# Rigorous derivation of a sixth order thin film equation

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Joint work with M. Bukal, University of Zagreb

# Thin film equation

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- $\beta > 0$ (typically $\beta = 1, 2, 3$).
- The sixth order thin film equation describes the free boundary problem with elastic interface:

$$\partial_t h + \operatorname{div}(h^3 \nabla \triangle^2 h) + \cdots = 0.$$

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- Our goal is to derive the sixth order thin film equation starting from the first principles.
- Therefore we have to understand relation between physical parameters - scaling.
- Outline of the talk:
  1. Formulation of $3D$ linear fluid-structure interaction problem.
  2. Apriori estimate and scaling assumptions.
  3. Convergence to the reduced model.
  4. Nonlinear (free boundary) case.

# Literature review

- Reynolds equation for thin (incompressible) viscous fluid - lubrication approximation (Reynolds 1886, Cimatti 1983, Bayada and Chambat 1986, Marušić-Paloka 2012, Hillairet and Kelai 2015, . . . )
- Fourth order thin film equation (Constantin et al. 1993; Oron, Davis and Bankoff 1997, Bertozzi 1998, Bernis and Friedman 1990, Myers 1998, Giacomelli and Otto 2003, Becker and Grün 2005, . . . )
- Derivation of reduced model in FSI (Čanić, Mikelić 2003, Panasenko and Stavre 2006, 2014, Čurković 2013)
- Nonlinear sixth order thin film equation - King '1986, Hosoi and Mahadevan 2004

# $3D$ linear fluid-structure interaction (FSI) problem

- Physical domain: $\Omega_{\varepsilon,h} = \Omega_\varepsilon \cup \Gamma_{01} \cup \Omega_h \subset \mathbb{R}^3$, where
  $\Omega_\varepsilon = (0,1)^2 \times (-\varepsilon, 0)$ - denotes the fluid domain,
  $\Gamma_{01} = (0,1)^2 \times \{0\}$ is the interface between the two phases,
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- Fin fluid velocity $\boldsymbol{v}$, presure $p$ and the structure displacement $\boldsymbol{u}$ such that

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\begin{aligned}
\varrho_f \partial_t \boldsymbol{v} - \operatorname{div} \sigma_f(\boldsymbol{v}, p) = \boldsymbol{f}\,, &\quad \Omega_\varepsilon \times (0,\infty)\,,\\
\operatorname{div} \boldsymbol{v} = 0\,, &\quad \Omega_\varepsilon \times (0,\infty)\,,\\
\varrho_s \partial_{tt} \boldsymbol{u} - \operatorname{div} \sigma_s(\boldsymbol{u}) = 0\,, &\quad \Omega_h \times (0,\infty)\,,\\
\partial_t \boldsymbol{u} = \boldsymbol{v}\,, &\quad \Gamma_{01} \times (0,\infty)\,,\\
(\sigma_f(\boldsymbol{v}, p) - \sigma_s(\boldsymbol{u}))\boldsymbol{e}_3 = 0\,, &\quad \Gamma_{01} \times (0,\infty)\,,
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(\sigma_f(\boldsymbol{v}, p) - \sigma_s(\boldsymbol{u}))\boldsymbol{e}_3 = 0\,, &\quad \Gamma_{01} \times (0,\infty)\,,
\end{aligned}
$$

- The fluid and structure stress tensors are given respectively by

$$
\sigma_f(\boldsymbol{v}, p) = 2\eta \operatorname{sym} \nabla \boldsymbol{v} - pI_3\,, \quad \sigma_s(\boldsymbol{u}) = 2\mu \operatorname{sym} \nabla \boldsymbol{u} + \lambda(\operatorname{div} \boldsymbol{u}) I_3\,,
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$$\mathcal{V}_F(0,T;\Omega_\varepsilon) = L^\infty(0,T;L^2(\Omega_\varepsilon;\mathbb{R}^3)) \cap L^2(0,T;V_F(\Omega_\varepsilon)),$$

$$V_F(\Omega_\varepsilon) = \left\{ \boldsymbol{v} \in H^1(\Omega_\varepsilon;\mathbb{R}^3) \ : \ \operatorname{div} \boldsymbol{v} = 0, \ \boldsymbol{v}|_{x_3=-\varepsilon} = 0, \ \boldsymbol{v} \text{ is } \omega\text{-periodic} \right\}$$

$$\mathcal{V}_S(0,T;\Omega_h) = W^{1,\infty}(0,T;L^2(\Omega_h;\mathbb{R}^3)) \cap L^2(0,T;H^1_{\rm per}(\Omega_h)),$$

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- Solution space:

$$\mathcal{V}(\Omega_{\varepsilon,h}) = \{(\boldsymbol{v},\boldsymbol{u}) \in \mathcal{V}_F(\Omega_\varepsilon) \times \mathcal{V}_S(\Omega_h) : \boldsymbol{v}|_{x_3=0} = \partial_t \boldsymbol{u}|_{x_3=0}\}.$$

- Test space:

$$\mathcal{W}(\Omega_{\varepsilon,h}) = \{(\phi,\psi) \in C^1_c([0,T); V_F(\Omega_\varepsilon) \times V_S(\Omega_h)) \ : \phi|_{x_3=0} = \psi|_{x_3=0}\}$$

# Weak formulation II

- Find $(\boldsymbol{v}^\varepsilon, \boldsymbol{u}^h) \in \mathcal{V}(\Omega_{\varepsilon,h})$ such that:

$$-\varrho_f \int_0^T \int_{\Omega_\varepsilon} \boldsymbol{v}^\varepsilon \cdot \partial_t \boldsymbol{\phi} + 2\eta \int_0^T \int_{\Omega_\varepsilon} \operatorname{sym} \nabla \boldsymbol{v}^\varepsilon : \operatorname{sym} \nabla \boldsymbol{\phi}$$
$$-\varrho_s \int_0^T \int_{\Omega_h} \partial_t \boldsymbol{u}^h \cdot \partial_t \boldsymbol{\psi}$$
$$+ \int_0^T \int_{\Omega_h} (2\mu \operatorname{sym} \nabla \boldsymbol{u}^h : \operatorname{sym} \nabla \boldsymbol{\psi} + \lambda \operatorname{div} \boldsymbol{u}^h \operatorname{div} \boldsymbol{\psi})$$
$$= \int_0^T \int_{\Omega_\varepsilon} \boldsymbol{f}^\varepsilon \cdot \boldsymbol{\phi} + \varrho_f \int_{\Omega_\varepsilon} \boldsymbol{v}_0^\varepsilon \cdot \boldsymbol{\phi}(0) + \varrho_s \int_{\Omega_h} \boldsymbol{u}_1^h \cdot \boldsymbol{\psi}(0).$$

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- Goal is to describe $(\boldsymbol{v}^\varepsilon, p^\varepsilon, \boldsymbol{u}^h)$ for small $\varepsilon$ and $h$. What is the relation between $\varepsilon$ and $h$?

# Basic energy estimate

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- Basic energy estimate:

$$\varrho_f \int_{\Omega_\varepsilon} |\boldsymbol{v}^{\varepsilon}(t)|^2 + \eta \int_0^t \int_{\Omega_\varepsilon} |\operatorname{sym} \nabla \boldsymbol{v}^{\varepsilon}(s)|^2 + \varrho_s \int_{\Omega_h} |\partial_t \boldsymbol{u}^h(t)|^2$$
$$+\mu \int_{\Omega_h} |\operatorname{sym} \nabla \boldsymbol{u}^h(t)|^2 + \frac{\lambda}{2} \int_{\Omega_h} |\operatorname{div} \boldsymbol{u}^h(t)|^2 \leq Ct\varepsilon \,,$$

- By applying Korn inequality we "lose" a power of $\varepsilon$.

# Improved estimate

- To avoid using Korn's inequality we use:

$$\int_{\Omega_\varepsilon} \nabla \boldsymbol{v}^\varepsilon : \nabla^T \boldsymbol{v}^\varepsilon \mathrm{d}\boldsymbol{x} = \int_{\Omega_\varepsilon} \partial_i v_j^\varepsilon \partial_j v_i^\varepsilon \mathrm{d}\boldsymbol{x}$$

$$= - \int_{\Omega_\varepsilon} v_j^\varepsilon \partial_j \underbrace{\partial_i v_i^\varepsilon}_{=0} \mathrm{d}\boldsymbol{x} + \int_{\partial\Omega_\varepsilon} v_j^\varepsilon \partial_j v_i^\varepsilon n_i \mathrm{d}S \,,$$

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- Combining above equality with higher order estimates we get:

$${}_f \int_{\Omega_\varepsilon} |\boldsymbol{v}^\varepsilon(t)|^2 + \frac{\eta}{2}\int_0^t \int_{\Omega_\varepsilon} |\nabla \boldsymbol{v}^\varepsilon|^2 + \varrho_s \int_{\Omega_h} |\partial_t \boldsymbol{u}^h(t)|^2$$
$$+ \int_{\Omega_h} \Big(\mu |\operatorname{sym}\nabla \boldsymbol{u}^h(t)|^2 + \frac{\lambda}{2} |\operatorname{div}\boldsymbol{u}^h(t)|^2\Big)\mathrm{d}\boldsymbol{x} \le Ct\varepsilon^3.$$

# Scaling assumptions

- Rescaling of the domain:

$$\boldsymbol{y} = (y', y_3) = (x', \frac{x_3}{\varepsilon}),\ \boldsymbol{y} \in \Omega_- := (0,1)^2 \times (-1,0),\ \boldsymbol{v}(\varepsilon)(\boldsymbol{y}) := \boldsymbol{v}^{\varepsilon}(\boldsymbol{x}),$$

$$\boldsymbol{z} = (y', y_3) = (x', \frac{x_3}{h}),\ \boldsymbol{z} \in \Omega_+ := (0,1)^2 \times (0,1),\ \boldsymbol{v}(h)(\boldsymbol{z}) := \boldsymbol{v}^{h}(\boldsymbol{x}),$$

$$\nabla \boldsymbol{v}^{\varepsilon}(\boldsymbol{x}) = \nabla_{\varepsilon} \boldsymbol{v}(\varepsilon)(\boldsymbol{y}), \qquad \nabla \boldsymbol{u}^{h}(\boldsymbol{x}) = \nabla_{h} \boldsymbol{u}(h)(\boldsymbol{z}), \quad \boldsymbol{x} \in \Omega_{\varepsilon,h},$$

$$\nabla_{\varepsilon} = (\partial_{y_1}, \partial_{y_2}, \frac{1}{\varepsilon}\partial_{y_3}),\ \nabla_{h} = (\partial_{z_1}, \partial_{z_2}, \frac{1}{h}\partial_{z_3})$$

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$$\nabla_{\varepsilon} = (\partial_{y_1}, \partial_{y_2}, \frac{1}{\varepsilon}\partial_{y_3}),\ \nabla_{h} = (\partial_{z_1}, \partial_{z_2}, \frac{1}{h}\partial_{z_3})$$

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- Fluid and structure domain thickness ratio: $\varepsilon = h^{\gamma}$.
- Rescaling of the structure parameters:

$$\varrho_s = \hat{\varrho}_s h^{-\kappa}, \quad \mu = \hat{\mu} h^{-\kappa}, \quad \lambda = \hat{\lambda} h^{-\kappa}$$

# Estimates on the reference domain

$$\varrho_f \varepsilon \int_{\Omega_-} |\boldsymbol{v}(\varepsilon)(t)|^2 + \frac{\eta h^\tau}{2} \varepsilon \int_0^t \int_{\Omega_-} |\nabla_\varepsilon \boldsymbol{v}(\varepsilon)|^2 + \varrho_s h^{-\kappa-2\tau+1} \int_{\Omega_+} |\partial_t \boldsymbol{u}(h)(t)|^2$$
$$+ h^{-\kappa+1} \int_{\Omega_+} \left( \mu |\operatorname{sym} \nabla_h \boldsymbol{u}(h)(t)|^2 + \frac{\lambda}{2} |\operatorname{div}_h \boldsymbol{u}(h)(t)|^2 \right) \leq C h^\tau \varepsilon^3.$$

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$$+ h^{-\kappa+1} \int_{\Omega_+} \Big( \mu |\operatorname{sym} \nabla_h \boldsymbol{u}(h)(t)|^2 + \frac{\lambda}{2} |\operatorname{div}_h \boldsymbol{u}(h)(t)|^2 \Big) \leq C h^\tau \varepsilon^3.$$

This implies the following asymptotic behavior:

$$\boldsymbol{v}(\varepsilon) \approx \varepsilon^2 \bar{\boldsymbol{v}} \quad p \approx \bar{p}, \quad \boldsymbol{u}(h) \approx h^{(3\gamma-1+\tau+\kappa)/2} \bar{\boldsymbol{u}}.$$

# Weak convergence

Rescaled weak formulation:

$$
\begin{aligned}
-\varrho_f h^{-\tau}\varepsilon^3 \int_0^T \int_{\Omega_-} \boldsymbol{v}(\varepsilon)\cdot\partial_t\boldsymbol{\phi} + 2\eta\varepsilon^3 \int_0^T \int_{\Omega_-} \operatorname{sym}\nabla_\varepsilon \boldsymbol{v}(\varepsilon) : \operatorname{sym}\nabla_\varepsilon\boldsymbol{\phi} \\
-\varepsilon \int_0^T \int_{\Omega_-} p(\varepsilon)\operatorname{div}_\varepsilon \boldsymbol{\phi} - \varrho_s h^{\delta-2\tau} \int_0^T \int_{\Omega_+} \boldsymbol{u}(h)\cdot\partial_{tt}\boldsymbol{\psi} \\
+h^\delta \int_0^T \int_{\Omega_+} (2\mu \operatorname{sym}\nabla_h \boldsymbol{u}(h) : \operatorname{sym}\nabla_h\boldsymbol{\psi} + \lambda \operatorname{div}_h \boldsymbol{u}(h)\operatorname{div}_h \boldsymbol{\psi}) \\
= \varepsilon \int_0^T \int_{\Omega_-} \boldsymbol{f}\cdot\boldsymbol{\phi},
\end{aligned}
$$

where $\delta = (3\gamma - 1 + \tau + \kappa)/2 + 1 - \kappa$.

# Identifying the limit

In order to get the plate equation we need $\delta = 1$. By taking the lubrication time, i.e. $\tau = -2\gamma$ we obtain relation:

$$\kappa = 3\gamma + 1.$$

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Therefore we get the plate equation for vertical displacement $w_3$:

$$-\int_0^T \int_\omega p\zeta_3 + \int_0^T \int_\omega \left( \frac{4\mu}{3} \nabla'^2 w_3 : \nabla'^2 \zeta_3 + \frac{4\mu\lambda}{3(2\mu + \lambda)} \Delta' w_3 \Delta' \zeta_3 \right) = 0.$$

# Identifying the limit

The limit equation for the fluid is the lubrication approximation:

$$\eta \int_0^T \int_{\Omega_-} \partial_3 v_\alpha \partial_3 \phi_\alpha - \int_0^T \int_{\Omega_-} p \partial_\alpha \phi_\alpha = \int_0^T \int_{\Omega_-} f_\alpha \phi_\alpha, \ \alpha = 1, 2.$$

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Finally, from the divergence-free condition and kinematic coupling condition we get:

$$\int_0^T \int_\omega \left( \partial_\alpha \int_{-1}^0 v_\alpha \, \mathrm{d}y_3 + \partial_t w_3 \right) \varphi \, \mathrm{d}y' \mathrm{d}t = 0 \, ,$$

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Combining these three equation together we obtain linear thin film equation:

$$\partial_t w_3 - \frac{8\mu(\mu+\lambda)}{3(2\mu+\lambda)} (\Delta')^3 w_3 + G_\alpha = 0, \ G_\alpha(y', t) = \partial_\alpha \int_{-1}^0 F_\alpha \, \mathrm{d}y_3.$$

# Problem formulation

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- Motivated by analysis of linear problem, we consider moving boundary problem with only vertical displacement and viscoelastic plate equation for the structure. We consider $2D$ case for simplicity.
- Let $\eta$ be the vertical displacement of the top boundary. We define the fluid domain with:

$$\Omega_\eta(t) = \{(x_1, x_2) : x_1 \in (0, L),\ x_2 \in (0, \eta(t, x_1))\},$$

$$\Omega_\eta(t) \times (0, T) := \bigcup_{t\in(0,T)} \Omega_\eta(t) \times \{t\}.$$

$$\begin{aligned}
\varrho_f \partial_t \boldsymbol{v} - \operatorname{div} \sigma_f(\boldsymbol{v}, p) = \boldsymbol{f}\,, &\quad \Omega_\eta(t) \times (0, T),\\
\operatorname{div} \boldsymbol{v} = 0\,, &\quad \Omega_\eta(t) \times (0, T)\\
\varrho_s \partial_{tt}\eta - C_v \partial_{x_1}^2 \partial_t \eta + C_e \partial_{x_1}^4 \eta = -\sigma_f(\boldsymbol{v}, p,)(t, x_1, \eta(t, x_1))\mathbf{n} \cdot \mathbf{e}_2, &\ (0, L) \times (0, T)\\
\boldsymbol{v}(t, x_1, \eta(t, x_1)) = (0, \partial_t \eta(t, x_1))\,, &\quad (0, L) \times (0, T).
\end{aligned}$$

# Scaling

- Existence of (global) weak solution (Chambolle, Desjardines, Esteban and Grandmont 2005, Grandmont 2008, BM, Čanić 2013). Existence of global strong solution $2D$ - Grandmont, Hillairet 2016.

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- Scaling assumptions:

$$\eta_0 = \varepsilon \hat{\eta}_0 \,,\, T \sim \frac{1}{\varepsilon^2} \,,\, C_v \sim \frac{1}{\varepsilon^2} \,,\, C_e \sim \frac{1}{\varepsilon}$$

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$$\eta_0 = \varepsilon\hat{\eta}_0\,,\, T \sim \frac{1}{\varepsilon^2}\,,\, C_v \sim \frac{1}{\varepsilon^2}\,,\, C_e \sim \frac{1}{\varepsilon}$$

- Energy estimate:

$$\frac{1}{2}\frac{d}{dt}\Big(\varrho_f\|\boldsymbol{v}\|^2_{L^2(\Omega_\eta(t))} + \varrho_s\|\partial_t\eta\|^2_{L^2} + \frac{\hat{C}_e}{\varepsilon}\|\partial_x^2\eta\|^2_{L^2}\Big)$$
$$+\mu\|\operatorname{sym}\nabla\boldsymbol{v}(t)\|^2_{L^2(\Omega_\eta(t))} + \frac{\hat{C}_v}{\varepsilon^2}\|\partial_t\partial_x\eta(t)\|^2_{L^2} \leq C\varepsilon^3.$$

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- Therefore $\boldsymbol{v}^\varepsilon \approx \varepsilon^2 U$.
- In the limit we obtain: $P = \partial^4 h$ - plate equation,
  $\frac{1}{h^2}\partial_2^2 U_1 = \partial_1 P$ - lubrication equation for the fluid velocity,
  $\frac{1}{h}\partial_2 U_2 = -\partial_1 U_1 + y_2 \frac{\partial_1 h}{h}\partial_2 U_1$ - divergence free condition.

# Identifying the limit

- From energy estimate we get $\eta^\varepsilon \approx \varepsilon h$.
- Therefore $\boldsymbol{v}^\varepsilon \approx \varepsilon^2 U$.
- In the limit we obtain: $P = \partial^4 h$ - plate equation,
  $\frac{1}{h^2}\partial_2^2 U_1 = \partial_1 P$ - lubrication equation for the fluid velocity,
  $\frac{1}{h}\partial_2 U_2 = -\partial_1 U_1 + y_2 \frac{\partial_1 h}{h}\partial_2 U_1$ - divergence free condition.
- By putting it all together we obtain the nonlinear sixth order thin film equation:

$$\partial_t h = \frac{h}{12}\partial_1(h^2\partial_1^5 h) + \frac{\partial_1 h}{6} h^2 \partial_1^5 h = \frac{1}{12}\Big(h^3\partial_1^6 h + 2\partial_1 h h^2 \partial_1^5 h + \partial_1 h h^2 \partial_1^5 h\Big)$$
$$= \frac{1}{12}\big(h^3\partial_1^6 h + 3\partial_1 h h^2 \partial_1^5 h\big) = \frac{1}{12}\partial_1\big(h^3\partial_1^5 h\big)$$

Thank you for your attention!