Hamiltonian formulation for wave-current interactions in stratified rotational flows

Calin Martin[1]*

Joint work with Adrian Constantin[1] and Rossen Ivanov[2]

[1] University of Vienna, Austria

[2] Dublin Institute of Technology, Ireland

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Introduction

The main motivation lies in the study of (tropical, equatorial) ocean dynamics that presents several challanges due to

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- fluctuations of the pycnocline give rise to internal waves that are coupled with surface waves
- the strength of the coupling varies–very often very large internal waves are hardly noticeable at the surface, an observable manifestation of this feature being the dead-water phenomenon

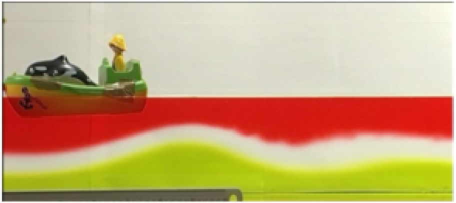

Figure: Dead-water occurs when a ship moves in still water and its displacement sets up small surface waves (some tens of cm). The coupling effect triggers concealed internal waves with amplitudes of up to tens of m. A scenario prone to bring about dead-water is when a low-salinity layer of water sits on top of more dense salt water.

The fluid domain

- adjacent to the bed there is a layer

$$\Omega^* := \{(x, y, t) : \; x \in \mathbb{R}, \, t \in \mathbb{R}, \, -h < y < \eta(x, t)\}$$

of constant density ρ

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  of constant density ρ
- adjacent to the free surface there is a layer

$$\Omega_1^* := \{(x, y, t) : \; x \in \mathbb{R}, \, t \in \mathbb{R}, \, \eta(x, t) < y < h_1 + \eta_1(x, t)\},$$

  of constant density $\rho_1 < \rho$ (stable stratification), that is,

$$\hat{\rho} = \begin{cases} \rho & \text{in } \Omega^* \\ \rho_1 & \text{in } \Omega_1^* \end{cases}$$

The fluid domain

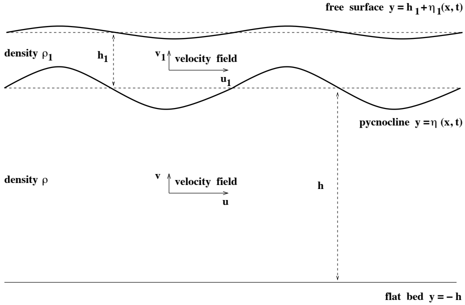

Figure: A cross section of the fluid domain, showing the relevant physical variables

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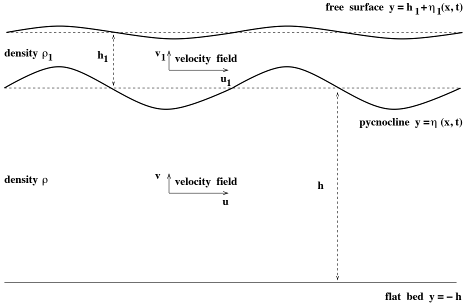

Figure: A cross section of the fluid domain, showing the relevant physical variables

$x \to \eta(x,t), x \to \eta_1(x,t)$ – periodic functions (of period L) with

$$\int_0^L \eta(x,t)dx = \int_0^L \eta_1(x,t) = 0, \quad \text{for all } t.$$

Equations of motion and boundary conditions

- Momentum conservation

$$\begin{cases} \hat{u}_t + \hat{u}\hat{u}_x + \hat{v}\hat{u}_y &= -\hat{\frac{1}{\rho}} P_x, \\ \hat{v}_t + \hat{u}\hat{v}_x + \hat{v}\hat{v}_y &= -\hat{\frac{1}{\rho}} P_y - g, \end{cases} \quad \text{in} \quad \Omega^* \cup \Omega_1^*,$$

with $(\hat{u}(x, y, t), \hat{v}(x, y, t))$–velocity field,
$P(x, y, t)$–pressure,
g–acceleration of gravity.

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- Mass conservation

$$\hat{u}_x + \hat{v}_y = 0 \quad \text{in} \quad \Omega^* \cup \Omega_1^*.$$

Boundary conditions

- the dynamic boundary condition

$$P = P_{atm} \quad \text{on} \quad y = h_1 + \eta_1(x,t)$$

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- kinematic boundary conditions (fluid particles do not cross boundaries)

$$v_1 = \eta_{1,t} + u_1 \eta_{1,x} \quad \text{on} \quad y = \eta_1(x,t) + h_1 \,,$$

$$v_1 = \eta_t + u_1 \eta_x \quad \text{and} \quad v = \eta_t + u \eta_x \quad \text{on} \quad y = \eta(x,t),$$

$$v = 0 \quad \text{on} \quad y = -h \,,$$

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- the continuity of the pressure accross the interface $y = \eta(x,t)$

$$P_a = P_b \quad \text{at} \quad y = \eta(x,t)\,,$$

Remarks

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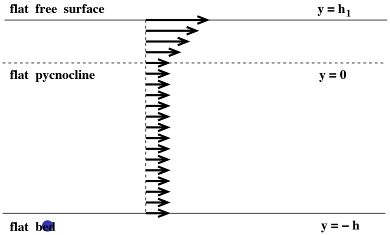

Figure: Sketch of a typical tropical oceanic background flow, $(\hat{u}(y), 0)$, in the absence of waves, depicting a favourable tidal current. A jump discontinuity occurs across the pycnocline, with $u_1(0) > u(0) = u(-h)$ relating the limits $u_1(0)$ and $u(0)$ from above and below the flat interface $y = 0$.

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- the proper function space setting: smooth η and η_1 with

$$u_1,\ v_1 \in C^\infty(\Omega_1^*) \cap C(\overline{\Omega_1^*})$$

and

$$u,\ v \in C^\infty(\Omega^*) \cap C(\overline{\Omega^*}).$$

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- to capture the wave-currents interactions one needs the vorticity

$$\hat{\gamma} := \hat{u}_y - \hat{v}_x\,, \quad \hat{\gamma} := \begin{cases} \gamma \in \mathbb{R},\ \text{in } \Omega^* \\ \gamma_1 \in \mathbb{R},\ \text{in } \Omega_1^*. \end{cases}$$

The stream functions

Aim: decompose the motion into a "current" component–average velocity–and periodic component that fluctuates around this average.

The equation of mass conservation entails the existence of $\hat{\psi}$ such that

$$\begin{cases} u = \psi_y \quad \text{and} \quad v = -\psi_x \quad \text{in} \quad \Omega, \\ u_1 = \psi_{1,y} \quad \text{and} \quad v_1 = -\psi_{1,x} \quad \text{in} \quad \Omega_1. \end{cases}$$

It holds ψ and ψ_1 are periodic (of period L) and

$$\Delta\psi = \gamma \quad \text{in} \quad \Omega, \qquad \Delta\psi_1 = \gamma_1 \quad \text{in} \quad \Omega_1.$$

There is $\hat{\psi} \in C(\overline{\Omega \cup \Omega_1})$ with

$$\hat{\psi} = \begin{cases} \psi \text{ in } \Omega \\ \psi_1 \text{ in } \Omega_1 \end{cases}$$

The (generalized) velocity potentials

There are velocity potentials φ in Ω and φ_1 in Ω_1 with

$$\begin{cases} u = \varphi_x + \gamma y & \text{and} \quad v = \varphi_y & \text{in} \quad \Omega, \\ u_1 = \varphi_{1,x} + \gamma_1 y & \text{and} \quad v_1 = \varphi_{1,y} & \text{in} \quad \Omega_1. \end{cases}$$

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Remarks:

- there is no $\hat{\varphi}$ extending φ and φ_1 accross $y = \eta(x,t)$!
- φ and φ_1 are not periodic!.
- In fact,

$$\varphi(x+L, y, t) = \varphi(x, y, t) + \kappa, \ (x, y) \in \Omega,$$

$$\varphi_1(x+L, y, t) = \varphi_(x, y, t) + \kappa_1, \ (x, y) \in \Omega_1,$$

with

$$\kappa := \frac{1}{L}\int_0^L \left[u(x, -h, t) + \gamma h\right] dx,$$

$$\kappa_1 := \frac{1}{L}\int_0^L \left[u_1(x, \eta(x,t), t) + v_1(x, \eta(x,t), t)\, \eta_x(x,t)\right] dx$$

It turns out that κ and κ_1 are independent of t.

- Denoting

$$\tilde{\varphi}(x,y) := \varphi(x,y) - \kappa x$$

and

$$\tilde{\varphi}_1(x,y) := \varphi_1(x,y) - \kappa_1 x$$

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- In fact

$$\gamma y + k = U(y,t) := \frac{1}{L}\int_0^L u(x,y,t)\,dx$$

$$\gamma_1 y + k_1 = U_1(y,t) := \frac{1}{L}\int_0^L u_1(x,y,t)\,dx$$

One ca recover the gov. eqs. and the boundary cond. by means of $\psi, \psi_1, \tilde{\varphi}, \tilde{\varphi}_1, \eta, \eta_1$.

- the momentum and mass cons., the dyn. cond on the surface and the kin. cond. on the bed can be written

$$P = P_{atm} - \rho_1 \Big[\tilde{\varphi}_{1,t} + \frac{1}{2}|\nabla\psi_1|^2 - \gamma_1\psi_1 + gy \\ + \gamma_1 \psi_1(0,\ h_1 + \eta_1(0,t),\ t) \Big] \quad \text{in} \quad \Omega_1 ,$$

$$P = P_{atm} - \rho\Big[\tilde{\varphi}_t + \frac{1}{2}|\nabla\psi|^2 - \gamma\psi + gy\Big] \\ -\rho_1\gamma_1\Big[\psi_1(0, h_1 + \eta_1(0,t), t) - \psi_1(0, \eta(0,t), t)\Big] \\ -\rho\gamma\, \psi(0, \eta(0,t), t) \text{ in } \Omega .$$

- the dyn. bd. cond on the surface can be written as

$$\tilde{\varphi}_{1,t}+\frac{1}{2}|\nabla\psi_1|^2-\gamma_1\chi_1+g(h_1+\eta_1)=0 \quad \text{on} \quad y=h_1+\eta_1(x,t)$$

with

$$\chi_1(x,t):=\psi_1(x,h_1+\eta_1(x,t),t)-\psi_1(0,h_1+\eta_1(0,t),t)\,,$$

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$$\chi_1(x,t):=\psi_1(x,h_1+\eta_1(x,t),t)-\psi_1(0,h_1+\eta_1(0,t),t),$$

- the continuity of the pressure across the interface is

$$\rho\left[(\tilde{\varphi}_t)_s+\frac{|\nabla\psi|_s^2}{2}-\gamma\chi+g\eta\right]$$
$$=\rho_1\left[(\tilde{\varphi}_{1,t})_s+\frac{|\nabla\psi_1|_s^2}{2}-\gamma_1\chi+g\eta\right] \quad \text{on} \quad y=\eta(x,t),$$

with

$$\begin{aligned}\chi(x,t) :&= \psi(x,\eta(x,t),t)-\psi(0,\eta(0,t),t)\\ &= \psi_1(x,\eta(x,t),t)-\psi_1(0,\eta(0,t),t)\end{aligned}$$

• the kinematic boundary conditions on the free surface and interface can be written as

$$\chi(x,t) = -\int_0^x \eta_t(l,t)\,dl\,, \qquad \chi_1(x,t) = -\int_0^x \eta_{1,t}(l,t)\,dl\,.$$

The Hamiltonian formulation

Aim: collapse the dynamical variables into suitable one-dim. representations, that is, write the governing eqs. as

$$\omega_t = J\,\frac{\delta H}{\delta \omega},$$

$t \mapsto \omega(t)$–path in a Hilbert space $\mathcal{H}$ equipped with an inner product $(\cdot,\cdot)$, $H : \mathcal{D} \subset \mathcal{H} \to \mathbb{R}$,
J is a skew-adjoint (pseudo-)differential operator which defines a Poisson bracket by

$$\{F_1, F_2\} = \Big(\frac{\delta F_1}{\delta \omega}, J\frac{\delta F_2}{\delta \omega}\Big),$$

provided that the *Jacobi identity*
$\{\{F_1, F_2\},\ F_3\} + \{\{F_2, F_3\},\ F_1\} + \{\{F_3, F_1\},\ F_2\} = 0$ holds.
$\frac{\delta H}{\delta \omega}$–variational derivative of H with respect to ω

$$\lim_{\varepsilon\to 0} \frac{H(\omega + \varepsilon\,\omega_0) - H(\omega)}{\varepsilon} = \Big(\omega_0\,,\ \frac{\delta H}{\delta \omega}\Big) \text{ for } \omega_0 \in \mathcal{D}.$$

Identify H, J, ω!

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Let H be the total energy of the flow.

$$H = \int \int_{\Omega \cup \Omega_1} \hat{\rho} \left\{ \frac{\hat{u}^2 + \hat{v}^2}{2} + gy \right\} dydx \,,$$

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that is

$$H = \int_0^L \int_{-h}^{\eta(x,t)} \rho\, \frac{u^2+v^2}{2}\, dydx' + \int_0^L \int_{\eta(x,t)}^{h_1+\eta_1(x,t)} \rho_1\, \frac{u_1^2+v_1^2}{2}\, dydx' \tag{1}$$

$$+ \int_0^L \int_{-h}^{\eta(x,t)} g\rho y\, dydx' + \int_0^L \int_{\eta(x,t)}^{h_1+\eta_1(x,t)} g\rho_1 y\, dydx'\,,$$

In terms of $\tilde{\varphi}, \tilde{\varphi}_1, \eta, \eta_1$...

$$\frac{\rho}{2}\int_0^L\int_{-h}^{\eta}|\nabla\tilde{\varphi}|^2\,dydx+\rho\gamma\int_0^L\int_{-h}^{\eta}y\tilde{\varphi}_x\,dydx+\frac{\rho\gamma^2}{6}\int_0^L(\eta^3+h^3)\,dx$$

$$+\rho\kappa\int_0^L\int_{-h}^{\eta}\tilde{\varphi}_x\,dydx+\frac{\rho\gamma\kappa}{2}\int_0^L(\eta^2-h^2)\,dx+\frac{\rho k^2}{2}\int_0^L(\eta+h)\,dx$$

$$+\frac{\rho_1}{2}\int_0^L\int_{\eta}^{h_1+\eta_1}|\nabla\tilde{\varphi}_1|^2\,dydx+\rho_1\gamma_1\int_0^L\int_{\eta}^{h_1+\eta_1}y\tilde{\varphi}_{1,x}\,dydx$$

$$+\frac{\rho_1\gamma_1^2}{6}\int_0^L\left((h_1+\eta_1)^3-\eta^3\right)dx$$

$$+\rho_1\kappa_1\int_0^L\int_\eta^{h_1+\eta_1}\tilde{\varphi}_{1,x}\,dydx+\frac{\rho_1\gamma_1\kappa_1}{2}\int_0^L\Big((h_1+\eta_1)^2-\eta^2\Big)\,dx$$

$$+\frac{\rho_1 k_1^2}{2}\int_0^L(h_1+\eta_1-\eta)\,dx$$

$$+\frac{\rho g}{2}\int_0^L(\eta^2-h^2)\,dx+\frac{\rho_1 g}{2}\int_0^L\Big((h_1+\eta_1)^2-\eta^2\Big)\,dx\,.$$

It turns out that the governing equations admit the following nearly-Hamiltonian formulation [1]

$$\begin{cases} \eta_{1,t} = \dfrac{\delta H}{\delta \xi_1}\,, & \xi_{1,t} = -\dfrac{\delta H}{\delta \eta_1} + \rho_1 \gamma_1 \chi_1\,, \\ \eta_t = \dfrac{\delta H}{\delta \xi}\,, & \xi_t = -\dfrac{\delta H}{\delta \eta} + (\rho\gamma - \rho_1\gamma_1)\chi\,, \end{cases}$$

where ξ and ξ_1 involve traces of the velocity potentials on $y = \eta(x,t)$ and $y = h_1 + \eta_1(x,t)$.

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where ξ and ξ_1 involve traces of the velocity potentials on $y = \eta(x,t)$ and $y = h_1 + \eta_1(x,t)$.

More precisely,

$$\begin{cases} \xi := \rho\Phi - \rho_1 \Phi_1, \\ \xi_1 := \rho_1 \Phi_2 . \end{cases}$$

where

$$\begin{cases} \Phi(x,t) = \tilde{\varphi}(x, \eta(x,t), t), \\ \Phi_1(x,t) = \tilde{\varphi}_1(x, \eta(x,t), t), \\ \Phi_2(x,t) = \tilde{\varphi}_1(x, h_1 + \eta_1(x,t), t). \end{cases}$$

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  Wahlen: *A Hamiltonian formulation for water waves with constant vorticity*. Lett. Math. Phys. (2007).

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From nearly-Hamiltonian to Hamiltonian formulation

The change of variables

$$\begin{cases} z = \xi + \dfrac{\rho\gamma - \rho_1\gamma_1}{2}\displaystyle\int_0^x \eta(l,t)\,dl\,, \\ z_1 = \xi_1 + \dfrac{\rho_1\gamma_1}{2}\displaystyle\int_0^x \eta_1(l,t)\,dl\,, \end{cases}$$

transforms the nearly-Hamiltonian system

$$\begin{cases} \eta_{1,t} = \dfrac{\delta H}{\delta \xi_1}\,, & \xi_{1,t} = -\dfrac{\delta H}{\delta \eta_1} + \rho_1\gamma_1\chi_1\,, \\ \eta_t = \dfrac{\delta H}{\delta \xi}\,, & \xi_t = -\dfrac{\delta H}{\delta \eta} + (\rho\gamma - \rho_1\gamma_1)\chi\,, \end{cases}$$

into the Hamiltonian one

$$\begin{cases} \eta_{1,t} = \dfrac{\delta H}{\delta z_1}\,, & z_{1,t} = -\dfrac{\delta H}{\delta \eta_1}\,, \\ \eta_t = \dfrac{\delta H}{\delta z}\,, & z_t = -\dfrac{\delta H}{\delta \eta}\,, \end{cases}$$

Details of proof

- $\nabla \cdot (\theta_1 \nabla \theta_2) = (\nabla \theta_1) \cdot (\nabla \theta_2)$, for harmonic functions θ_1, θ_2:

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- thus,

$$\delta \left(\int_0^L \int_{-h}^{\eta} |\nabla \tilde{\varphi}|^2 \, dydx \right) =$$

$$\int_0^L \int_{-h}^{\eta} \delta(|\nabla \tilde{\varphi}|^2)\, dydx + \int_0^L |\nabla \tilde{\varphi}|_s^2 \, \delta\eta \, dx\,.$$

$$\int_0^L \int_{-h}^{\eta} \delta(|\nabla\tilde{\varphi}|^2)\,dydx = 2\int_{\partial D} \delta\tilde{\varphi}\,\nabla\tilde{\varphi}\cdot\boldsymbol{n}$$

$$= 2\int_0^L [\tilde{\varphi}_y - \eta_x\tilde{\varphi}_x]_s(\delta\tilde{\varphi})_s\,dx$$

$$= 2\int_0^L [\tilde{\varphi}_y - \eta_x\tilde{\varphi}_x]_s[\delta\Phi - (\tilde{\varphi}_y)_s\delta\eta]\,dx$$

Similarly...

$$\delta\left(\int_0^L\int_\eta^{h_1+\eta_1}|\nabla\tilde{\varphi}_1|^2\,dydx\right)$$

$$=2\int_0^L[\tilde{\varphi}_{1,y}-\eta_{1,x}\tilde{\varphi}_{1,x}]_{s_1}\left[\delta\Phi_2-(\tilde{\varphi}_{1,y})_{s_1}\delta\eta_1\right]dx$$

$$-2\int_0^L[\tilde{\varphi}_{1,y}-\eta_x\tilde{\varphi}_{1,x}]_s[\delta\Phi_1-(\tilde{\varphi}_{1,y})_s\delta\eta]\,dx$$

$$+\int_0^L|\nabla\tilde{\varphi}_1|^2_{s_1}\delta\eta_1\,dx-\int_0^L|\nabla\tilde{\varphi}_1|^2_s\delta\eta\,dx\,.$$

Hamiltonian formulation with geophysical effects

Take into consideration the rotational speed of Earth, $\omega \approx 7.29 \times 10^{-5}$ rad/s. The governing equations (in the equatorial f-plane approximation, of which the mass conservation is

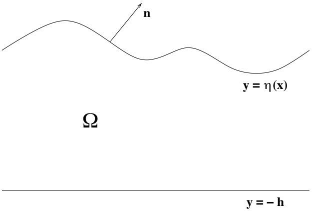
$$\begin{cases} \hat{u}_t + \hat{u}\hat{u}_x + \hat{v}\hat{u}_y + 2\omega\hat{v} & = & -\frac{1}{\hat{\rho}}P_x, \\ \hat{v}_t + \hat{u}\hat{v}_x + \hat{v}\hat{v}_y - 2\omega\hat{u} & = & -\frac{1}{\hat{\rho}}P_y - g, \end{cases} \quad \text{in} \quad \Omega^* \cup \Omega_1^*,$$

also admit the nearly-Hamiltonian formulation

$$\begin{cases} \eta_{1,t} = \dfrac{\delta H}{\delta \xi_1}, & \xi_{1,t} = -\dfrac{\delta H}{\delta \eta_1} + \rho_1(\gamma_1 + 2\omega)\chi_1, \\ \eta_t = \dfrac{\delta H}{\delta \xi}, & \xi_t = -\dfrac{\delta H}{\delta \eta} + (\rho(\gamma + 2\omega) - \rho_1(\gamma_1 + 2\omega))\chi, \end{cases}$$

The Hamiltonian by means of the Dirichlet-Neumann Operator

Write H in terms of η, η_1, ξ and ξ_1. Consider the domain $\Omega(\eta)$



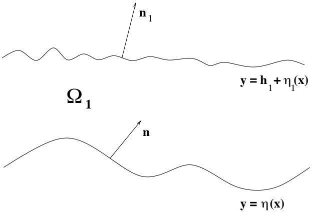
For $\Phi, \eta \in C^{\infty}_{per}[0, L]$ denote with $\tilde{\varphi}$ the (unique) L-periodic solution of the BVP

$$\begin{cases} \Delta\tilde{\varphi} = 0 & \text{in} \quad \Omega(\eta)\,, \\ \tilde{\varphi} = \Phi & \text{on} \quad y = \eta(x), \\ \tilde{\varphi}_y = 0 & \text{on} \quad y = -h\,, \end{cases}$$

set

$$G\Phi := \sqrt{1+\eta_x^2}\,\frac{\partial\tilde{\varphi}}{\partial n}\Big|_{y=\eta(x)}\,.$$

Similarly, consider the domain $\Omega_1(\eta, \eta_1)$



Given η, η_1, Φ_1, $\Phi_2 \in C^\infty_{per}[0, L]$ denote with $\tilde{\varphi}_1$ the (unique) L-periodic solution of the BVP

$$\begin{cases} \Delta\tilde{\varphi}_1 = 0 & \text{in} \quad \Omega_1(\eta, \eta_1)\,, \\ \tilde{\varphi}_1 = \Phi_1 & \text{on} \quad y = \eta(x)\,, \\ \tilde{\varphi}_1 = \Phi_2 & \text{on} \quad y = h_1 + \eta_1(x)\,, \end{cases}$$

set

$$G_1(\Phi_1, \Phi_2) := \begin{pmatrix} -\sqrt{1+\eta_x^2}\, \dfrac{\partial \tilde{\varphi}_1}{\partial n}\Big|_{y=\eta(x)} \\ \sqrt{1+\eta_{1,x}^2}\, \dfrac{\partial \tilde{\varphi}_1}{\partial n_1}\Big|_{y=h_1+\eta_1(x)} \end{pmatrix}.$$

Then

$$H = \frac{1}{2}\int_0^L \begin{pmatrix}\xi\\ \xi_1\end{pmatrix}^T \begin{pmatrix} G_{11}B^{-1}G & -GB^{-1}G_{12} \\ -G_{21}B^{-1}G & -\frac{\rho}{\rho_1}G_{21}B^{-1}G_{12}+\frac{1}{\rho_1}G_{22}\end{pmatrix}\begin{pmatrix}\xi\\ \xi_1\end{pmatrix} dx$$
$$+\int_0^L \mu B^{-1}(\rho_1 G\xi+\rho G_{12}\xi_1)\,dx - \frac{\rho\rho_1}{2}\int_0^L \mu B^{-1}\mu\,dx$$
$$-\int_0^L (\gamma\eta+\kappa)\xi\eta_x\,dx - \int_0^L [\gamma_1(h_1+\eta_1)+\kappa_1]\xi_1\eta_{1,x}\,dx$$
$$+\frac{\rho\gamma^2}{6}\int_0^L(\eta^3+h^3)\,dx+\frac{\rho_1\gamma_1^2}{6}\int_0^L\left((h_1+\eta_1)^3-\eta^3\right)dx$$
$$+\frac{\rho(\gamma\kappa+g)}{2}\int_0^L(\eta^2-h^2)\,dx+\frac{\rho_1(\gamma_1\kappa_1+g)}{2}\int_0^L[(\eta_1+h_1)^2-\eta^2]\,dx$$
$$+\frac{\rho\kappa^2 hL}{2}+\frac{\rho_1\kappa_1^2 h_1 L}{2},$$

with $B := B(\eta,\eta_1) = \rho_1 G + \rho G_{11}$.

Using the scale $\varepsilon = a/h_1$ (a–typical wave amplitude), the Hamiltonian expansion

$$H(\eta, \xi, \eta_1, \xi_1) = \sum_{j=2}^{\infty} \varepsilon^j H^{(j)}(\eta, \xi, \eta_1, \xi_1).$$

explicit expressions for $H^{(2)}$, and the Hamiltonian formulation, Constantin and Ivanov[2] derived the linearized equations

[2] A. Constantin and R. Ivanov, *A Hamiltonian approach to wave-current interactions in two-layer fluids*, Physics of fluids

- $$\begin{aligned}\xi_t = &-[g(\rho-\rho_1)+\rho\gamma\kappa-\rho_1\gamma_1\kappa_1]\eta\\ &-\frac{\rho\rho_1(\kappa-\kappa_1)^2 D}{\rho_1\tanh(hD)+\rho\coth(h_1D)}\eta-\Gamma\partial_x^{-1}\eta_t\\ &-\kappa\xi_x+\frac{\rho_1(\kappa-\kappa_1)}{\rho_1+\rho\coth(h_1D)\coth(hD)}\xi_x\\ &-\frac{(\kappa-\kappa_1)\rho}{\rho_1\tanh(hD)\sinh(h_1D)+\rho\cosh(h_1D)}\xi_{1,x}\end{aligned}$$

$$\begin{aligned}\eta_t = &\frac{D\tanh(hD)\coth(h_1D)}{\rho\coth(h_1D)+\rho_1\tanh(hD)}\xi\\ &+\frac{D\tanh(hD)\operatorname{csch}(h_1D)}{\rho\coth(h_1D)+\rho_1\tanh(hD)}\xi_1\\ &-\kappa\eta_x+\frac{\rho_1(\kappa-\kappa_1)}{\rho_1+\rho\coth(h_1D)\coth(hD)}\eta_x\end{aligned}$$

- $$\xi_{1,t} = -\gamma_1 h_1 \xi_{1,x} - \rho_1 (g + \gamma_1^2 h_1)\eta_1 - \Gamma_1 \partial_x^{-1} \eta_{1,t} - \kappa_1 \xi_{1,x} - \rho_1 \gamma_1 \kappa_1 \eta_1$$

$$\eta_{1,t} = -\gamma_1 h_1 \eta_{1,x} + \frac{D \tanh(hD) \operatorname{csch}(h_1 D)}{\rho \coth(h_1 D) + \rho_1 \tanh(hD)} \xi + \frac{D\left(\tanh(hD) \coth(h_1 D) + \frac{\rho}{\rho_1} \right)}{\rho \coth(h_1 D) + \rho_1 \tanh(hD)} \xi_1 - \kappa_1 \eta_{1,x} - \frac{(\kappa - \kappa_1)\rho}{\rho_1 \tanh(hD) \sinh(h_1 D) + \rho \cosh(h_1 D)} \eta_x$$

The upshots of these equations were

- a KdV type equation for the interface η
- formulas for η_1 and for the velocities u and u_1
- formulas for wave speeds (dispersion relations) corresponding to internal waves whose amplitudes are much bigger than the amplitudes of the surface waves

Conclusions and further perspectives

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- By performing a variable transformation it was then shown that the system actually has canonical Hamiltonian structure with canonical phase space variables $(\eta, \zeta, \eta_1, \zeta_1)$.
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  The linearization leads to dispersion relations, equations for the interface, free surface.
- Further to the obtained results, new scenarios (like a linear density distribution, the presence of a non flat bed) could be interesting for future studies.