Relation between Distributional and Leray-Hopf Solutions to the Navier-Stokes Equations

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Preliminary Considerations

Cauchy problem for the Navier-Stokes equations:

$$\left.\begin{aligned} &\partial_t v + v\cdot\nabla v = \Delta v - \nabla p \\ &\operatorname{div} v = 0 \end{aligned}\right\} \text{ in } \mathbb{R}^3\times(0,\infty)$$

$$v(x,0) = v_0(x) \quad x\in\mathbb{R}^3 .$$

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$$v(x,0) = v_0(x) \quad x\in\mathbb{R}^3 .$$

Leray-Hopf solutions: For any $v_0 \in L^2_\sigma(\mathbb{R}^3)$ there is

$$v \in L^\infty(0,T;L^2_\sigma)\cap L^2(0,T;H^1)\,, \quad \text{all } T>0$$

$$\lim_{t\to 0^+}\|v(t)-v_0\|_2 = 0$$

$$\int_{\mathbb{R}^3}|v(x,t)|^2 + 2\int_0^t\int_{\mathbb{R}^3}|\nabla v(x,\tau)|^2 \le \int_{\mathbb{R}^3}|v_0(x)|^2\,,$$

all $t\in[0,T]$.

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$$v \in L^{\frac{2s}{s-3}}(0,T;L^s)\,,\;\; s>3\,, \qquad \text{(B)}\,,$$

or

$$v \in C([0,T];L^3)\,, \qquad \text{(C)}$$

Uniqueness/Regularity (Prodi-Serrin-Ladyzhenskaya)

Main Results

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- v is any (distributional) solutions to the Navier-Stokes equations with initial data in L^2_σ (assumed in a suitable sense) ;
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Then v is necessarily a Leray-Hopf solution.

A Remark

In general, classes defined by (A), (B) or (C) do not contain nor are they contained in the Leray-Hopf's.

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$$\int_{\mathbb{R}^3} |v(x,t)|^2 \,, \quad \int_0^t \int_{\mathbb{R}^3} |\nabla v(x,\tau)|^2 \,,$$

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Our result implies, in particular, that this cannot happen if only the kinetic energy of the initial datum is finite.

Main Results: Precise Statement

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Definition. Let $v_0 \in L^2_\sigma$. Then $v \in L^2_{\text{loc}}(\mathbb{R}^3 \times [0,T))$ is a *distributional solution* (corresponding to v_0) if

$$\int_0^T \int_{\mathbb{R}^3} v \cdot (\partial_t \varphi + \Delta\varphi + v \cdot \nabla\varphi) = -\int_{\mathbb{R}^3} v_0 \cdot \varphi(0)$$

$$\int_0^T \int_{\mathbb{R}^3} v \cdot \nabla\phi = 0\,,$$

for all

$$\varphi \in \mathcal{D}_T := \{\varphi \in C_0^\infty(\mathbb{R}^3 \times [0,T)) : \ \operatorname{div}\varphi = 0\}\,,$$

and $\phi \in C_0^\infty(\mathbb{R}^3 \times (0,T))$.

Main Results: Precise Statement

Theorem 1. *Let* v *be a distributional solution satisfying one of the following conditions:*

$$v \in L^4(0,T;L^4)\,;$$

$$v \in L^{\frac{2s}{s-3}}(0,T;L^s)\,,\, s > 3\,;$$

$$v \in C([0,T];L^3)\,.$$

Then v *is a Leray-Hopf solution.*

Main Results: Precise Statement

Theorem 2. *Let v be a distributional solution satisfying one of the following conditions:*

$$v \in L^{\frac{2s}{s-3}}(\delta, T; L^s)\,,\, s > 3\,;$$

$$v \in C([\delta, T]; L^3)\,,$$

all "small" $\delta > 0$, and also

$$\lim_{t\to 0^+} \int_{\mathbb{R}^3} (v(x,t) - v_0(x)) \cdot \Phi(x) dx = 0\,, \quad \textit{all } \Phi \in L^2_\sigma.$$

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Theorem 3. *Let* v *be a distributional solution satisfying*

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Corollary 2. *If* v *is a distributional solution satisfying one of the conditions*

$$v \in L^{\frac{2s}{s-3}}(0,T;L^s)\,, s > 3\,; \quad v \in C([0,T];L^3)\,,$$

then v *is the unique Leray-Hopf solution corresponding to* v_0*, and also* $v \in C^\infty((0,T] \times \mathbb{R}^3)$.

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Corollary 3. *If v is a distributional solution such that*

$$v \in L^4(\delta, T; L^4)\,, \quad \textit{all "small" } \delta > 0\,,$$

and $\|v(t) - v_0\|_2 \to 0$, as $t \to 0^+$, then v satisfies the energy equality, and also $v \in C([0,T]; L^2)$.

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Corollary 4. *If v is a distributional solution satisfying one of the conditions for all "small" $\delta > 0$*

$$v \in L^{\frac{2s}{s-3}}(\delta, T; L^s)\,, s > 3\,; \; v \in C([0, \delta, T]; L^3)\,,$$

and $v(t) \to v_0$ weakly in L^2_σ as $t \to 0^+$, then v is a Leray-Hopf solution, so that $v \in C^\infty((0,T] \times \mathbb{R}^3)$.

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2. Investigation in either class

$$v \in L^{\frac{2s}{s-3}}(0,T;L^s)\,,\ s>3\,;\ \ v \in C([0,T];L^3)\,,$$

falls, instead, in the framework of "very weak" (or "mild") solutions, and has been performed extensively.

Some Remarks

FOIAS (1961):

$$v \in L^2(0,T;H^1) \cap L^r(0,T;L^s),\ s>3\ ,\ r>\frac{2s}{s-3};\ v_0 \in L^2_\sigma.$$

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FOIAS (1961):

$$v \in L^2(0,T;H^1) \cap L^r(0,T;L^s),\, s > 3\,,\, r > \frac{2s}{s-3};\, v_0 \in L^2_\sigma.$$

FABES, JOHNS & RIVIERE (1972):

$$v \in L^{\frac{2s}{s-3}}(0,T;L^s)\,,\, s > 3\,,\, v_0 \in L^2_\sigma \cap L^s.$$

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FABES, JOHNS & RIVIERE (1972):

$$v \in L^{\frac{2s}{s-3}}(0,T;L^s)\,,\, s>3\,,\, v_0 \in L^2_\sigma \cap L^s.$$

KATO (1984), GIGA (1986):

$$v \in C([0,T];L^3) \cap L^{\frac{2s}{s-3}}(0,T;L^s)\,,\, s>3\,,\, v_0 \in L^2_\sigma \cap L^3\,.$$

$$\lim_{t\to 0} \left(t^{\frac{s-3}{2s}} \|v(t)\|_s \right) = 0\,.$$

Some Remarks

3. BUCKMASTER & VICOL (2017/2018), LUO (2018) show non-uniqueness in the class of distributional solutions defined by

$$v \in C([0,T]; H^{\beta}),\, \beta > 0 \text{ "small"}, \text{ and } v_0 \in L^2_\sigma\,.$$

having, therefore, infinite dissipation.

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having, therefore, infinite dissipation.

Our results imply, in particular, that any possible non-uniqueness result in that class could only hold with $\beta < 1/2$.

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4. It is probable that the borderline assumption

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continues to ensure that the distributional solution v is in the Leray-Hopf class.

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5. Our results continue to hold in (1) bounded or (2) exterior domain (no-slip conditions), and (3) in a torus (periodic boundary conditions). The proof in cases (1), (2) becomes much more technical.

Basic Ideas

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Step 1. For the given distributional solution v, consider the linear Cauchy problem

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$$u(x,0) = v_0(x) \quad x \in \mathbb{R}^3 .$$

By standard Galerkin method one shows the existence of a solution

$$u \in L^\infty(0,T; L^2_\sigma) \cap L^2(0,T; H^1) ,$$

$$u(t) \to v_0 \text{ in } L^2 .$$

Basic Ideas

Set

$$w := v - u$$

then

$$\int_0^T \int_{\mathbb{R}^3} w \cdot (\partial_t \varphi + \Delta \varphi + v \cdot \nabla \varphi) = 0$$

$$\int_0^T \int_{\mathbb{R}^3} w \cdot \nabla \phi = 0 \,,$$

for all

$$\varphi \in \mathcal{D}_T := \{\varphi \in C_0^\infty(\mathbb{R}^3 \times [0,T)) : \ \operatorname{div} \varphi = 0\} \,,$$

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Step 2. Replace

$$\partial_t \varphi + \Delta \varphi + v \cdot \nabla \varphi \,, \quad \varphi \in \mathcal{D}_T$$

with arbitrary $f \in C_0^\infty(\mathbb{R}^3 \times (0,T))$.

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In which case, we get

$$\int_0^T \int_{\mathbb{R}^3} w \cdot f = 0\,, \quad \text{for all } f \in C_0^\infty(\mathbb{R}^3 \times (0,T))\,,$$

implying that $v \equiv u$ is a Leray-Hopf solution.

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This step can be achieved if v is in one of the classes (A), (B) or (C).

Implementation of Step 2, under the assumption

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"Maximal regularity" class ($p, q \in (1,\infty)$):

$$\mathcal{W}^{p,q} := \left\{\psi \in W^{1,p}(0,T;L^q_\sigma) \cap L^p(0,T;W^{2,q})\right\}.$$

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Lemma 1. (Technical) *Test space $\mathcal{D}_T$ is dense in*

$$\left\{\psi \in \mathcal{W}^{p_1,q_1} \cap \mathcal{W}^{p_2,q_2} : \psi(T) = 0\right\},$$

for any $p_i, q_i \in (1,\infty)$, $i = 1,2$.

Basic Ideas

Recall that $(r := 2s/(s-3))$

$$w := v - u \in L^r(0,T;L^s) + L^\infty(0,T;L^2_\sigma)\,.$$

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Lemma 2. (Embeddings + Hölder) *The linear form*

$$\psi \in \mathcal{W}^{1,2} \cap \mathcal{W}^{r',s'} \to \int_0^T \int_{\mathbb{R}^3} v \cdot \nabla\psi \cdot w \in \mathbb{R}$$

is continuous ($r' := r/(r-1)$, $s' := s/(s-1)$.)

Basic Ideas

As a consequence,

$$\int_0^T \int_{\mathbb{R}^3} w \cdot (\partial_t \varphi + \Delta \varphi + v \cdot \nabla \varphi) = 0$$

for all

$$\varphi \in \mathcal{D}_T \,,$$

implies

$$\int_0^T \int_{\mathbb{R}^3} w \cdot \quad (\partial_t \psi + \Delta \psi + v \cdot \nabla \psi) = 0$$

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implies

$$\int_0^T \int_{\mathbb{R}^3} w \cdot \overbrace{(\partial_t \psi + \Delta \psi + v \cdot \nabla \psi)}^{= f \in C_0^\infty(\mathbb{R}^3 \times (0,T))} = 0$$

for all

$$\psi \in W^{1,2} \cap \mathcal{W}^{r',s'} \,, \text{ with } \psi(T) = 0 \,.$$

Lemma 3. *Let*

$$\alpha \in L^r(0,T;L^s_\sigma) \equiv L^{r,s}\,, \quad r := \frac{2s}{s-3}\,.$$

For any $F \in C_0^\infty(\mathbb{R}^3 \times (0,T))$, *the problem*

$$\left.\begin{array}{l} \partial_t \Psi - \alpha \cdot \nabla \Psi = \Delta \Psi - \nabla p + F \\ \operatorname{div} \Psi = 0 \end{array}\right\} \text{ in } \mathbb{R}^3 \times (0,T)\,,$$

$$\Psi(x,0) = 0 \quad x \in \mathbb{R}^3\,,$$

has at least one solution $\Psi \in \mathcal{W}^{1,2} \cap \mathcal{W}^{r',s'}$,.

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Estimate in $\mathcal{W}^{r',s'}$. For $\lambda > 0$, set

$$\zeta = \mathrm{e}^{-\lambda t}\Psi\,, \quad \rho = \mathrm{e}^{-\lambda t}p\,,$$

and

$$G = \mathrm{e}^{-\lambda t}F + \alpha\cdot\nabla\zeta\,.$$

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We get the Stokes problem

$$\left.\begin{array}{l}\partial_t\zeta + \lambda\,\zeta = \Delta\zeta - \nabla\rho + G\\ \mathrm{div}\,\zeta = 0\end{array}\right\}\ \text{in}\ \mathbb{R}^3 \times (0,T)\,,$$

$$\zeta(x,0) = 0 \quad x \in \mathbb{R}^3\,,$$

Basic Ideas

Maximal regularity (λ "large"):

$$\begin{aligned} \|\zeta\|_{\mathcal{W}^{r',s'}} + \lambda\|\zeta\|_{L^{r',s'}} &\leq c\,\|G\|_{L^{r',s'}} \\ &\leq c\,\left(\|\alpha\cdot\nabla\zeta\|_{L^{r',s'}} + \|F\|_{L^{r',s'}}\right)\,. \end{aligned}$$

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Split $\alpha = \alpha_1 + \alpha_2$ with

$$\|\alpha_1\|_{L^{r,s}} < \varepsilon\,,\quad \alpha_2 \in L^{\infty,\infty}\,.$$

We get (Hölder + embeddings + Ehrling)

$$\begin{aligned}\|\alpha\cdot\nabla\zeta\|_{L^{r',s'}} &\le \varepsilon\|\nabla\zeta\|_{L^{\frac{r}{r-2},\frac{s}{s-2}}} + \|\alpha_2\|_{L^{\infty,\infty}}\|\nabla\zeta\|_{L^{r',s'}} \\ &\le c\,\varepsilon\,\|\zeta\|_{\mathcal{W}^{r',s'}} + c_\varepsilon\|\alpha_2\|_{L^{\infty,\infty}}\|\zeta\|_{L^{r',s'}}\,.\end{aligned}$$

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Therefore:

$$\|\zeta\|_{\mathcal{W}^{r',s'}} \le c\,\|F\|_{L^{r',s'}}\,.$$

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Test the original equation with Ψ, $\Delta\Psi$, and $\partial_t\Psi$:

$$\tfrac{1}{2}\frac{d}{dt}\|\Psi\|_2^2 + \|\nabla\Psi\|_2^2 = \int_{\mathbb{R}^3} F\cdot\Psi$$

$$\tfrac{1}{2}\frac{d}{dt}\|\nabla\Psi\|_2^2 + \|\Delta\Psi\|_2^2 = \int_{\mathbb{R}^3} (\alpha\cdot\nabla\Psi - F)\cdot\Delta\Psi$$

$$\tfrac{1}{2}\frac{d}{dt}\|\nabla\Psi\|_2^2 + \|\partial_t\Psi\|_2^2 = -\int_{\mathbb{R}^3} (\alpha\cdot\nabla\Psi - F)\cdot\partial_t\Psi$$

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Assumption on α + Sobolev lead to

$$\|\Psi\|_{\mathcal{W}^{1,2}} \leq c\,\|F\|_{L^{2,2}}\,.$$

QED

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Proof of Leray-Hopf property.

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For a given $f \in C_0^\infty(\mathbb{R}^3 \times (0,T))$ we use the previous lemma with

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Thus, $\Psi(T-t) \equiv \psi(t) \in \mathcal{W}^{1,2} \cap \mathcal{W}^{r',s'}$, and solves the final-value problem

$$\left.\begin{array}{l} \partial_t \psi + v \cdot \nabla \psi + \Delta \psi = \nabla p + f \\ \operatorname{div} \psi = 0 \end{array}\right\} \text{ in } \mathbb{R}^3 \times (0,T)\,,$$

$$\psi(x,T) = 0 \quad x \in \mathbb{R}^3\,,$$

Basic Ideas

Recalling that $w := v - u$ satisfies

$$\int_0^T \int_{\mathbb{R}^3} w \cdot (\partial_t \psi + \Delta\psi + v \cdot \nabla\psi) = 0$$

for all

$$\psi \in W^{1,2} \cap \mathcal{W}^{r',s'}, \text{ with } \psi(T) = 0\,,$$

we get

$$\int_0^T \int_{\mathbb{R}^3} (v-u) \cdot f = 0\,, \text{ for all } f \in C_0^\infty(\mathbb{R}^3 \times (0,T))$$

that is $v \equiv u$. **QED**

Basic Ideas

Sketch of the proof under the alternate assumption $v \in L^{\frac{2s}{s-3}}(\delta, T; L^s)$, $s > 3$; $v \longrightarrow v_0$ weakly in L^2.

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$$v \in L^{\frac{2s}{s-3}}(\delta, T; L^s)\,,\ s > 3\,; \quad v \longrightarrow v_0 \text{ weakly in } L^2.$$

Let

$$\chi_\delta(t) = \begin{cases} 1 & \text{if } t \geq 2\delta \\ 0 & \text{if } t \leq \delta \end{cases}\,, \quad |\chi'_\delta| \leq \frac{c}{\delta}\,.$$

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One can show $(w := v - u)$

$$\int_\delta^T \chi_\delta \int_{\mathbb{R}^3} w\cdot(\partial_t \psi + \Delta\psi + v\cdot\nabla\psi) = -\int_\delta^{2\delta} \chi'_\delta \int_{\mathbb{R}^3} w \cdot \psi\,,$$

for all $\psi \in W^{1,2} \cap \mathcal{W}^{r',s'}\,,\ \psi(T) = 0\,.$

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Reasoning as in the case $\delta = 0$,

$$\int_\delta^T \chi_\delta \int_{\mathbb{R}^3} w \cdot f = -\int_\delta^{2\delta} \chi_\delta' \int_{\mathbb{R}^3} w \cdot \psi \,, \; f \in C_0^\infty(\mathbb{R}^3 \times (0,T)) \,.$$

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because

$$\chi_\delta' \sim \delta^{-1} \,, \; w \to 0 \text{ weakly in } L^2 \,, \; \psi \in C([0,T]; L^2) \,.$$

QED

Entirely similar arguments can be employed to draw the same conclusions if either

$$v \in L^4(0,T;L^4) \quad [\text{resp. } v \in L^4(\delta,T;L^4)]\,,$$

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Thank you