theory

- Consider two phase spaces M_1, M_2, with J_1 defining ω_1 on M_1
- A transformation

$$\tau : M_1 \to M_2 \,, \qquad v \mapsto w = \tau(v)$$

gives a Hamiltonian $H_2(w) = H_2(\tau(v)) = H_1(v)$ defined on M_2

- The Hamiltonian vector field X_{H_1} transforms as

$$\dot{w} = \partial_v \tau(v) \dot{v} = \partial_v \tau(v) J_1 \mathrm{grad}\,_v H_1(v)$$
$$\mathrm{grad}\,_v H_1(v) = (\partial_v \tau)^T \mathrm{grad}\,_w H_2(\tau(v))$$

The vector field $X_{H_1} = J_1 \mathrm{grad}\,_v H_1(v)$ becomes

$$\dot{w} = \partial_v \tau(v) J_1 (\partial_v \tau)^T \, \mathrm{grad}\,_w H_2(\tau(v))$$

calculus of transformations

- Use $J = \partial_v \tau(v) J_1 (\partial_v \tau)^T$ to define a symplectic form on M_2
- When M_2 already has a symplectic structure represented by J_2 and $w = \tau(v)$ is such that

$$J_2 = \partial_v \tau(v) J_1 (\partial_v \tau)^T$$

  then τ is called canonical.
- When $M_1 = M_2$ and $J_1 = J_2$ are given by Darboux coordinates, these are the traditional canonical transformations.

examples of transformations

- Phase space for surface water waves, let $M = L^2(\mathbb{R}^{d-1})^2$

$$v = \begin{pmatrix} \eta \\ \xi \end{pmatrix} \in M\,, \qquad \langle v_1 | v_2 \rangle = \int \eta_1 \eta_2 + \xi_1 \xi_2 \, dx$$

$$J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} \qquad \text{Darboux coordinates}$$

- Amplitude scaling: Define $\tau : v \to w$

$$w = \begin{pmatrix} \eta' \\ \xi' \end{pmatrix} = \begin{pmatrix} \alpha\eta \\ \beta\xi \end{pmatrix} = \tau(v)\,,$$

- The Jacobian

$$\partial_v \tau = \begin{pmatrix} \alpha I & 0 \\ 0 & \beta I \end{pmatrix}$$

thus the induced symplectic form is given by

$$J_1 = \partial_v \tau J \partial_v \tau^T = \alpha\beta J$$

Amplitude scaling

- Use scaling to introduce a small parameter

$$\begin{pmatrix} \varepsilon^2 \eta' \\ \varepsilon \xi' \end{pmatrix} = \begin{pmatrix} \eta \\ \xi \end{pmatrix} , \qquad \varepsilon \ll 1$$

- Then $J_1 = \varepsilon^{-3} J$ and the Hamiltonian has a series expansion in ε

$$H_1 = \int \tfrac{1}{2}\varepsilon^2 \xi' G^{(0)} \xi' + \tfrac{g}{2}\varepsilon^4 \eta'^2 \, dx + \sum_{j \geq 3} \tfrac{1}{2} \int \varepsilon^{2+2j} \xi' G^{(j-2)}(\eta') \xi' \, dx$$

- Up to order $\mathcal{O}(\varepsilon^4)$

$$\varepsilon^2 H_1^{(2)} + \varepsilon^4 H_1^{(4)} = \varepsilon^2 \left(\int \tfrac{1}{2} \xi' G^{(0)} \xi' \, dx \right) + \varepsilon^4 \left(\int \tfrac{g}{2} \eta'^2 + \tfrac{1}{2} \xi' G^{(1)}(\eta') \xi' \, dx \right)$$

where $G^{(1)}(\eta') = D_x \eta' D_x - G^{(0)} \eta' G^{(0)}$

spatial scaling

- A small parameter introduced by spatial scaling $x \mapsto X := \varepsilon x$ resulting in

$$\tau : v(x) \mapsto w(X) = v(X/\varepsilon) = \tau(v)(X)$$

- The Jacobian of this transformation

$$\partial_v \tau(v) V(X) = \frac{d}{ds}|_{s=0} \Big(v(X/\varepsilon) + sV(X/\varepsilon) \Big) = V(X/\varepsilon)$$

- The transpose is less obviously $(\partial_v \tau)^T V(x) = \varepsilon^{d-1} V(\varepsilon x)$ since

$$\begin{aligned} \langle V_1 | \partial_v \tau V_2 \rangle &= \int_{\mathbb{R}^{d-1}} V_1(X) V_2(X/\varepsilon), dX \\ &= \int_{\mathbb{R}^{d-1}} V_1(\varepsilon x) V_2(x)\, \varepsilon^{d-1} dx = \langle (\partial_v \tau)^T V_1 | V_2 \rangle \end{aligned}$$

spatial scaling

- Lemma

  Fourier multiplier operators subject to spatial scaling

$$(m(D_x)v)(x) = \frac{1}{(2\pi)^{d-1}} \int\int_{\mathbb{R}^{2(d-1)}} e^{ik\cdot(x-x')} m(k) v(x')\, dx' dk$$

satisfy $\tau(m(D_x)v)(X) = (m(\varepsilon D_X)\tau(v))(X)$

- The Dirichlet – Neumann operator $G^{(0)}(D_x) = |D_x| \tanh(h|D_x|)$ is transformed to

$$G^{(0)}(\varepsilon D_X) = \varepsilon |D_x| \tanh(\varepsilon h|D_x|) \sim \varepsilon^2 h |D_X|^2 - \frac{\varepsilon^4 h^3}{3} |D_X|^4 + \ldots$$

- and the Hamiltonian becomes

$$\begin{aligned} H_2 \quad = \quad & \varepsilon^4 \int_{\mathbb{R}^{d-1}} \Big(\tfrac{1}{2} \xi (h|D_X|^2 \xi + \tfrac{g}{2} \eta^2 \Big) \\ & + \frac{\varepsilon^2}{2} \Big(\xi(-\frac{h^3}{3} |D_X|^4 \xi) + \xi D_X \cdot \eta D_X \xi \Big) \frac{dX}{\varepsilon^{d-1}} + \varepsilon^{9-d} R_2^{(2)} \end{aligned}$$

surface elevation - velocity coordinates

- Let $d = 2$ at this point. Transform $w = (\eta, u) = \tau(v) = (\eta, \partial_X \xi)$, with Jacobian

$$\partial_v \tau(v) = \begin{pmatrix} I & 0 \\ 0 & \partial_X \end{pmatrix}$$

  and the new

$$J_2 = \partial_v \tau J (\partial_v \tau)^T = \begin{pmatrix} 0 & -\partial_X \\ -\partial_X & 0 \end{pmatrix}$$

  In this one recognizes the *Boussinesq* symplectic form

- Phrasing the Hamiltonian in these variables

$$H_2 = \varepsilon^3 \int_{\mathbb{R}^1} \left(\tfrac{h}{2} u^2 + \tfrac{g}{2} \eta^2 \right) + \tfrac{\varepsilon^2}{2} \left(\tfrac{-h^3}{3} (\partial_X u)^2 + \eta u^2 \right) dX + \mathcal{O}(\varepsilon^7) \ ,$$

  the leading term gives precisely the Boussinesq system of Kaup and Sachs.

Boussinesq system of Kaup and Sachs

- **Boussinesq system**
  For $x \in \mathbb{R}^1$ the Hamiltonian is

$$H_{Boussinesq}(u, \eta) = \int \frac{h}{2}u^2 + \frac{g}{2}\eta^2 - \varepsilon^2 \frac{h^3}{6}(\partial_X u)^2 + \frac{\varepsilon^2}{2}\eta u^2 \, dX$$

The symplectic form is given by $J_2 = \begin{pmatrix} 0 & -\partial_X \\ -\partial_X & 0 \end{pmatrix}$

$$\partial_t \eta = -\partial_X \big(hu + \varepsilon^2 \frac{h^3}{3} \partial_X^2 u + \varepsilon^2 \eta u \big)$$
$$\partial_t u = -\partial_X \big(g\eta + \frac{\varepsilon^2}{2} u^2 \big) \qquad (4)$$

The original version of the Boussinesq equation is one where the term ηu^2 is replaced by u^3, which was studied by Zakharov.

Both cases of Zakharov and of Kaup and Sachs are completely integrable nonlinear PDEs, studied via the inverse spectral/scattering transform

moving reference frame

- Coordinate systems which move with c the characteristic speed $v'(x,t) := v(x-tc,t)$
- The momentum

$$I(\eta,\xi) := \int_{\mathbb{R}} \eta(x)\partial_x\xi(x)\,dx = \varepsilon^3 \int_{\mathbb{R}} u\eta\,dX$$

  gives flow by constant unit speed translation $\varphi_s^I(v)(x) = v(x-s)$. It Poisson commutes with H, $\{I,H\} = 0$
- Therefore the Hamiltonian flow of $H(v) + cI(v)$ is the flow of $H(v)$ observed in a coordinate frame translating with velocity c
  The Hamiltonian is

$$\begin{aligned}
&H_3 := H_2 + \sqrt{gh}I = \\
&= \varepsilon^3 \int_{\mathbb{R}} \tfrac{1}{2}\Big(hu^2 + 2\sqrt{gh}\,u\eta + g\eta^2\Big) + \tfrac{\varepsilon^2}{2}\Big(\tfrac{-h^3}{3}(\partial_X u)^2 + \eta u^2\Big)dX \\
&= \varepsilon^3 \int_{\mathbb{R}} \tfrac{1}{2}(\sqrt{h}u + \sqrt{g}\eta)^2 + \tfrac{\varepsilon^2}{2}\Big(\tfrac{-h^3}{3}(\partial_X u)^2 + \eta u^2\Big)dX + \mathcal{O}(\varepsilon^7)
\end{aligned}$$

characteristic coordinates

- A common situation with $A, B > 0$

$$H_3^{(2)} = \tfrac{1}{2} \int_{\mathbb{R}} Au^2 + B\eta^2 \, dX$$

- Find characteristic coordinates in which

$$J_3 := \partial_v \tau \begin{pmatrix} 0 & -\partial_X \\ -\partial_X & 0 \end{pmatrix} (\partial_v \tau)^T = \begin{pmatrix} \partial_X & 0 \\ 0 & -\partial_X \end{pmatrix}$$

  and it should transform the Hamiltonian to normal form

$$H_3^{(2)} = \tfrac{1}{2} \int_{\mathbb{R}} \sqrt{AB}(r^2 + s^2) \, dX$$

- In the case of surface water waves

$$\begin{pmatrix} r \\ s \end{pmatrix} = \begin{pmatrix} \sqrt[4]{\frac{g}{4h}} & -\sqrt[4]{\frac{h}{4g}} \\ \sqrt[4]{\frac{g}{4h}} & \sqrt[4]{\frac{h}{4g}} \end{pmatrix} \begin{pmatrix} \eta \\ u \end{pmatrix}$$

characteristic coordinates

- The relevant Hamiltonian approximation which is valid up to $\mathcal{O}(\varepsilon^5)$ is given by

$$H_3 = H_2 + \sqrt{gh}I = \varepsilon^3 \int_{\mathbb{R}} \sqrt{gh}s^2\, dX$$
$$+\varepsilon^5 \int_{\mathbb{R}} \tfrac{-h^3}{6}\sqrt{\tfrac{g}{4h}}(\partial_X r - \partial_X s)^2 + \tfrac{1}{\sqrt{32}}\sqrt[4]{\tfrac{g}{h}}(r^3 - r^2 s - r s^2 + s^3)\, dX$$

- Restrict this Hamiltonian to the hypersurface $M_1 := \{s = 0\} \subseteq M$, which is a symplectic subspace of M
- Denoting the transformed Hamiltonian by H_4 and $J_4 := \partial_x$

$$H_4 = \varepsilon^5 \int_{\mathbb{R}} -\tfrac{h^3}{6}\left(\sqrt{\tfrac{g}{4h}}\right)(\partial_X r)^2 + \tfrac{1}{4\sqrt{2}}\sqrt[4]{\tfrac{g}{h}} r^3\, dX + \mathcal{O}(\varepsilon^7)$$

which is the KdV Hamiltonian up to order $\mathcal{O}(\varepsilon^5)$

Korteweg deVries scaling limit

- Taking the above Hamiltonian H_4 and rescaling $\partial_t = \varepsilon^3 \partial_\tau$ to slowly varying timescales $\varepsilon^3 t = \tau$

$$
\begin{aligned}
\partial_\tau r &= \partial_X \mathrm{grad}_{\,r} H_4 \\
&= \partial_X (c_1 \partial_X^2 r + c_2 r^2)
\end{aligned}
$$

  where

$$
c_1 = \frac{h^3}{3}\sqrt{\frac{g}{4h}}\,, \qquad c_2 = \frac{3}{4\sqrt{2}}\sqrt[4]{\frac{g}{h}}
$$

- The KdV scaling regime of water waves is a situation of a singular limit. This was studied in the papers Craig (1985) and T. Kano & T. Nishida (1986). The key point is to provide an existence theorem for water waves that is valid for time intervals of at least $|\tau| \sim \mathcal{O}(1)$ in this scaling regime.

  Further progress appears in G. Schneider & E. Wayne (2000) on the decomposition of general water wave initial data into two KdV components, one traveling left and a second traveling right.

The NLS and its higher order corrections

- The nonlinear Schrödinger equation (NLS)

$$i\partial_t u = a_1 \partial_x^2 u + a_2 |u|^2 u \,, \qquad u(0,x) = u_0(x)$$

  and its Hamiltonian

$$H(u,\overline{u}) = \int a_1 |\partial_x u|^2 - \frac{a_2}{2} |u|^4 \, dx$$

- The NLS arises as a description of the evolution of the envelope of a near-monochromatic solutions

$$\eta(t,x) = e^{ik_0 \cdot x} Y(t,X) + c.c. \,, \qquad X = \varepsilon x$$

  Such descriptions arise in Benjamin – Feir instabilities and in some theories of rogue waves
- The assumptions constitute an Ansatz along with a scaling. Our task, to give a rigorous justification of the Ansatz and scaling regime

Hamiltonian derivation of the NLS

- The water waves Hamiltonian

$$H(\eta,\xi) = \frac{1}{2}\int \xi G^{(0)}\xi + g\eta^2\,dx + \sum_{m\geq 3} H^{(m)}(\eta,\xi)$$

  Study the case $x \in \mathbb{R}^1$ and $h = +\infty$ so that $G^{(0)}\xi = |D_x|\xi$

- Canonical transformation to complex symplectic coordinates

$$z(x) := \frac{1}{\sqrt{2}}\big(a(D_x)\eta + ia^{-1}(D_x)\xi\big)$$

  where $a(D_x) = \sqrt[4]{g/|D_x|}$

- In these coordinates the linearized equations are encoded

$$H^{(2)} = \int \bar{z}\omega(D_x)z\,dx\,, \qquad \omega(D_x) := \sqrt{g|D_x|}$$

  This describes a harmonic oscillator in complex variables

$$\partial_t z = i\omega(D_x)z$$

Several averaging lemmas

- Introduce the modulational Ansatz through the transformation

$$z(x) := \varepsilon u(X) e^{ik_0 \cdot x}$$

- Lemma (commutation with Fourier multipliers)

  Let $m(D_x)$ *be a smooth Fourier multiplier. For* $X = \varepsilon x$ *and for* $f(X) \in \mathcal{S}$ *Schwartz class*

$$\begin{aligned} m(D_x)(e^{ik_0 \cdot x} f(X)) &= e^{ik_0 \cdot x} m(k_0 + \varepsilon D_X) f(X) \\ &= e^{ik_0 \cdot x} \big(m(k_0) + \varepsilon \partial_k m(k_0) \cdot D_X + \dots \big) f(X) \end{aligned}$$

- Lemma (averaging lemma)

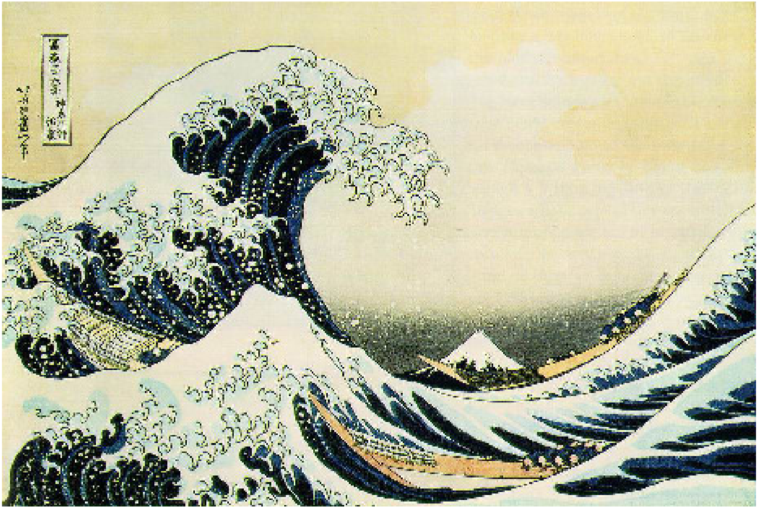
  Let $g(x)$ *be periodic with average* $E(g)$*. For any* $f(X) \in \mathcal{S}$ *and any* N

$$\int g(X/\varepsilon) f(X)\, dX = E(g) \int f(X)\, dX + \mathcal{O}(\varepsilon^N)$$

- The Hamiltonian for $u(X)$ in the modulational Ansatz

$$H(u,\overline{u}) = \int \omega(k_0)|u|^2\,dX + \frac{\varepsilon}{2}\partial_k\omega(k_0)\cdot\left(\overline{u}D_Xu + u\overline{D_Xu}\right)dX$$
$$+\frac{\varepsilon^2}{2}\int \partial_k^2\omega(k_0)|D_Xu|^2 + \frac{3}{4}\omega(k_0)^{-2}|u|^4\,dX + \mathcal{O}(\varepsilon^3)$$

  The first and second terms are the L^2 mass and the momentum, respectively. They are conserved quantites of NLS flow and thus can be set to constant. The third term is the NLS Hamiltonian.
- The paper of Craig, C. Sulem & P.L. Sulem (1992) studied the NLS Ansatz and scaling regime for water waves, it however did not give a proof of existence time $T \sim \mathcal{O}(\varepsilon^{-2})$. This is addressed by N. Totz & S. Wu (2011) and in W. Düll, G. Schneider & C.E. Wayne (2016)
- The Dysthe equations represent a higher order approximation to the water wave equations in a modulational regime. A Hamiltonian derivation is given in Craig, P. Guyenne & C. Sulem (2010). However a rigorous limit theorem is still lacking



Thank you