Lecture 3: The initial value problem for water waves

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Outline

History of the study of water waves

The initial value problem

Shallow water regime

Korteweg de Vries regime

Energy estimates and choice of coordinates

History of study of water waves

- **Pierre – Simon Laplace** (1813) *Une masse fluide pesante, primitivement en repos, et d'une profondeur indéfinie, a été mise en mouvement par l'effet d'une cause donnée. On demande, au bout d'un temps déterminé, la forme de la surface extérieure du fluide et la vitesse de chacune des molécules situées à cette même surface.*
- **John Scott Russell** (1834) Observations of solitary waves in the Union Canal outside of Edinburgh
- **G.B. Airy** (1845) Argued that linear theory does not produce waves of permanent form (solitary waves)
- **G.G. Stokes** (1847) Described water wave motion in terms of the linear wave equation
- **J. Boussinesq** (1872) A nonlinear theory gives rise to solitary waves (Boussinesq system)
- **D.J. Korteweg & G. de Vries** (1894) Further justification for the existence of solitary waves (Korteweg - deVries equations)

History of mathematical analysis

- **Steady wave motion**
  - **T. Levi-Civita** (1924) in two dimensions, nonlinear periodic traveling waves over deep water $h = +\infty$
  - **D. Struik** (1925) nonlinear periodic traveling waves in finite depth $0 < h < +\infty$
  - **H. Lewy** (1952) a priori analyticity of C^1 steady waves
  - **M.A. Lavrientiev** (1943) and **K.-O. Friedrichs & D.H. Hyers** (1954) existence of the solitary wave solution family
  - **C. Amick, E. Fraenkel & J. Toland** (1982) proved the Stokes conjecture

- **Initial value problem**
  - **V.I. Nalimov** (1974) Local well posedness in two dimensions for Sobolev data for $h = +\infty$, and **H. Yosihara** (1982) for the case $0 < h < +\infty$
  - **L.V. Ovsyannikov** (1973) and **T. Kano & T. Nishida** (1979) Local well posedness for analytic data for $0 < h < +\infty$, and rigorous justification of the shallow water scaling limit

A question of P. Lax

- The equations of water waves

$$\partial_t \eta = G(\eta)\xi$$
$$\partial_t \xi = -g\eta - \frac{1}{2(1+|\partial_x\eta|^2)}\Big(|\partial_x\xi|^2 - (G(\eta)\xi)^2 - 2(\partial_x\eta\cdot\partial_x\xi)G(\eta)\xi + \big(|\partial_x\eta|^2|\partial_x\xi|^2 - (\partial_x\eta\cdot\partial_x\xi)^2\big)\Big)$$

  expressed in terms of the variables $(\eta(t,x), \xi(t,x))$ and the Dirichlet – Neumann operator $G(\eta)\xi(x) = \partial_y\varphi - \partial_x\eta\cdot\partial_x\varphi$

- This is a quasilinear system of equations.
  The term containing g the acceleration of gravity is lower order

  Question: Why does the sign of gravity play such an important role in the well-posedness of the initial value problem?

Hyperbolic equations with multiple characteristics

- Linearized equations about $(\eta, \xi) = 0$, setting $(Y, \Xi) := (\delta\eta, \delta\xi)$

$$\begin{pmatrix} \partial_t Y \\ \partial_t \Xi \end{pmatrix} = \begin{pmatrix} 0 & G^{(0)} \\ -g & 0 \end{pmatrix} \begin{pmatrix} Y \\ \Xi \end{pmatrix} := M \begin{pmatrix} Y \\ \Xi \end{pmatrix}$$

- The Dirichlet – Neumann operator is first order, when the bottom is flat it is given by a Fourier multiplier operator

$$G^{(0)}\Xi = |D| \tanh(h|D|)\Xi \ , \qquad \partial_x = iD$$

  The principal symbol of the RHS is

$$M_1 := \begin{pmatrix} 0 & G^{(0)}(k) \\ 0 & 0 \end{pmatrix}$$

  which has multiple eigenvalues $\lambda(k) = 0$, hence it is not strictly hyperbolic. When $g < 0$ the full symbol $M(k)$ has complex eigenvalues, while for $g > 0$ they remain real

Parametrix for the linearized equations

- The full symbol

$$M(k) = M_1(k) + M_0(k) = \begin{pmatrix} 0 & G^{(0)}(k) \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ -g & 0 \end{pmatrix}$$

  When $g = 0$ the fundamental solution is given by the oscillatory kernel

$$\exp(tM_1(k)) = \begin{pmatrix} 1 & tG^{(0)}(k) \\ 0 & 1 \end{pmatrix}$$

  This evolution has one derivative loss:
  $(\eta_0, \xi_0) \in H^r \times H^s \mapsto (H^r + H^{s-1}) \times H^s$

- Set $\omega(k) = \sqrt{|gk| \tanh(h|k|)}$ the dispersion relation. When $g > 0$

$$\exp(tM(k)) = \begin{pmatrix} \cos(t\omega(k)) & \frac{\omega(k)}{g} \sin(t\omega(k)) \\ -\frac{g}{\omega(k)} \sin(t\omega(k)) & \cos(t\omega(k)) \end{pmatrix}$$

  This evolution operator maps $H^r \times H^{r+1/2} \mapsto H^r \times H^{r+1/2}$

Linearized equations are ill posed for $g < 0$

- However when $g < 0$ the parametrix takes the following form

$$\exp(tM(k)) = \begin{pmatrix} \cosh(t\omega(k)) & \frac{\omega(k)}{g}\sinh(t\omega(k)) \\ \frac{g}{\omega(k)}\sinh(t\omega(k)) & \cosh(t\omega(k)) \end{pmatrix}$$

  which has the property that it is unbounded as a map from any H^r to any H^{-s}

- Denote by C^ω_ρ the space of functions $f(x)$ bounded and analytic in a complex neighborhood of $\mathbb{R}^1$ of width ρ.

  The parametrix $\exp(tM(k))$ is bounded from C^ω_ρ to C^ω_σ for $\sigma < \rho$, as well as on certain Gevrey class scales

Shallow water regime: Analytic initial data

- One method to address the lack of strict hyperbolicity is to work with data $(\eta, \xi) \in C_a^\omega$

Theorem (Kano & Nishida (1979))

Consider the water wave equations scaled in the shallow water regime $X = \varepsilon x$*. Given initial data* $(\eta_0, \xi_0) \in B_a$ *and* $\varepsilon < \varepsilon_0$ *sufficiently small, then there exists a time interval* $[-T, +T]$ *independent of* ε*, and an analytic solution* $(\eta_\varepsilon(t,X), \xi_\varepsilon(t,X)) \in C_{a(t)}^\omega$*. Furthermore as* $\varepsilon \to 0$ *this solution converges to a solution of the shallow water equations.*

Proof: The proof of this theorem uses the Nirenberg – Nishida abstract version of the Cauchy – Kovalevsky theorem

Global Sobolev initial data

- Nalimov's original theorem uses Lagrangian variables as there is a key but subtle cancellation that allows one to overcome the problem of multiple characteristics in energy estimates. This is also used in Yosihara (1982), Craig (1985), S. Wu (1997), Schneider & Wayne (2000), and further work in Sobolev spaces.
- It turns out that almost any initial data gives rise to a solution of the water waves equations, at least for short time intervals. To allow for overturning wave profiles Lagrangian coordinates are also useful

Theorem (S. Wu (1997) $d = 2, h = +\infty$)

Given Sobolev data for the initial value problem, posed in Lagrangian coordinates, such that the initial free surface is given by a simple chord - arc curve, then there exists $T > 0$ *and a solution over the time interval* $[-T, +T]$.

S. Wu extended this result to $d = 3$ in (1999) using quaternionic coordinates

Shallow water regime: Sobolev initial data

- The problem of existence for water waves, for $x \in \mathbb{R}^{d-1}$ and for possibly large initial data given in a Sobolev space $(\eta_0, \xi_0) \in H^r$. Alinhac's 'good' variables are used for coercive energy estimates in Eulerian coordinates

Theorem (D. Lannes (2000) initial value problem for ZCS)

Consider $d \geq 1$ and $r = r(d)$ sufficiently large. Given inital data $(\eta_0, \xi_0) \in H^r$ suppose that the Taylor condition holds on the boundary

$$-\nabla p \cdot N \geq c_0 > 0$$

There exists a solution $(\eta(t,x), \xi(t,x)) \in C([-T,+T], H^r)$ for a time interval $[-T,+T]$, where T depends only upon c_0 and $\|(\eta_0, \xi_0)\|_r$

Furthermore Lannes makes a thorough study of scaling regimes related to the shallow water case, including situations with a variable bathymetry (four small parameters in all). This includes the Green – Naghdi equations.

- For flat or infinite $h(x)$ the Taylor condition automatically holds

Korteweg de Vries scaling regime

- Basic questions concern the KdV scaling regime, $0 < h < +\infty$:
  - The KdV evolution describes slow evolution of a wave front, on a time scale $\tau = \varepsilon^3 t$.
    To compare solutions of the water wave equations to the KdV we require a long existence time $T \sim \mathcal{O}(\varepsilon^{-3})$
  - Where do we derive the differential operator ∂_X^3 from a first order equation?
  - Initial data for water waves $(\eta_0(x), \xi_0(x))$ however for KdV $r_0(x)$

- Theorem (Craig (1985))

  Given initial data $(\eta_0(\varepsilon x), \xi_0(\varepsilon x))$ *such that* $(\eta_0, \xi_0) \in H^r \times H^{r+1/2}$ *a solution to the water wave equations exists for* $|t| \leq T_\varepsilon \sim \mathcal{O}(\varepsilon^{-3})$
  If additionally

$$G^{(0)}\xi_0 = \partial_x \eta_0 + \varepsilon^2 \Big(c_1 \partial_x^3 \eta_0 + 2c_2 \eta_0 \partial_x \eta_0 \Big) + \mathcal{O}(\varepsilon^3)$$

  then the solution $\eta(\tau, X; \varepsilon)$ *converges in* H^s *(*$s < r$*) to a solution of the KdV equation.*

Korteweg de Vries scaling regime

- Kano & Nishida prove a similar theorem on the KdV limit of water waves
- General solutions of the equations of water waves decompose into left- and right-propagating components

Theorem (G. Schneider & E. Wayne (2000))

Let $(\eta_0, \xi_0) \in H^r \times H^{r+1/2}$ *and* $c = \sqrt{gh}$. *Consider data for the water wave equations in the long wave scaling* $(\eta_0(\varepsilon x), \xi_0(\varepsilon x))$. *Then under reparametrization of the free surface a solution of the water wave equations decomposes into two components*

$$(r(\tau, X - c\tau), s(\tau, X + c\tau))$$

for large time $\tau \sim \mathcal{O}(1)$ *(that is* $t = \tau/\varepsilon^3 \sim \mathcal{O}(\varepsilon^{-3})$*)*

Boussinesq equations

- It is also natural to ask about the validity of the Boussinesq system as an approximation to water waves

  For example, it is consistent. Given a solution $(\eta(t,x), \xi(t,x)) \in H^r \times H^{r+1/2}$ of the equations of water waves on $[-T, +T]$, appropriately scaled for the Boussinesq regime, then

  $$\|\partial_t(\eta, \xi) - J grad\, H_{Boussinesq}(\eta, \xi)\|_s \leq C\varepsilon^4$$

  where $s < r - 6$ and $C = C(\|(\eta, \xi)\|_r)$
  However this does not mean that the Boussinesq equation has solutions that are good approximations for water waves
- Ill-posedness of the Boussinesq system

Theorem (P. Deift, C. Tomei & E. Trubowitz (1982))

The inverse spectral/scattering transform defines coordinates in which the Boussinesq system is completely integrable. Phase space contains three infinite dimensional, infinite co-dimensional submanifolds $M^s \oplus M^c \oplus M^u$ such that for data (η_0, ξ_0) which is not on any M^ℓ a solution does not exist forward nor backwards in time.

A question of coordinates

- Analysis of the equations of motion for water waves depends in a sensitive way upon coordinates

  Nalimov (1974) works in Lagrangian coordinates in order to exploit a subtle cancellation for his energy estimates

  Lannes (2000) works in Eulerian coordinates, in Alinhac's 'good variables' in order to have a coercive energy estimate
- For reasons that will be evident in lecture 4, I would like to work in coordinates that preserve the Hamiltonian structure of the water waves problem

The variational equation for a Hamiltonian system

- A Hamiltonian system for (η, ξ) with Hamiltonian $H(\eta, \xi)$ takes the following form

$$\begin{pmatrix} \partial_t \eta \\ \partial_t \xi \end{pmatrix} = \begin{pmatrix} \partial_\xi H \\ -\partial_\eta H \end{pmatrix}$$

  Denote its solution map by $\varphi_t(\eta, \xi)$

- The variational equation describes the evolution of solutions $(Y, \Xi) := (\delta\eta, \delta\xi)$ of the linearized equations

$$\begin{pmatrix} \partial_t Y \\ \partial_t \Xi \end{pmatrix} = \begin{pmatrix} \partial_\eta \partial_\xi H & \partial_\xi^2 H \\ -\partial_\eta^2 H & -\partial_\eta \partial_\xi H \end{pmatrix} \begin{pmatrix} Y \\ \Xi \end{pmatrix}$$

  This is also a Hamiltonian system, with nonautonomous Hamiltonian

$$H_1(Y, \Xi) = \frac{1}{2} \left\langle \begin{pmatrix} Y \\ \Xi \end{pmatrix}, \begin{pmatrix} \partial_\eta^2 H & \partial_\eta \partial_\xi H \\ \partial_\eta \partial_\xi H & \partial_\xi^2 H \end{pmatrix} (\varphi_t(\eta, \xi)) \begin{pmatrix} Y \\ \Xi \end{pmatrix} \right\rangle$$

Variational Hamiltonian for water waves

- Express the horizontal and vertical components of the fluid velocity on the free surface

$$v = \frac{\partial_x \xi - \partial_x \eta G(\eta)\xi}{1 + |\partial_x \eta|^2}, \qquad w = \frac{G(\eta)\xi + \partial_x \eta \cdot \partial_x \xi}{1 + |\partial_x \eta|^2}$$

  Then the variational Hamiltonian takes the form

$$\begin{aligned} H_1(Y, \Xi) &= \frac{1}{2} \int \Xi(\partial_\xi^2 H)\Xi + 2\Xi(\partial_\eta \partial_\xi H)Y + Y(\partial_\eta^2 H)Y \, dx \\ &= \frac{1}{2} \int \Big[\Xi G(\eta)\Xi + (g + w\partial_x v)Y^2 + (wY)G(\eta)(wY) \\ &\qquad + 2\big((vY)\partial_x \Xi - (wY)G(\eta)\Xi\big) \Big] \, dx \end{aligned}$$

  It can be used as a basis for energy estimates for the water wave
- This can be expressed in terms of Alinhac good variables $(Y, \Omega = \Xi - wY)$

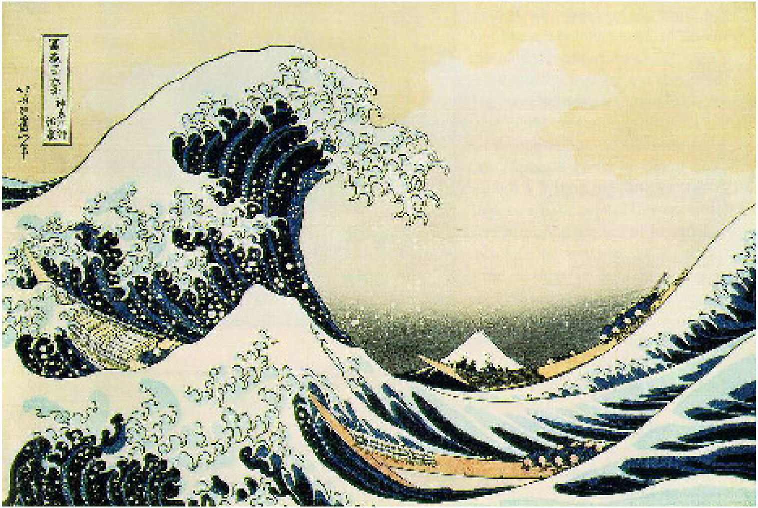
$$H_1(Y, \Xi) = \frac{1}{2} \int \Big[\Omega G(\eta)\Omega + (g + v\partial_x w)Y^2 + 2(vY)\partial_x \Omega \Big] \, dx$$

Generalizations

- The initial value problem with surface tension and the KdV limit
  G. Schneider & E. Wayne (2002), T. Iguchi (2007)
- Fluids with vorticity
  D. Christodoulou & H. Lindblad (2000), T. Iguchi (2007)
- Overturning wave (splash) singularities
  A. Castro, D. Córdoba, C. Fefferman, F. Gancedo & J. Gómez - Serrano (2013)

Global solutions in $\mathbb{R}^d$ for small data

- Global solutions for small initial data in the 3-dimensional case
  S. Wu (2011), P. Germain, N. Masmoudi & J. Shatah (2012)
- Global solutions for small initial data in the 2-dimensional case
  T. Alazard & J.-M. Delort (2013), A. Ionescu & F. Pusateri (2015)
- Global solutions for small initial data for gravity - capillary waves
  Y. Deng, A. Ionescu, B. Pausader & F. Pusateri (2016)



Thank you