# Continuity and two dimensional Euler equation with $L^1$ vorticity

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# Motivation

Consider the $2D$-Euler equation

$$\begin{cases} \partial_t \omega + u \cdot \nabla \omega & = & 0\,, \\ u & = & \mathcal{K} * \omega\,, \end{cases} \qquad (1)$$

with the Biot-Savart kernel $\mathcal{K}$

$$\mathcal{K}(x) = \frac{x^{\perp}}{2\pi |x|^2}\,, \qquad \text{where } (x_1, x_2)^{\perp} = (-x_2, x_1)\,.$$

Remark: $\nabla \cdot u = 0$.

# Question in Lopes-Mazzucato-Lopes 2006

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Are solutions $\omega \in L^\infty([0,T]; L^p(\mathbb{R}^2))$ with $1 \leq p < 2$ obtained as a vanishing viscosity limit renormalized?

# Distributional solution

**Definition (Weak solutions)**

Let $\omega \in L^\infty([0;T); L^p(\mathbb{R}^2))$ for some $p \geq \frac{4}{3}$ and $u = K * \omega$. We say $\omega$ is a *weak-solution* of (1) if $\forall \phi \in C_c^\infty([0,T) \times \mathbb{R}^2)$, we have

$$\int_0^T \int_{\mathbb{R}^2} \phi_t \omega + \nabla\phi \cdot u\omega \, dx \, dt + \int_{\mathbb{R}^2} \phi(x,0)\omega_0(x) \, dx = 0 \, .$$

In addition we require *mild growth conditions* for $u$.

**Remark** 1: When $\omega_0 \in L^p(\mathbb{R}^2)$ for $p \geq \frac{4}{3}$ distributional solution are well defined: $u\omega \in L^1_{loc}$ .
**Remark** 2: Uniqueness of weak solutions only for (nearly) *bounded* vorticity (Yudovich, Vishik)

# Renormalized solutions

**Definition (Renormalized solution for the linear transport equation)**

A function $\omega \in L^\infty([0,T];L^0)$ is a *renormalized solution* of $\omega_t + u \cdot \nabla\omega = 0\,$, if (in the sense of distribution)

$$\partial_t \beta(\omega) + u \cdot \nabla \beta(\omega) = 0\,,$$

for all $\beta \in C^1 \cap L^\infty$.

$L^0$ denotes the set of all measurable functions such that $|\{|\omega|> \lambda\}|$ if finite for all $\lambda > 0$

- Renormalized solution make sense even when one cannot define distributionally the product $u\omega$.
- DiPerna-Lions theory assures existence, uniqueness of renormalized solution in Sobolev class.
- Distributional and renormalized solutions coincide when $u \in L^1(W^{1,p})$ with $\nabla \cdot u \in L^1(L^\infty)$ and $\omega \in L^\infty L^q$ with $\frac{1}{p} + \frac{1}{q} = 1$.

# Viscosity solutions

Definition (viscosity solutions)

We call $\omega$ a viscosity solution to the Euler equations if

$$\omega = \lim_{\nu \downarrow 0} \omega^{\nu} \,,$$

where $\omega^{\nu}$ is the curl of some divergence-free velocity field and solve the Navier-Stokes vorticity equation with viscosity $\nu$

$$\partial_t \omega^{\nu} + (u \cdot \nabla) \omega^{\nu} \quad = \quad \nu \Delta \omega^{\nu}$$

**Remark 1**: Physically meaningful approximation
**Remark 2**: Anomalous dissipation of enstrophy $\rightarrow 2D$ Turbulence

# Connection between weak, renormalized and viscosity solutions

Let $\omega_0 \in L^p_c(\mathbb{R}^2)$.

When $p \geq 2$ every distributional solution is renormalized solutions of the transport equation with velocity $u = K * \omega$. [Lopes, Lopes&Mazuccato, '06]

When $1 < p < 2$ viscosity solutions to the Euler vorticity are renormalized solutions [Crippa&Spirito, '15]

The arguments rely on Calderón-Zygmund estimate for the Biot-Savart kernel:

$$\|\nabla u\|_{L^p} \leq \|\omega\|_{L^p} \qquad \text{when } p \in (1, \infty)$$

# What happens when $\omega_0 \in L^1$?

For $p = 1$ the Calderón-Zygmund estimate fails $\rightsquigarrow$ $u$ does not have Sobolev regularity $\rightsquigarrow$ DiPerna-Lions theory does not apply.

**Theorem 1**

For initial vorticities in $L^1$, viscosity solutions to the Euler vorticity are renormalized solutions.

Crucial tool to prove Theorem 1: uniqueness for the linear continuity equation with velocity field $\nabla u = K * g$ with $K$ singular kernel and $g \in L^1$.

# Uniqueness for continuity equation with singular velocity

Let $\rho$ be the solution

$$\begin{cases} \partial_t \rho + \nabla \cdot (u\rho) &= 0 \qquad \text{in } [0,T] \times \mathbb{R}^n\,, \\ \rho(0,\cdot) &= \rho_0 \qquad \text{in } \mathbb{R}^n\,, \end{cases} \tag{2}$$

with

- $u \in L^{p,\infty}([0,T],\Omega)$
- $\nabla \cdot u \in L^1([0,T];L^\infty(\mathbb{R}^n))$
- $\nabla u = K * g$ where $g = g(x,t) \in L^1$

  $K$ smooth in $\mathbb{R}^n \setminus \{0\}$, $|K| \lesssim \frac{1}{|x|^n}$, $|\nabla K| \lesssim \frac{1}{|x|^{n+1}}$
- $\rho_0 \in L^1([0,T];L^\infty(\mathbb{R}^n))$

Under these hypotheses on $u$ and $\rho_0$.

**Theorem 2: Uniqueness**

Then the Cauchy problem (2) has a unique distributional solution $\rho$ in the class $L^\infty((0,T); L^\infty \cap L^1(\mathbb{R}^n))$.

Under these hypotheses on $u$ and $\rho_0$.

**Theorem 2: Uniqueness**

Then the Cauchy problem (2) has a unique distributional solution $\rho$ in the class $L^\infty((0,T); L^\infty \cap L^1(\mathbb{R}^n))$.

In this setting the existence and uniqueness of Lagrangian solutions was established by Bouchut&Crippa in 2013.

**Corollary 1**

Distributional and renormalized solutions are equivalent.

Under these hypotheses on $u$ and $\rho_0$.

**Theorem 2: Uniqueness**

Then the Cauchy problem (2) has a unique distributional solution $\rho$ in the class $L^\infty((0,T); L^\infty \cap L^1(\mathbb{R}^n))$.

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**Corollary 1**

Distributional and renormalized solutions are equivalent.

**Corollary 2**

If $\nabla \cdot u = 0$ and $\rho_0$ be in $L^0$ . Then there exists a unique renormalized solution to the Cauchy problem (2).

Moreover, the notions of Lagrangian and renormalized solutions are equivalent.

**Method**: Combine

- Estimate for $\eta = \rho_1 - \rho_2$ : $\fint \eta = 0$ in the *Kantorovich-Rubinstein distance*

  (A' la [Brenier, Otto & Seis et.al. '11/'13]):

$$\mathcal{D}_c(\eta) := \mathcal{D}_c(\eta, 0) := \mathcal{D}_c(\eta_+, \eta_-) := \inf_{\pi \in \Pi} \iint \underbrace{c(|x-y|)}_{\text{cost function}} \, d\pi_{\text{opt}}(x,y)$$

  with $c(|x-y|) = \log\left(\frac{|x-y|}{\delta} + 1\right)$.

- Harmonic analysis tool: estimate for difference quotient of the type

  (A' la [Crippa & De Lellis '08], [Crippa & Bouchut '13]):

$$\frac{|u(x)-u(y)|}{|x-y|} \lesssim M\nabla u(x) + M\nabla u(y)$$

  where $M$ is the maximal function $Mf(x) = \sup_{r>0} \frac{1}{|B(x,r)|} \int_{B(x,r)} |f(y)| \, dy$.

# Stability estimate

The main ingredient for uniqueness is the following stability estimate:

Let $\rho_1$ and $\rho_2$ be solutions of the continuity equation with velocity field $u_1$ and $u_2$ and initial data $\rho_{0,1}$ and $\rho_{0,2}$ respectively and set $\eta = \rho_1 - \rho_2$. Then there exists for every $\varepsilon > 0$ a finite constant $C_\varepsilon$ such that for every $\delta > 0$ it holds

$$\sup_{0\le t\le T} \mathcal{D}_\delta(\eta_t) \lesssim \mathcal{D}_\delta(\eta_0) + \varepsilon\|\rho\|_{L^1}\left[1+\log\left(\frac{1}{\delta\varepsilon}\left(\frac{\|\rho\|_{L^1}}{\|\rho\|_{L^\infty}}\right)^{1-\frac{1}{p}}||u||_{L^{p,\infty}}\right)\right] + \|\rho\|_{L^\infty(L^2)}C_\varepsilon$$

where

$$\mathcal{D}_\delta(\eta) := \mathcal{D}_\delta(\eta_+,\eta_-) := \inf_{\pi\in\Pi(\eta_+,\eta_-)} \iint \log\left(\frac{|x-y|}{\delta}+1\right) d\pi_{\mathsf{opt}}(x,y)$$

# Stability ⇒ Uniqueness

**Uniqueness proof:** Let $\rho_1$ and $\rho_2$ be two solutions of

$$\begin{cases} \partial_t \rho + \nabla \cdot (u\rho) = 0 & \text{in } [0,T] \times \mathbb{R}^n\,, \\ \rho(0,\cdot) = \rho_0 & \text{in } \mathbb{R}^n\,, \end{cases}$$

By the stability estimate applied to $\eta := \rho_1 - \rho_2$ we have

$$\sup_{0 \le t \le T} \mathcal{D}_\delta(\eta_t) \lesssim \varepsilon \left[1 + \log\left(\frac{1}{\delta\varepsilon}\right)\right] + C_\varepsilon$$

Then

$$\frac{\mathcal{D}_\delta(\eta)}{|\log \delta|} \xrightarrow{\delta \to 0} 0\,.$$

Therefore

$$\eta = 0 \quad \forall t \qquad \Rightarrow \qquad \boxed{\rho_1 = \rho_2}$$

# Argument for the stability estimate

Using the absolute continuity of the map $t \to \mathcal{D}_\delta(\rho)$ and the marginal conditions we can write

$$\frac{d}{dt}\mathcal{D}_\delta(\eta_t) \leq \iint (u(t,x)\cdot\nabla\phi_{\mathsf{opt}}(x) - u(t,y)\cdot\nabla\phi_{\mathsf{opt}}) d\pi_{\mathsf{opt}}(x,y)$$

where

$$\nabla\phi_{\mathsf{opt}}(x) = \nabla\phi_{\mathsf{opt}}(y) = c'(|x-y|)\frac{x-y}{|x-y|} \quad \text{for } \pi_{\mathsf{opt}} - \text{ almost all } (x,y).$$

Then

$$\frac{d}{dt}\mathcal{D}_\delta(\eta_t) \leq \iint \frac{|u(t,x)-u(t,y)|}{\delta + |x-y|} d\pi_{\mathsf{opt}}\,.$$

Observe that

$$\frac{|u(t,x)-u(t,y)|}{|x-y|} \leq M_\sigma \nabla u(t,x) + M_\sigma \nabla u(t,y)$$

where

$$M_\sigma \nabla u(t,x) = M_\sigma (K*g)(t,x) = \sup_{\varepsilon>0} \left| \int_{\mathbb{R}^d} \sigma\left(\frac{x-y}{\varepsilon}\right)(K*g)(y)\,dy \right|$$

where $\sigma$ smooth convolution kernel.

Then we can estimate the integrand as

$$\frac{|u(t,x)-u(t,y)|}{\delta+|x-y|} \lesssim \min\left\{ \frac{|u(t,x)|+|u(t,y)|}{\delta}, M_\sigma(K*g)(t,x) + M_\sigma(K*g)(t,y) \right\} .$$

**Apply** interpolation inequality

$$||f||_{L^1} \leq ||f||_{L^{1,\infty}} \left[ 1 + \log \left( \frac{||f||_{L^{p,\infty}}}{||f||_{L^{1,\infty}}} \right) \right]$$

to

$$f = \min \left\{ \frac{|u(t,x)| + |u(t,y)|}{\delta}; M_\sigma(K * g)(t,x) + M_\sigma(K * g)(t,y) \right\}$$

**On the one hand**

$$||f||_{L^{p,\infty}} \lesssim \frac{1}{\delta} ||u||_{L^{p,\infty}} \lesssim \frac{1}{\delta}$$

**On the other hand**

$$||f||_{L^{1,\infty}} \lesssim ||M_\sigma(K * g)||_{L^{1,\infty}} \overset{!}{\lesssim} ||g||_{L^1} \qquad \text{(cancellations)}$$

**Obtain:**

$$\int_0^T \iint \min\left\{\frac{|u(t,x)|+|u(t,y)|}{\delta}; M_\sigma(K*g)(t,x)+M_\sigma(K*g)(t,y)\right\} d\pi_{\mathrm{opt}}\, dt$$

$$\lesssim \|g\|_{L^{1,\infty}}\left[1+\log\left(\frac{\|u\|_{L^{p,\infty}}}{\delta\|g\|_{L^{1,\infty}}}\right)\right]$$

**Not enough! Decompose**

$$g \in L^1 \qquad \Rightarrow \qquad g = g_1 + g_2$$

where

$$||g_1||_{L^1} \leq \varepsilon, \qquad \operatorname{supp} g_2 \subset A_\varepsilon, \qquad ||g_2||_{L^2} \leq C_\varepsilon$$

**and finally deduce:**

$$\sup_{0\leq t\leq T} \mathcal{D}_\delta(\eta_t) \lesssim \underbrace{\varepsilon\left[1+\log\left(\frac{\|u\|_{L^{p,\infty}}}{\delta\varepsilon}\right)\right]}_{\text{from } g_1} + \underbrace{C_\varepsilon}_{\text{from } g_2} + \mathcal{D}_\delta(\eta_0)\,.$$

Thank you for your attention!!!