Czech Academy of Sciences

Czech Technical University in Prague

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Nečas Center for Mathematical Modelling

Waves in Flows

Global Existence of Solutions with non-decaying initial data 2d(3d)-Navier-Stokes ibvp in half-plane(space)

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Università degli Studi della Campania “L. Vanvitelli”

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P.M. AND S. SHIMIZU,

Global existence of solutions to 2-D Navier-Stokes flow with non-decaying initial data in half-plane, in press on J. Diff. Equations.

Introduction

We consider the Navier-Stokes problem

$$u_t + u \cdot \nabla u + \nabla \pi_u = \Delta u, \ \nabla \cdot u = 0, \ \text{in } (0,T) \times \Omega,$$
$$u = 0 \text{ on } (0,T) \times \partial\Omega, \ u(0,x) = u_0(x) \text{ on } \{0\} \times \Omega, \tag{1}$$

where u is the kinetic field and π_u is the pressure field

$u \cdot \nabla u := u_k \frac{\partial}{\partial x_k} u,$

Ω can be $\mathbb{R}^n_+$ or $\mathbb{R}^n$, $n = 2, 3$.

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However, very recently, for $p \in (2, \infty]$ the study of a L^p-setting is considered.

- There are physical and mathematical motivations for an L^p-theory with $p > 2$, connected with fact that the L^2-norm cannot be assumed finite.
- We do not go into the physics matter of these questions.
- From the mathematical point view, at least the problem of the uniqueness is not recent.
- After introducing the question, I give some results concerning the L^∞-theory for non decaying data.

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- After introducing the question, I give some results concerning the L^∞-theory for non decaying data.

Solutions with non decaying data

We start remarking that the denomination of *solutions with non decaying data* is not ours.

The first time appears in the paper
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Subsequently non decaying is substituted by non decreasing in the paper
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Results related to the global existence in time

Global existence is achieved

- by regular solutions for 2D-Navier-Stokes problem:
  L^2-theory, that is $u_0 \in L^2$, for all domains,
  non decaying data, $u_0 \in L^\infty$, there are some restrictions on the unbounded domains,
- by weak solutions for 3D-Navier-Stokes problem:
  L^2-theory, that is $u_0 \in L^2$, for all domains,
  L^3-theory, that is $u_0 \in L^3$, we can assume $(0,T) \times \mathbb{R}^3$, $(0,T) \times \mathbb{R}^3_+$ and $(0,T) \times \Omega$, $\Omega \equiv$exterior domains,
  non decaying data, $u_0 \in L^\infty$, open problem.

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Two different approaches to the solutions with non decaying data

Two different approaches in order to prove the well posedeness of non decaying solutions:

- one looks for solutions in $\mathbb{L}^2_{\ell oc}$, a suitable space of functions,
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An $L^2_{\ell oc}$-setting in place of the $L^2(\mathbb{R}^n)$-setting

$L^2_{\ell oc}$-setting.

At least in the case of local (in time) existence, a good answer to the problem is given by Lemarié-Rieusset in

Recent development in the Navier-Stokes problem, CHAPMAN & HALL/CRC (2002).

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Set

$$\|u\|^2_{\mathbb{L}^2_{\ell oc}(\mathbb{R}^3)} := \sup_{x_0 \in \mathbb{R}^3} \int_{|x-x_0|<1} |u(x)|^2 dx \,.$$

Definition 1 (local Leray solution)

For all $u_0 \in \mathbb{L}^2_{\ell oc}(\mathbb{R}^3)$ with $\nabla \cdot u_0 = 0$, a field $u : (0,T) \times \mathbb{R}^3 \to \mathbb{R}^3$ is said a weak solution to the Navier-Stokes Cauchy problem associated to u_0 if

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- *the CKN-energy inequality holds for u,*
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Teorema 1 (Lemarié-Rieusset)

For all $u_0 \in \mathbb{L}^2_{\ell oc}(\mathbb{R}^3)$ with $\nabla \cdot u_0 = 0$, there exist a positive real number T and a local Leray solution u on $(0, T) \times \mathbb{R}^3$ for the Navier-Stokes Cauchy problem.

Teorema 2 (L-R)

For all $u_0 \in E_2(\mathbb{R}^3)$ with $\nabla \cdot u_0 = 0$, then the solution given in Theorem 1 is defined for all $t > 0$, it belongs to $L^\infty((0, T), E_2)$, for all $T < \infty$, and $\lim_{t \to 0} \|u(t) - u_0\|_{\mathbb{L}^2_{\ell oc}(\mathbb{R}^3)} = 0$.

In the above theorem

$$E_2 := \text{completion of } C_0^\infty(\Omega) \text{ in } \mathbb{L}^2_{\ell oc}(\mathbb{R}^3)\,.$$

One gets

$$u \in E_2 \;\Leftrightarrow\; u \in L^2_{\ell oc}(\mathbb{R}^3) \;\text{ and }\; \lim_{x_0 \to \infty} \int_{|x - x_0| < 1} |u(x)|^2 dx = 0\,.$$

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An $L^2_{\ell oc}$-setting in place of the $L^2(\mathbb{R}^n_+)$-setting

The extension of the above results to the IBVP in $(0, T) \times \mathbb{R}^n_+$, $n = 2, 3$, together with interesting connections with the blow up of the solutions are given in some very recent papers:

Maekawa Y., Miura H., and Prange C., *Local energy weak solutions for the Navier-Stokes equations in the half-space*, ArXiv:1711.044862v2.

Maekawa Y., Miura H., and Prange C., *Estimates for the Navier-Stokes equations in the half-space for non localized data*, ArXiv:1711.01651.

Prange C., *Infinite energy solutions to the Navier-Stokes equations in the half-space and applications*, ArXiv, 1803.02068.

The matter of the talk: L^∞-setting, the state of the art

We arrive at the chief step of the talk, that is the L^∞-setting for non decaying data.

- Recall that non decaying data means that *a priori* for $t > 0$ the solution can be bounded and not converging at infinity.
- This makes the difference not only with respect to a L^p-theory, but also with respect to the metric of E_2 introduced by Lemarié-Rieusset.
- I start with some references on the topic. It is suitable to distinguish between the question of the existence and of the uniqueness.

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L^∞, questions of uniqueness

Apart of the pioneering work on the existence for the Cauchy problem by Knightly (1972), the problem of the uniqueness goes back to the papers by Galdi & Rionero (1983) and Galdi & Maremonti (1986).
From (1999) the uniqueness for the Cauchy problem in the class of non-decaying solutions has been considered in several papers: Giga, Inui & Matsui (1999); Giga, Inui, J.Kato & Matsui (2001); Solonnikov (2003), J.Kato (2003); Kukavica (2003); Maremonti (2009 and 2011).

Actually, for the Cauchy problem we can give an explicit example of non-uniqueness of the null solution in any space dimension $n \geq 2$.
A typical example (Muratori 1971) is the following:

$$v = (a(t), 0, ..., 0), \quad \pi_v = -a'(t)x_1, \quad t \in (0, T),$$

$a \in W^{1,1}(0, T)$. Then, uniqueness is violated by selecting by means of any non null $a(t)$ with $a(0) = 0$. Examples of non uniqueness also hold for the IBVP in half-space and in exterior domains.

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The loss of uniqueness is due to the fact that the solutions have the pressure field does grow "sufficiently fast" at large (spatial) distances.
In particular, we have

$$\pi_v = O(|x|),\ \text{as } |x| \to \infty,\ \text{somehow equivalently, } \nabla\pi_v \text{ is bounded.}$$

The digression on the uniqueness, apart from the specific interest, is also important in order to understand why some authors look for nondecaying solutions "taking care of the behavior" of the pressure field.

This is our point of view, actually we prove that

$$\text{for some } \mu \in (0,1), \quad |\pi_v(t,x)| = P(t)O(|x|^\mu), \quad t > 0,\ x \in \mathbb{R}^2_+ .$$

An analogous estimate holds for regular 3D-solutions (of course, *a priori* on some $(0, T)$).
For the 3D weak solutions the estimate for the pressure field is not pointwise.

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The digression on the uniqueness, apart from the specific interest, is also important in order to understand why some authors look for nondecaying solutions "taking care of the behavior" of the pressure field.

This is our point of view, actually we prove that

$$\text{for some } \mu \in (0,1), \quad |\pi_v(t,x)| = P(t)O(|x|^\mu)\,, \quad t > 0,\ x \in \mathbb{R}^2_+\,.$$

An analogous estimate holds for regular 3D-solutions (of course, *a priori* on some $(0, T)$).
For the 3D weak solutions the estimate for the pressure field is not pointwise.

L^∞-setting for the existence

Also for the existence I would like to give a short “reference’s history”.
The problem of the existence with non-decaying data has been considered by several authors.

- Cauchy problem local existence, the first contributions go back to the paper: 3D Knightly (1972);
  $n \geq 2$, Giga, Inui & Matsui (1999); Giga, Matsui & Sawada (2001);
  3D Lemarié-Rieusset (2002);
- half-space, IBVP local existence,
  $n \geq 2$, Solonnikov (2003); $n \geq 3$ Maremonti (2009);
- exterior domains, IBVP local existence, $n \geq 3$,
  Galdi, Maremonti & Zhou (2012); Maremonti (2014), Abe (2017).

But a special interest deserves the case of global existence.

In this connection it is important to distinguish the **2D** problem from the **3D** problem.

L^∞-setting for the existence, a digression for the MMT

A natural setting of the problem is the function space $L^\infty((0,T)\times\Omega)$. Not only the well posedness of the problem is important, but it is also important to give a time pointwise estimate of $\|v(t)\|_\infty$.

It is important to stress that at least in the two dimensional case the questions can be considered, roughly speaking, independent.

L^∞-setting for the existence, a digression on the MMT

However, the best pointwise estimate up to day is obtained by Gallay (2014):

$$\|v(t)\|_{L^\infty(\mathbb{R}^2)} \leq c\|v_0\|_{L^\infty(\mathbb{R}^2)}(1 + ct\|v_0\|_{L^\infty(\mathbb{R}^2)}), \text{ for all } t > 0, \tag{2}$$

and it is based on a result by Zelik (2013) who deduces local estimates in the spaces $\mathbb{L}^2_{\ell oc}(\mathbb{R}^2)$ and $E_2(\mathbb{R}^2)$ of the solutions to the Cauchy problem.

L^∞-setting for the existence

A natural investigation is the case of the IBVP in half-plane and in exterior domains $\Omega\ (\subset \mathbb{R}^2)$.

In this sense a first contribution is given by Abe (*Math. Nachr.* 2016) for the IBVP in exterior domains $\Omega\ (\subset \mathbb{R}^2)$, $v_0 \in L^\infty(\Omega)$, but in this case it is just a local existence result.

Subsequently, for 2D-global existence, Abe (*ARMA* 2018) considers the case of the half-plane but under the assumption $v_0 \in L^\infty(\mathbb{R}^2_+)$ and $\|\nabla v_0\|_{L^2(\mathbb{R}^2_+)} < \infty$.

The latter assumption in the half-plane implies a decay from the viewpoint of Hardy's inequality.

L^∞-setting for the existence

Jointly with S.Shimizu, we give two contributions to the questions left open by Abe in his papers.

In order of time, we solve the problem of the global existence in exterior domains $\Omega \subseteq \mathbb{R}^2$ (*JMFM*, 2017).

Subsequently, we solve the case of the half-plane, (*JDE* in press).

L^∞-setting for the existence

For the sake of completeness, I like to recall that for the 2D-IBVP of (1), Ω exterior, we get

Teorema 3

Let $v_0 \in L^\infty(\Omega)$ be divergence free in the weak sense, and $v_0 \cdot \nu = 0$. Then there exists a unique solution (v, π_v) to problem (1) such that, for all $\delta > 0$,

$$\text{for all } T > 0,\ v \in C_w([0,T); L^\infty(\Omega)), v, v_t, D^2 v, \nabla \pi_v \in C((\delta, T) \times \Omega)$$

and, for a suitable $\varepsilon, \mu \in (0,1)$, smooth functions F and h, we get

$$|\pi_v(t,x)| \leq c t^{-\mu} |x|^\varepsilon F(\|v_0\|_\infty, h(t)),\ \text{for all } (t,x) \in (0,T) \times \Omega,$$

for $\Omega \equiv \mathbb{R}^2$, $\varepsilon = 2\mu \in (0,1)$ arbitrary.

Theorem 3: the difficulty to consider the IBVP in $\mathbb{R}^2_+$

The approach used for the exterior domain does not work in the case of the half-plane, more in general for the domains with unbounded boundaries. The reason is that we are not able to realize a “certain extension V” sufficiently regular and with compact support.

This leads to a change of the proof.
The new strategy of proof allows us to solve the half-plane case, but it also allows us to prove existence of **weak solutions** for the **3D** Cauchy and for the IBV problem in a **3D** half-space by means of special data.

Statement of the results in the case of $\mathbb{R}^n$ and $\mathbb{R}^n_+$ for $n = 2, 3$

We introduce some notations:

$$\Omega \text{ can be } \mathbb{R}^n \text{ or } \mathbb{R}^n_+ \,, n = 2, 3\,, \quad \dot{W}^{1,q}(\Omega) := \{u : \|\nabla u\|_q < \infty\},$$
$$J^q_0(\Omega) := \text{completion of } \mathscr{C}_0(\Omega) \text{ in } \dot{W}^{1,q}(\Omega)\,.$$

Teorema 4 (n=2)

Let $u_0 \in L^\infty(\Omega) \cap J^q_0(\Omega)$, $q \in (2, \infty)$. Then there exists a unique solution (u, π_u) to problem (1) such that

$$\begin{aligned} &\text{for all } T > 0,\ u \in C([0, T) \times \overline{\Omega})\,,\\ &\forall\, \delta > 0,\ u \in C(\delta, T; C^2(\overline{\Omega}))\,,\ u_t, \nabla\pi_u \in C((\delta, T) \times \overline{\Omega})\\ &\text{up to a function } c(t),\ |\pi(t,x)| \le P(t)|x|^\mu,\ \mu \in (0,1),\ \text{for all } t > 0 \text{ and } x \in \Omega, \end{aligned} \tag{3}$$

where, for all $\delta > 0$ and $T > 0$, $c(t), P(t) \in C(\delta, T) \cap L^s(0, T)$ for a suitable $s > 1$.
If $u_0 \in L^\infty(\Omega) \cap J^2_0(\Omega)$, the claim $(4)_1$ becomes $u \in L^\infty((0, T) \times \Omega)$ and

$$\text{for all } q \ge 1 \text{ and } R > 0, \quad \lim_{t \to 0} \|u(t) - u_0\|_{L^q(|x| \le R,\, x_2 > 0)} = 0\,.$$

Statement of the results in the case of $\mathbb{R}^n$ and $\mathbb{R}^n_+$ for $n = 2, 3$

We introduce some notations:

$$\Omega \text{ can be } \mathbb{R}^n \text{ or } \mathbb{R}^n_+ \,, n = 2, 3 \,, \quad \dot{W}^{1,q}(\Omega) := \{u : \|\nabla u\|_q < \infty\},$$
$$J^q_0(\Omega) := \text{completion of } \mathscr{C}_0(\Omega) \text{ in } \dot{W}^{1,q}(\Omega)\,.$$

Teorema 4 (n=2)

Let $u_0 \in L^\infty(\Omega) \cap J^q_0(\Omega)$, $q \in (2, \infty)$. *Then there exists a unique solution* (u, π_u) *to problem* (1) *such that*

$$\begin{gathered} \textit{for all } T > 0,\; u \in C([0,T) \times \overline{\Omega})\,, \\ \forall\, \delta > 0,\; u \in C(\delta, T; C^2(\overline{\Omega}))\,,\; u_t, \nabla\pi_u \in C((\delta, T) \times \overline{\Omega}) \\ \textit{up to a function } c(t),\; |\pi(t,x)| \le P(t)|x|^\mu, \mu \in (0,1),\; \textit{for all } t > 0 \textit{ and } x \in \Omega, \end{gathered} \tag{3}$$

where, for all $\delta > 0$ *and* $T > 0$, $c(t), P(t) \in C(\delta, T) \cap L^s(0, T)$ *for a suitable* $s > 1$.

If $u_0 \in L^\infty(\Omega) \cap J^2_0(\Omega)$, *the claim* $(4)_1$ *becomes* $u \in L^\infty((0, T) \times \Omega)$ *and*

$$\textit{for all } q \ge 1 \textit{ and } R > 0, \quad \lim_{t \to 0} \|u(t) - u_0\|_{L^q(|x| \le R,\, x_2 > 0)} = 0\,.$$

Comparing Theorem 4

For $q = 2$, Theorem 4 just reproduces the result given by Abe (ARMA 2018) for decaying data.

Different is the comparison with the results by Maekawa & al. (2018). Actually the initial data are not comparable in the sense that

- since our data is more regular, it belongs to $\mathbb{L}^2_{\ell oc}(\Omega)$, but *a priori* does not admit limit at infinity, hence it is a non decaying data;
- in the case of Maekawa & al. the initial data is assumed in E_2, so that it is weaker, but it suitably tends to zero at infinity. Hence it is a decaying data.

Statement of Theorem 4 in 3D

J^2 :=completion of $\mathscr{C}_0$ in L^2, $J^{1,2}$:=completion of $\mathscr{C}_0$ in $W^{1,2}$,

$J^q_0(\Omega) :=$ completion of $\mathscr{C}_0(\Omega)$ in $\dot{W}^{1,q}(\Omega)$.

Teorema 5 (n=3, $\Omega \equiv \mathbb{R}^3_+$ or $\Omega \equiv \mathbb{R}^3$)

Let $u_0 \in L^\infty(\Omega) \cap J^q_0(\Omega)$, $q \in (3, \infty)$. Then there exists a field $u : (0, \infty) \times \Omega \to \mathbb{R}^3$ which is a solution in the distributional sense to problem (1). *Moreover, set k the greatest integer less than or equal to $\log_2 \frac{q}{2}$, we get*

$$u = U + \sum_{\ell=1}^{k+1} v^\ell + w\,, \text{ for all } T > 0\,,$$

$$\pi_u := \pi_U + \sum_{\ell=1}^{k+1} \pi_{v^\ell} + \pi_w\,, \text{ for all } T > 0\,,$$

where, for all $\delta > 0$,

$$U, v^\ell \in C([0,T) \times \overline{\Omega}) \cap C(\delta, T; C^2(\overline{\Omega})),\ U_t, v^\ell_t, \nabla\pi_U, \nabla\pi_{v^\ell} \in C((\delta,T) \times \overline{\Omega})\,,$$

$$w \in L^\infty(0,T; J^2(\Omega)) \cap L^2(0,T; J^{1,2}(\Omega)),\ \pi_w \in L^{\frac{5}{4}}((0,T) \times \Omega)\,.$$

Finally, the solution u is strongly continuous to the initial data:

$$\lim_{t\to 0} \|u(t) - u_0\|_\infty = 0\,. \qquad (4)$$

After I give the comment on property (5).

Statement of Theorem 4 in 3D

$J^2 :=$completion of $\mathscr{C}_0$ in L^2, $\quad J^{1,2} :=$completion of $\mathscr{C}_0$ in $W^{1,2}$,
$J^q_0(\Omega) :=$ completion of $\mathscr{C}_0(\Omega)$ in $\dot{W}^{1,q}(\Omega)$.

Teorema 5 (n=3, $\Omega \equiv \mathbb{R}^3_+$ or $\Omega \equiv \mathbb{R}^3$)

Let $u_0 \in L^\infty(\Omega) \cap J^q_0(\Omega)$, $q \in (3, \infty)$. *Then there exists a field* $u : (0, \infty) \times \Omega \to \mathbb{R}^3$ *which is a solution in the distributional sense to problem* (1). *Moreover, set* k *the greatest integer less than or equal to* $\log_2 \frac{q}{2}$, *we get*

$$u = U + \sum_{\ell=1}^{k+1} v^\ell + w\,, \quad \textit{for all } T > 0\,,$$

$$\pi_u := \pi_U + \sum_{\ell=1}^{k+1} \pi_{v^\ell} + \pi_w\,, \quad \textit{for all } T > 0\,,$$

where, for all $\delta > 0$,

$$U, v^\ell \in C([0,T) \times \overline{\Omega}) \cap C(\delta, T; C^2(\overline{\Omega})),\ U_t, v^\ell_t, \nabla\pi_U, \nabla\pi_{v^\ell} \in C((\delta, T) \times \overline{\Omega})\,,$$

$$w \in L^\infty(0, T; J^2(\Omega)) \cap L^2(0, T; J^{1,2}(\Omega)),\ \pi_w \in L^{\frac{5}{4}}((0,T) \times \Omega)\,.$$

Finally, the solution u is strongly continuous to the initial data:

$$\lim_{t \to 0} \|u(t) - u_0\|_\infty = 0\,. \tag{4}$$

After I give the comment on property (5).

Comparing Theorem 5

Comparing our result with the results by Lemarié-Rieusset (2002), there are the same remarks developed for the result by Maekawa & al..
That is the initial data are not comparable in the sense that

- since our data is more regular, it belongs to $\mathbb{L}^2_{\ell oc}(\Omega)$, but *a priori* does not admit limit at infinity, hence it is a non decaying data;
- in the case of Lemarié-Rieusset (just Cauchy problem) the initial data is assumed in E_2, so that it is weak but tends to zero at infinity, hence it is a decaying data.

The idea of the proof

The arguments related to the proof are the same in the 2D and 3D cases. Firstly we introduce a finite set of functions, each is solution of a suitable Navier-Stokes **linearized problem**.

The number of the functions depends on q.

Actually, by k we mean the greatest integer less than or equal to $\log_2 \frac{q}{2}$. The first element of the set is the solution to the Stokes problem:

$$\begin{aligned} & U_t + \nabla \pi_U = \Delta U, \quad \nabla \cdot U = 0 \ \text{ in } (0,T) \times \Omega\,, \\ & U = 0 \text{ on } (0,T) \times \partial\Omega\,, \quad U = u_0 \text{ on } \{0\} \times \Omega\,. \end{aligned} \tag{5}$$

Then, for $k \geq 1$, we set for $h = 1, \ldots, k$

$$\begin{aligned} & v^h_t - \Delta v^h + \nabla \pi_{v^h} = -\sum_{\ell=0}^{h-1} w^\ell \cdot \nabla v^h - \sum_{\ell=0}^{h-1} v^h \cdot \nabla w^\ell - F^h\,, \\ & \nabla \cdot v^h = 0 \quad \text{in } (0,T) \times \Omega, \\ & v^h = 0 \quad \text{on } (0,T) \times \partial\Omega, \quad v^h = 0 \quad \text{on } \{0\} \times \Omega, \end{aligned} \tag{6}$$

with $\Omega \subseteq \mathbb{R}^2$, $w_0 := U$, $w^\ell := v^\ell$ for $\ell = 1, \ldots, h-1$ and $F^h := v^{h-1} \cdot \nabla v^{h-1}$.

The idea of the proof

The data force $F^h = v^{h-1} \cdot \nabla v^{h-1}$ is smooth and enjoys integrability properties as follows:

- $h = 1$, $F^1 = U \cdot \nabla U \in L^q$, with $\|F^1\|_q \leq c\|u_0\|_\infty \|\nabla u_0\|_q$, it generates v^1,
- $h = 2$, $F^2 := v^1 \cdot \nabla v^1 \in L^{\frac{q}{2}} \cap L^q$,
  with $\|F^2\|_{\frac{q}{2}} \leq \|v^1\|_q \|\nabla v^1\|_q$ and $\|F^2\|_q \leq \|v^1\|_\infty \|\nabla v^1\|_q$, it generates v^2,
- $\ldots$
- $F^h := v^{h-1} \cdot \nabla v^{h-1} \in L^{\frac{q}{2^h}} \cap L^q$
  with $\|F^h\|_{\frac{q}{2^h}} \leq \|v^{h-1}\|_{\frac{q}{2^{h-1}}} \|\nabla v^{h-1}\|_{\frac{q}{2^{h-1}}}$ and $\|F^h\|_q \leq \|v^{h-1}\|_\infty \|\nabla v^{h-1}\|_q$,
  it generates v^h
- $\ldots$
- and so on until to step k.

The idea of the proof

Finally, to the step k we work with the nonlinear problem

$$\begin{aligned} & w_t - \Delta w + \nabla \pi_w = -w \cdot \nabla w - \sum_{\ell=0}^{k} w^\ell \cdot \nabla w - w \cdot \nabla \sum_{\ell=0}^{k} w^\ell - F\,, \\ & \nabla \cdot w = 0 \text{ in } (0,T) \times \Omega, \\ & w = 0 \text{ on } (0,T) \times \partial\Omega, \quad w = 0 \text{ on } \{0\} \times \Omega, \end{aligned} \tag{7}$$

where $w^0 := U$, $w^\ell := v^\ell, \ell = 1, \ldots, k$, and

$$F := v^k \cdot \nabla v^k \in L^2(0,T; L^2(\Omega)) \cap L^\infty(0,T; L^q(\Omega)).$$

It is easy to imagine that

$$u = U + \sum_{\ell=1}^{k} v^\ell + w\,, \text{ for all } T > 0\,,$$

$$\pi_u := \pi_U + \sum_{\ell=1}^{k} \pi_{v^\ell} + \pi_w\,, \text{ for all } T > 0\,,$$

is a solution to (1) in a suitable sense.

The idea of the proof

If $n = 2$, the above construction starts from some $q > 2$ and ensures a regular solution (u, π_u).
If $n = 3$, the above construction starts from some $q > 3$, and, since in the last step (w, π_w) is a L^2-weak solution, also (u, π_u) is a weak solution.

A remark on Theorem 5

In the statement we claim the limit property (5), that is

$$\lim_{t \to 0} \|u(t) - u_0\|_\infty = 0\,.$$

The limit property does not have to surprise.

Actually, at least on some interval $(0, T_0)$ we can state the existence of a regular solution $(\widetilde{u}, \pi_{\widetilde{u}})$. This is consequence of the fact that the nonlinear step for "w" admits a smooth solution, say $(\widetilde{w}, \pi_{\widetilde{w}})$, on some $(0, T_0)$. Since $U, v^h, \widetilde{w} \in C([0, T_0); J^{1,q}(\Omega))$, for some $q > 3$, we get the regularity such that

$$\lim_{t \to 0} \|\widetilde{u}(t) - u_0\|_\infty = 0\,.$$

We can compare the weak solution (w, π_w) with the regular solution $(\widetilde{w}, \pi_{\widetilde{w}})$, and they coincide on $(0, T_0)$. This explains the limit property.

The last remark: we are not able to employ the proof in the case of an exterior domain $\Omega \subset \mathbb{R}^3$, because we do not know the following estimates for solutions to the Stokes problem:

$$q \neq 2, \ \|\nabla U(t)\|_{L^q(\Omega)} \leq c(t)\|\nabla U_0\|_{L^q(\Omega)}, \ t > 0 \,, \tag{8}$$

with $c(t)$ smooth function.
In the case of $\mathbb{R}^n$ or $\mathbb{R}^n_+$ we obtain (9) by using the representation formula of the solutions and $c(t) = const$.