Lecture 4: Birkhoff normal forms

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Outline

Two ODEs

Water waves with and without surface tension

Birkhoff normal forms

Resonances

Formal Birkhoff normal forms

Mapping properties of normal forms transformations

Critique

Contrast two ODEs

- Quadratic case

$$\dot{z} = z^2 \ , \qquad z(0) = \varepsilon$$

$$z(t) = \frac{\varepsilon}{1 - \varepsilon t} \ , \qquad T = \frac{1}{\varepsilon}$$

- Cubic case

$$\dot{w} = w^3 \ , \qquad w(0) = \varepsilon$$

$$w(t) = \sqrt{\frac{\varepsilon^2}{1 - 2\varepsilon^2 t}} \ , \qquad T = \frac{1}{2\varepsilon^2}$$

- The general time of existence does not change when these ODE are replaced by

$$\dot{z} = i\omega z + z^2 \ , \qquad \dot{w} = i\omega w + w^3$$

Free surface water waves

- Incompressible and irrotational flow

$$\nabla \cdot u = 0 \, , \qquad \nabla \wedge u = 0$$

which is a potential flow in the fluid domain $S(\eta)$

$$u = \nabla \varphi \, , \quad \Delta \varphi = 0$$

Fluid domain $S(\eta)$: $-h < y < \eta(x,t)$, $\quad x \in \mathbb{R}^{d-1}$
Bottom boundary conditions $\partial_N \varphi = 0$

- Free surface conditions on $y = \eta(x,t)$

$$\begin{aligned} \partial_t \eta &= \partial_y \varphi - \partial_x \eta \cdot \partial_x \varphi && \text{kinetic BC} \\ \partial_t \varphi &= -g\eta - \tfrac{1}{2}|\nabla \varphi|^2 && \text{Bernoulli condition} \end{aligned}$$

Free surface water waves with surface tension

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Bottom boundary conditions $\partial_N \varphi = 0$

- Free surface conditions on $y = \eta(x,t)$ with surface tension

$$\begin{aligned} \partial_t \eta &= \partial_y \varphi - \partial_x \eta \cdot \partial_x \varphi \quad \text{kinetic BC} \\ \partial_t \varphi &= -g\eta - \tfrac{1}{2}|\nabla\varphi|^2 + \sigma\kappa(\eta) \quad \text{Bernoulli condition} \end{aligned}$$

where $\kappa(\eta)$ is the mean curvature of the free surface

Zakharov's Hamiltonian

- The energy functional

$$\begin{aligned} H &= K + P \\ &= \int_x \int_{y=-h}^{\eta(x)} \tfrac{1}{2}|\nabla\varphi|^2 \, dydx + \int_x \tfrac{g}{2}\eta^2 + \sigma\big(\sqrt{|\partial_x\eta|^2+1} - 1\big)\, dx \end{aligned}$$

- Zakharov's choice of variables

$$z := (\eta(x), \xi(x)) \,, \qquad \text{where} \quad \xi(x) := \varphi(x, \eta(x))$$

That is $\varphi = \varphi[\eta, \xi](x, y)$

- Expressed in terms of the Dirichlet – Neumann operator $G(\eta)$

$$H(\eta, \xi) = \int \tfrac{1}{2}\xi G(\eta)\xi + \tfrac{g}{2}\eta^2 + \sigma\big(\sqrt{|\partial_x\eta|^2+1} - 1\big)\, dx$$

Taylor expansion of $G(\eta)$

- Given by the expression

$$G(\eta)\xi = \sum_{j\geq 0} G^{(j)}(\eta)\xi$$

  where each $G^{(j)}(\eta)$ is homogeneous of degree j in η

- Explicitely, for $D_x := -i\partial_x$

$$G^{(0)}\xi(x) = |D_x|\tanh(h|D_x|)\xi(x)$$
$$G^{(1)}(\eta)\xi(x) = D_x \cdot \eta D_x - G^{(0)}\eta G^{(0)}\xi(x)$$
$$G^{(2)}(\eta)\xi(x) = \tfrac{1}{2}(G^{(0)}\eta^2 D_x^2 + D_x^2\eta^2 G^{(0)} - 2G^{(0)}\eta G^{(0)}\eta G^{(0)})\xi(x)$$

- Accordingly, the Hamiltonian is analytic, with expansion

$$H(\eta,\xi) = \int_{\mathbb{R}^{d-1}} \tfrac{1}{2}\xi G^{(0)}\xi + \tfrac{g}{2}\eta^2 + \tfrac{\sigma}{2}|\partial_x\eta|^2\,dx + \sum_{j\geq 3} H^{(j)}(\eta,\xi)$$

$$H^{(j)}(\eta,\xi) = \tfrac{1}{2}\int_{\mathbb{R}^{d-1}} \xi G^{(j-2)}(\eta)\xi + \sigma S_j|\partial_x\eta|^j\,dx$$

Taylor expansion of the Hamiltonian

- From the analyticity of $G(\eta)$ and an explicit Taylor expansion of $\sqrt{1+|\partial_x\eta|^2}$

$$\begin{aligned} H &= H^{(2)} + H^{(3)} + H^{(4)} + \dots \\ &= \frac{1}{2}\int \xi G^{(0)}\xi + g\eta^2\,dx \;+\; \sum_{m\geq 3}\frac{1}{2}\int \xi G^{(m-2)}(\eta)\xi + \sigma S_m|\partial_x\eta|^m\,dx \end{aligned}$$

- The water wave equations linearized about $(\eta,\xi)=0$ are

$$\partial_t\begin{pmatrix}\eta\\ \xi\end{pmatrix} = J\,\mathrm{grad}_{\eta,\xi}H^{(2)}(\eta,\xi)$$

namely

$$\begin{aligned} \partial_t\eta &= |D_x|\tanh(h|D_x|)\xi \\ \partial_t\xi &= -g\eta + \sigma\partial_x^2\eta \end{aligned}$$

A harmonic oscillator with frequencies

$$\omega(k) = \sqrt{(g+\sigma|k|^2)|k|\tanh(h|k|)}$$

Birkhoff normal form

- Normal form - transform the equations to retain only essential nonlinearities

$$\tau : z = \begin{pmatrix} \eta \\ \xi \end{pmatrix} \mapsto w$$

  in a neighborhood $B_R(0) \subseteq H^r$
- Conditions:
  1. The transformation τ is canonical, so the new equations are

  $$\partial_t w = J\, grad\, \overline{H}(w)\ , \qquad \overline{H}(w) = H(\tau^{-1}(w))$$

  2. The transformed Hamiltonian is

  $$\overline{H}(w) = H^{(2)}(w) + \left(Z^{(3)} + \cdots + Z^{(M)}\right) + \overline{TR}^{(M+1)}$$

  where each $Z^{(m)}$ retains only resonant terms $\{H^{(2)}, Z^{(m)}\} = 0$
- The new Hamiltonian $\overline{H}(w)$ is conserved by the flow $\bar{\varphi}_t(w)$

Significance of the normal form

- This transformation procedure and reduction to Birkhoff normal form is part of the theory of averaging for dynamical systems
- Consider $x \in \mathbb{T}^{d-1}$ with periodic boundary conditions on a torus
  Fourier transform variables and complex symplectic coordinates

$$(\eta_k, \xi_k) := \frac{1}{|\mathbb{T}^{d-1}|^{1/2}} \int_{\mathbb{T}^{d-1}} e^{-ik\cdot x}(\eta(x), \xi(x))\, dx$$

$$z_k := \frac{1}{\sqrt{2}}\left(Q_k \eta_k + i Q_k^{-1} \xi_k\right) , \qquad Q_k = \sqrt[4]{\frac{g + \sigma |k|^2}{|k| \tanh(h|k|)}}$$

  Action angle variables $z_k = \sqrt{I_k} e^{i\Theta_k}$ $\quad I_k = |z_k|^2$

- A Hamiltonian $h(I)$ in action variables alone is integrable

$$\partial_t \Theta = \partial_I h(I) , \qquad \Theta(t) = \Theta(0) + t \partial_I h(I)$$
$$\partial_t I = -\partial_\Theta h(I) = 0 , \qquad I(t) = I(0)$$

  Such flows conserve each I_k, hence every Sobolev norm

$$\|z(t)\|_r^2 = \sum_k \langle k \rangle^{2r} |z_k(t)|^2 = \sum_k \langle k \rangle^{2r} I_k = \|z(0)\|_r^2$$

Reasons to study the normal form

- Long - time existence If $Z^{(3)} + \cdots + Z^{(M)}$ are integrable, then the existence time for small initial data $w(t) \in B_\varepsilon(0) \in H^s$ is $T_\varepsilon \sim \mathcal{O}(\varepsilon^{-(M-1)})$
- Zakharov's theory of wave turbulence. The reduction of a Hamiltonian PDE to its resonant manifold is a normal forms transformation. In Zakharov's notation,

$$a(x) := \frac{1}{\sqrt{2}}\Big(Q(D_x)\eta(x) + iQ^{-1}(D_x)\xi(x)\Big) \mapsto b(x)$$

- In KAM theory the Arnold condition depends upon the normal form
- special solutions. Resonant terms $Z^{(3)} + \ldots Z^{(M)}$ describe an averaged system, which often has particular solutions of interest.

  Example: Wilton ripples and three wave resonances

Resonant terms

- Cubic resonances: It is well known in the folklore of fluid dynamics that when surface tension $\sigma = 0$ there are no *three wave interactions*. Namely

$$\omega_{k_1} \pm \omega_{k_2} \pm \omega_{k_3} = 0 \,, \qquad k_1 + k_2 + k_3 = 0$$

implies that at least one of $k_1, k_2, k_3 = 0$. This means that formally $Z^{(3)} = 0$

- Quartic resonances: Four wave interactions are nontrivial In deep water $h = +\infty$ with $d = 2$ and frequencies $\omega_k = \sqrt{g|k|}$

$$\omega_{k_1} \pm \omega_{k_2} \pm \omega_{k_3} \pm \omega_{k_4} = 0 \,, \qquad k_1 + k_2 + k_3 + k_4 = 0$$

has integrable solutions

$$\omega_{k_1} + \omega_{k_2} - \omega_{k_3} - \omega_{k_4} = 0 \,, \quad \{k_1, k_2\} = \{k_3, k_4\}$$

and nonintegrable *Benjamin* - *Feir* resonant interactions

$$k_1 : k_2 : k_3 : k_4 = n^2 : (n+1)^2 : n^2(n+1)^2 : -(n^2+n+1)^2$$

$$\omega_1 : \omega_2 : \omega_3 : \omega_4 = n : -(n+1) : -n(n+1) : (n^2+n+1)$$

Formal fourth order Birkhoff normal form

Theorem (Dyachenko, Lvov & Zakharov (1994), WC & Worfolk (1995))

The Benjamin - Feir resonant interaction terms vanish, specifically $c_{k_1 k_2 k_3 k_4} = 0$ *whenever*

$$k_1 : k_2 : k_3 : k_4 = n^2 : (n+1)^2 : n^2(n+1)^2 : -(n^2+n+1)^2)$$

$$\omega_1 : \omega_2 : \omega_3 : \omega_4 = n : -(n+1) : -n(n+1) : (n^2+n+1)$$

Set $I_1(k) = \frac{1}{2}(z_k \bar{z}_k + z_{-k}\bar{z}_{-k})$ *and* $I_2(k) = \frac{1}{2}(z_k \bar{z}_k - z_{-k}\bar{z}_{-k})$.
The formal Birkhoff normal form up to quintic residual is integrable

$$\begin{aligned}\overline{H}^{+} = & \sum_k \omega_k I_1(k) - \frac{1}{2\pi}\sum_k |k|^3 \big(I_1(k)^2 - 3 I_2(k)^2\big) \\ & + \frac{4}{\pi}\sum_{|k_4|<|k_1|} I_2(k_1) I_2(k_4) + \overline{R}^{(5)}(w) \\ = & H^{(2)}(I) + \overline{H}^{(4)}(I) + \overline{R}^{(5)}(w)\end{aligned}$$

Mapping properties of normal form transformations

- A PDE question: *mapping properties* of the transformation τ: is the transformation well defined, on which Banach spaces

- Theorem (Properties of $\tau^{(3)}$) C. Sulem & WC (2016)

  Fix $d = 2$, $h = +\infty$ and $s > 3/2$. There exists $R_0 > 0$ such that for any $R < R_0$, the transformation $\tau^{(3)}$ is defined on $B_R(0) \subseteq H^s_\eta \oplus H^s_\xi$

$$\tau^{(3)} : B_R(0) \to B_{2R}(0) \qquad (\tau^{(3)})^{-1} : B_{R/2}(0) \to B_R(0)$$

- Other results on the transformation of three-wave interactions for water waves
  - WC (1996) Canonical normal forms on scales of analytic spaces
  - S. Wu (2009) Transformations in Lagrangian coordinates
  - P. Germain N. Masmoudi & J. Shatah (2012) geometric coordinates
  - J. Hunter, M. Ifrim & D. Tataru (2014) ad hoc methods in holomorphic cordinates
  - A. Ionescu & F. Pusateri (2015) global solutions for $x \in \mathbb{R}^1$

Fourth order resonance relations

Define the energy space $E^r := H^r_\eta \oplus H^{r+1/2}_\xi$
Quartet interactions are indexed by

$$\{(k_1, k_2, k_3, k_4) \in \mathbb{Z}^4 \ : \Sigma^4_{j=1} k_j = 0\}$$

Due to Theorem 1 the resonant set is

$$\mathcal{R} = \{k_1 k_4, k_2 k_3 > 0 : k_1 + k_2 = 0 = k_3 + k_4 \text{ or } k_1 + k_3 = 0 = k_2 + k_4\}$$

A quasihomogeneous neighborhood of $\mathcal{R}$ is a set of near-resonant modes

$$C^+_{\mathcal{R}} := \{(k_1, k_2, k_3, k_4) \in \mathbb{Z}^4 : \Sigma^4_{j=1} k_j = 0 \quad \text{satisfying}$$
$$|k_1 + k_2| < (|k_1| + |k_2|)^{1/4} \text{ and } |k_3 + k_4| < (|k_3| + |k_4|)^{1/4}\}$$

The neighborhood $C^-_{\mathcal{R}} \subseteq \mathbb{Z}^4$ is similar with $k_2 \leftrightarrow k_3$

Fourth order partial Birkhoff normal form

Theorem (WC & Sulem (2016))

Let $Q \subseteq \mathbb{Z}^4$ *be a set of quartet interactions, such that*

$$Q \backslash B_\rho(0) \cap C^\pm_{\mathcal{R}} = \emptyset \qquad \rho < +\infty$$

is symmetric under $(k \leftrightarrow -k)$, $(k_2 \leftrightarrow k_3)$ *and* $(k_1 \leftrightarrow k_4)$.
Then for $r > 3/2$ *there exists a canonical transformation* $\tau_Q^{(4)}$ *on* $B_R(0) \subseteq E^r$ *such that*

$$\tau_Q^{(4)} : H^{(2)} + \overline{H}^{(4)} + \overline{R}^{(5)} \to \widetilde{H} = H^{(2)} + \widetilde{Z}^{(4)} + \widetilde{R}^{(5)}$$

such that $supp\, \tilde{Z}^{(4)} \subseteq C^\pm_{\mathcal{R}}$. *For* $(k_1, k_2, k_3, k_4) \in \mathcal{R}$ *then*

$$\widetilde{Z}^{(4)}_{k_1,k_2,k_3,k_4} = Z^{(4)}_{k_1,k_2,k_3,k_4}(I)$$

Surface tension $\sigma > 0$

Theorem (C. Sulem & WC (2015))

In the case of positive surface tension, with $0 < h \leq +\infty$*, a similar statement holds for* $r > 1$*, namely* $\partial_z \tau^{(3)} : B_R(0) \subseteq H_\eta^{r+1} \oplus H_\xi^{r+1/2} \to H_\eta^{r+1} \oplus H_\xi^{r+1/2}$*. However it is possible that* $Z^{(3)}$ *is nonzero (Wilton ripples).*

NB Furthermore $\tau^{(3)}$ is smooth on a scale of Hilbert spaces. That is, in the case with surface tension the Jacobian maps energy spaces

$$\partial_z \tau^{(3)} : H_\eta^{r+1/2} \oplus H_\xi^{r} \to H_\eta^{r+1/2} \oplus H_\xi^{r}$$

In the case w/o surface tension the Jacobian maps

$$\partial_z \tau^{(3)} : H_\eta^{s-1} \oplus H_\xi^{s-1} \to H_\eta^{s-1} \oplus H_\xi^{s-1}$$

NB: The transformation mixes the domain η and the potential ξ.

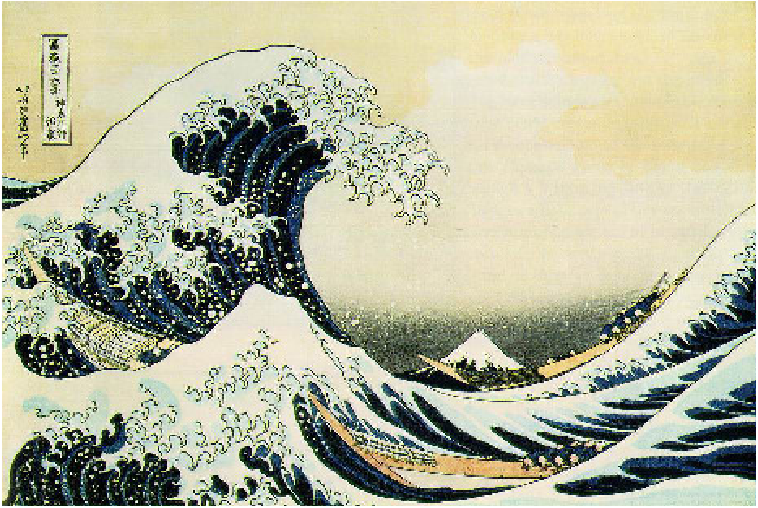
Critique

- This talk in on work in progress for several reasons
  Existence theorems depend upon solving the initial value problem in special variables.
  Nalimov (1971) shows that this is possible in Lagrangian coordinates, as in the work of S. Wu
  In Eulerian coordinates such proofs depend upon Alinhac's 'good variables', as in the work of D. Lannes
  Question: is there a systematic symmetrization for the water wave equations, that is independent of coordinates

- Mapping properties of normal forms in the case $\sigma = 0$ and $0 < h < +\infty$ are not included

- In the case of a variable bathymetry $h(x)$, periodic for example, the dispersion relation is replaced by the Bragg frequencies $\omega_h(k)$

- Properties of the normal forms transformations $\tau^{(M)}$ on energy spaces $H^s_\eta \oplus H^{s+1/2}_\xi$



Thank you