Divergence-free positive symmetric tensors and fluid dynamics

Denis Serre
École Normale Supérieure de Lyon, France
denis.serre@ens-lyon.frl

Abstract

We consider $d \times d$ tensors $A(x)$ that are symmetric, positive semi-definite, and whose row-wise divergence vanishes identically. We establish sharp inequalities for the integral of $(\det A)^{\frac{1}{d-1}}$. The latter quantity failing to be concave over $\mathbf{Sym}_d^+$, unlike $(\det A)^{\frac{1}{d}}$, this result does not follow from Jensen's inequality. It is likely to imply a Div-quasiconcavity property for $(\det A)^{\frac{1}{d-1}}$, after Fonseca & Müller's terminology.

We consider two cases, whether the tensor is periodic according to a lattice, or it is defined over a convex domain. These estimates are sharp enough to yield a rather broad equality case. The proof involves an auxiliary fonction that solves a Monge–Ampère equation ; in one case, it is given by Brenier's Theorem on optimal transport of measures.

These inequalities can be viewed as a "non-commutative" version of Gagliardo's inequality, the latter being the particular case of a diagonal tensor. Gagliardo's inequality is a useful trick in the proof of the Sobolev embeddings. They also contain as a particular case the isoperimetric inequality for convex bodies. Another motivation for this work comes from homogenization of periodic materials.

We apply the inequality with convex domain to the Cauchy problem for models of compressible inviscid fluids, when the total mass and energy are finite: Euler equations, Euler–Fourier, relativistic Euler, Boltzman, BGK, etc... We deduce an *a priori* estimate for a new quantity, namely the space-time integral of either $\rho^{\frac{1}{n}}p$, where ρ is the mass density, p the pressure and n the space dimension, or

$$\left(\int_{\mathbb{R}^n}^{\otimes(n+1)} f(\xi_0)\cdots f(\xi_n)\,[\mathrm{vol}\,(\mathrm{Simplex}(\xi_0,\dots,\xi_n))]^2\,d\xi_0\cdots d\xi_n\right)^{\frac{1}{n}}$$

where $f(\xi) = f(t, x, \xi)$ is the particle density in kinetic models. The latter estimate was known only in one space dimension (Bony's functional). In both situations, this provides a higher integrability than that given by mass and energy, though the price to pay is a time integration.

Keywords: Conservation laws ; gas dynamics ; functional inequalities.

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