Stationary Navier-Stokes problem in $2D$ exterior domains

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$$\begin{cases} -\nu\Delta\mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} + \nabla p = 0 & \text{in } \Omega, \\ div\mathbf{u} = 0 & \text{in } \Omega, \\ \mathbf{u} = \mathbf{h} & \text{on } \partial\Omega, \end{cases} \qquad (NS)$$

$\mathbf{u}$ – velocity of the fluid, p -pressure, $\mathbf{h}$ is an assigned vector field on $\partial\Omega$,

$$\Omega = \mathbb{R}^2 \setminus \bigcup_{i=1}^{N} \overline{\Omega}_i,$$

where Ω_i are N pairwise disjoint bounded Lipschitz domains.
Condition at infinity:

$$\lim_{|x|\to\infty} \mathbf{u}(x) = \mathbf{u}_\infty, \quad \mathbf{u}_\infty = \text{const.} \qquad (CI)$$

Starting from the pioneering papers by J. Leray

J. LERAY: Étude de diverses équations intégrales non linéaire et de quelques problèmes que pose l'hydrodynamique, *J. Math. Pures Appl.* **12** (1933), 1–82.

it is now customary to consider solutions to (NS) with the finite Dirichlet integral

$$\int_\Omega |\nabla \mathbf{u}|^2 < +\infty, \tag{D}$$

known also as *D–solutions*. A priory D-solutions are not obliged to satisfy (CI). Functions, satisfying (D), may even grow at infinity (as log).

Stokes paradox

The linearized equations

$$\begin{cases} -\nu\Delta\mathbf{u} + \nabla p = 0 & \text{in } \Omega, \\ \operatorname{div}\mathbf{u} = 0 & \text{in } \Omega, \\ \mathbf{u} = \mathbf{h} & \text{on } \partial\Omega, \\ \lim_{x\to\infty} \mathbf{u}(x) = \mathbf{u}_\infty, \end{cases} \tag{S}$$

have a solution if and only if

$$\int_{\partial\Omega} (\mathbf{h} - \mathbf{u}_\infty) \cdot \psi \, ds = 0,$$

for all densities ψ of the simple layer potentials constant on $\partial\Omega$. In particular, since $\int_{\partial\Omega} \psi \neq 0$, if $\mathbf{h}$ vanishes and $\mathbf{u}_\infty \neq 0$, then (S) is not solvable. For the Stokes equations, is equivalent to say that a solution to (S) constant on the boundary and vanishing at infinity does not exist.

The situation is different for the nonlinear problem (NS).
The first existence theorem for (NS) is due to D.R. Smith and R. Finn (1967)

R. FINN, D.R. SMITH: On the stationary solutions of the Navier–Stokes equations in two dimensions, *Arch. Ration. Mech. Anal.* **25** (1967), 26–39,

where it is proved that if $\mathbf{u}_\infty \neq 0$ and $|\mathbf{h} - \mathbf{u}_\infty|$ is sufficiently small, then there is a D–solution which converges uniformly to $\mathbf{u}_\infty$. This result is particularly meaningful since it rules out (at least for small data) for the non–linear system the *Stokes paradox*.

However, for large data and for $\mathbf{u}_\infty = 0$ the question of the existence of D–solution satisfying (CI) is open.

The problem of the asymptotic behavior at infinity of an arbitrary D–solution $\mathbf{u}$ to (NS) was tackled by D. Gilbarg & H. Weinberger

D. GILBARG, H.F. WEINBERGER: Asymptotic properties of Leray's solution of the stationary two–dimensional Navier–Stokes equations, *Russian Math. Surveys*, **29** (1974), 109–123;

D. GILBARG, H.F. WEINBERGER: Asymptotic properties of steady plane solutions of the Navier–Stokes equations with bounded Dirichlet integral, *Ann. Scuola Norm. Pisa (4)*, **5** (1978), 381–404

and Ch. Amick

C.J. AMICK: On Leray's problem of steady Navier-Stokes flow past a body in the plane, *Acta Math.*, **161** (1988), 71–130.

In G & W it is shown that

$$p(z) - p_0 = o(1) \qquad \text{as } r \to \infty,$$

i.e., pressure has a limit at infinity (one can choose, say, $p_0 = 0$) and

$$\begin{aligned} \mathbf{u}(z) &= o(\sqrt{\log r}), \\ \omega(z) &= o(r^{-3/4} \log^{1/8} r), \\ \nabla \mathbf{u}(z) &= o(r^{-3/4} \log^{9/8} r), \end{aligned}$$

where $r = |z|$ and

$$\omega = \partial_2 u_1 - \partial_1 u_2$$

is the vorticity.

If, in addition, $\mathbf{u}$ **is bounded**, then there is a constant vector $\mathbf{u}_0$ such that

$$\lim_{r\to+\infty}\int_0^{2\pi}|\mathbf{u}(r,\theta)-\mathbf{u}_0|^2 d\theta = 0, \qquad (*)$$

and

$$\omega(z) = o(r^{-3/4}),$$
$$\nabla\mathbf{u}(z) = o(r^{-3/4}\log r).$$

Moreover, if $\mathbf{u}_0 = 0$, then

$$\mathbf{u}(z)\to 0 \qquad \text{uniformly as } |z|\to\infty. \qquad (1)$$

In the case $\mathbf{u}_0 \neq 0$ there exists a sequence of radii $R_n \in (2^n, 2^{n+1})$, $n \geq n_0$, such that

$$\sup_{\theta\in[0,2\pi]}|\mathbf{u}(R_n,\theta)-\mathbf{u}_0|\to 0 \qquad \text{as } n\to\infty. \qquad (2)$$

Amick proved that if $\mathbf{u}$ **vanishes on the boundary**, then $\mathbf{u}$ **is bounded** and, hence, satisfies the G & W estimates. However, in this last case the solution could tend to zero at infinity ($\mathbf{u}_0$ could be zero) and even be the trivial one.

In the talk will be shown that any D-solution to (NS) **converges uniformly** to $\mathbf{u}_0$ as $|x| \to \infty$ (we do not know the relation between $\mathbf{u}_0$ and $\mathbf{u}_\infty$).

The first step is to prove that any D-solution **is bounded**.

THEOREM 1. *Let* $\mathbf{u}$ *be a* D*-solution* ($\int_\Omega |\nabla \mathbf{u}|^2 dx < \infty$) *to (NS) problem in the exterior domain* $\Omega \subset \mathbb{R}^2$*. Then* $\mathbf{u}$ *is uniformly bounded in* $\Omega_0 = \mathbb{R}^2 \setminus B_{R_0}$*, i.e.,*

$$\sup_{x \in \Omega_0} |\mathbf{u}(x)| < \infty,$$

where B_{R_0} *is a ball with a sufficiently large radius:* $\frac{1}{2} B_{R_0} \Supset \partial\Omega$*.*

Proof. By regularity results $\mathbf{u}$ is uniformly bounded on each bounded subset of $\Omega_0 = \mathbb{R}^2 \setminus B_{R_0}$ and is real analytical in Ω_0. Moreover, by G & W pressure is uniformly bounded in Ω_0:

$$\sup_{x \in \Omega_0} |p(x)| \leq C < +\infty.$$

Suppose that the Theorem is false. Then there exists a sequence of points $x_k \in \Omega_0$ such that

$$|x_k| \to +\infty \qquad \text{and} \qquad |\mathbf{u}(x_k)| \to +\infty. \tag{1}$$

Then

$$\Phi(x_k) \to +\infty,$$

where $\Phi = p + \frac{1}{2}|\mathbf{u}|^2$ is the total head pressure. Since $\mathbf{u}$ is a D-solution, there exists an increasing sequence on numbers $R_m < R_{m+1}$ such that $R_m \to \infty$ and

$$\int_{C_{R_m}} |\nabla \mathbf{u}| \, ds \to 0,$$

where $C_R := \{x \in \mathbb{R}^2 : |x| = R\}$. It implies that

$$\sup_{x \in C_{R_m}} |\mathbf{u}(x) - \bar{\mathbf{u}}_m| \to 0, \tag{2}$$

$\bar{\mathbf{u}}_m$ is the mean value of $\mathbf{u}$ on the circle C_{R_m}. Indeed, for u_j, by mean value theorem, there exists a point $\theta_j^* \in [0, 2\pi)$ such that

$$u_j(R_m, \theta_j^*) = (2\pi)^{-1} \int_0^{2\pi} u_j(R_m, \theta) d\theta = \bar{u}_{jm}, \quad j = 1, 2,$$

and

$$|u_j(R_m, \theta) - \bar{u}_{jm}| = |u_j(R_m, \theta) - u_j(R_m, \theta_j^*)| \leq \int_{\theta_j^*}^{\theta} \left|\frac{\partial u_j}{\partial \theta}\right| d\theta \leq \int_{C_{R_m}} |\nabla \mathbf{u}| \, ds \to 0.$$

Since Φ satisfies the elliptic equation

$$\Delta\Phi - \frac{1}{\nu}\mathrm{div}(\Phi\mathbf{u}) = \omega^2$$

where $\omega = \partial_2 u_1 - \partial_1 u_2$ is the vorticity, Φ satisfies the maximum principle, in particular, for any subdomain $\Omega_{m_1,m_2} = \{x : R_{m_1} < |x| < R_{m_2}\}$, with $\partial\Omega_{m_1,m_2} = C_{R_{m_1}} \cup C_{R_{m_2}}$ we have

$$\sup_{x\in\Omega_{m_1,m_2}} \Phi(x) = \sup_{x\in C_{R_{m_1}}\cup C_{R_{m_2}}} \Phi(x).$$

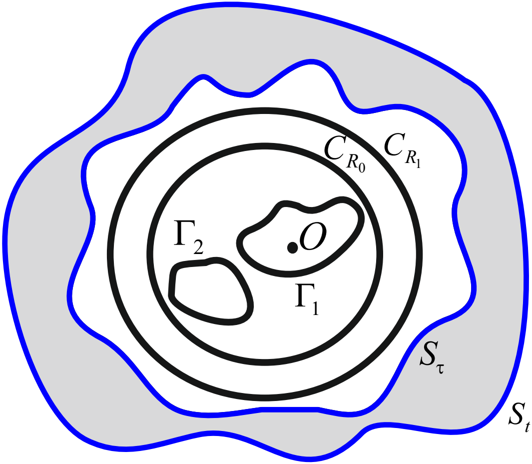
Relations (1), (2) imply that $|\bar{\mathbf{u}}_m| \to +\infty$; consequently,

$$\inf_{x\in C_{R_m}} \Phi(x) \to +\infty.$$

Then we could assume without loss of generality (choosing a subsequence) that

$$\sup_{x\in C_{R_m}} \Phi(x) < \inf_{x\in C_{R_{m+1}}} \Phi(x). \tag{3}$$

Recall that by Morse–Sard Theorem, applied to the analytical function Φ, for almost all values $t \in \Phi(\Omega_0)$ the level set $\{\Phi = t\}$ contains no critical points, i.e., $\nabla\Phi(x) \neq 0$ if $x \in \Omega_0$ and $\Phi(x) = t$. We call such values **regular**.
Take arbitrary regular value $t > t_* = \sup\limits_{x \in C_{R_1} \cup C_{R_0}} \Phi(x)$. Then by the implicit function theorem the level set $\{x \in \Omega_0 : \Phi(x) = t\}$ consists of a family of disjoint smooth curves which are separated (by construction) both from infinity and from the boundary $\partial\Omega_0 = C_{R_0}$. This implies that every connected component of this level set $\{\Phi = t\}$ is homeomorphic to a circle. We call these components **quasicircles**. For every regular $t > t_*$, there exists a quasicircle S separating C_{R_1} from infinity, i.e., C_{R_1} is contained in the bounded connected component of the open set $\mathbb{R}^2 \setminus S$. Because of the maximum principle, such quasicircle is unique. We denote it by S_t.



For $t_* < \tau_0 < t$, let $\Omega_{\tau_0,t}$ be a domain with $\partial\Omega_{\tau_0,t} = S_{\tau_0} \cup S_t$.

Integrating the identity

$$\Delta\Phi = \omega^2 + \frac{1}{\nu}\mathrm{div}(\Phi\mathbf{u})$$

over $\Omega_{\tau_0,t}$, we obtain

$$\int\limits_{S_t} |\nabla\Phi|\, ds - \int\limits_{S_{\tau_0}} |\nabla\Phi|\, ds = \int\limits_{\Omega_{\tau_0,t}} \omega^2 dx + \tfrac{1}{\nu}\int\limits_{S_t} \Phi\mathbf{u}\cdot\mathbf{n}\, ds - \tfrac{1}{\nu}\int\limits_{S_{\tau_0}} \Phi\mathbf{u}\cdot\mathbf{n}\, ds$$

$$= \int\limits_{\Omega_{\tau_0,t}} \omega^2 dx + \tfrac{1}{\nu}(t-\tau_0)F,$$

where $F = \int\limits_{C_{R_0}} \mathbf{u}\cdot\mathbf{n}$ is the total flux. Notice that by construction the unit normal $\mathbf{n}$ to the level set $S_t = \{x : \Phi(x) = t\}$ is equal to $\frac{\nabla\Phi}{|\nabla\Phi|}$, so that $\nabla\Phi\cdot\mathbf{n} = |\nabla\Phi|$ on S_t; analogously, $\nabla\Phi\cdot\mathbf{n} = -|\nabla\Phi|$ on S_τ. The further proof splits into two cases.

CASE I. The total flux in not zero: $F \neq 0$.
(a). Suppose $F > 0$. Then from

$$\int\limits_{S_t} |\nabla\Phi|\, ds = \int\limits_{S_{\tau_0}} |\nabla\Phi|\, ds + \int\limits_{\Omega_{\tau_0,t}} \omega^2 dx + \frac{1}{\nu}(t-\tau_0)F,$$

fixing τ_0, we obtain for sufficiently large t,

$$C_1 t \leq \int\limits_{S_t} |\nabla\Phi|\, ds \leq C_2 t \tag{4}$$

with C_1 and C_2 independent of t.
Denote by $\mathcal{R}$ the set of all regular values $t > t_*$, and put

$$E_t := \bigcup_{\tau\in[t,2t]\cap\mathcal{R}} S_\tau .$$

Applying the classical Coarea formula

$$\int_{E_t} f\,|\nabla\Phi|\,dx = \int_t^{2t}\left(\int_{S_\tau} f\,ds\right)d\tau$$

for $f = |\omega|$ and for $f = |\nabla\Phi|$, we obtain

$$\int_t^{2t}\left(\int_{S_\tau}|\omega|\,ds\right)d\tau = \int_{E_t}|\omega|\cdot|\nabla\Phi|dx \le \left(\int_{E_t}|\nabla\Phi|^2dx\right)^{\frac{1}{2}}\left(\int_{E_t}\omega^2dx\right)^{\frac{1}{2}}$$

$$= \left(\int_t^{2t}\left(\int_{S_\tau}|\nabla\Phi|\,ds\right)d\tau\right)^{\frac{1}{2}}\left(\int_{E_t}\omega^2dx\right)^{\frac{1}{2}} \le \varepsilon t,$$

where $\varepsilon \to 0$ as $t \to \infty$. We used here (4) and $\int_{E_t}\omega^2dx \to 0$ as $t \to \infty$, since the Dirichlet integral is finite.

Then from the mean value theorem follows that there exists a value $\tau \in [t, 2t] \cap \mathcal{R}$ such that

$$\int_{S_\tau} |\omega| ds \le 2\varepsilon.$$

Since the pressure is uniformly bounded, $|\mathbf{u}| \sim \sqrt{2\tau}$ on S_τ for large τ ($p + \frac{1}{2}|\mathbf{u}|^2 = \tau$ on S_τ). Using the identity

$$\nabla\Phi = -\nu\nabla^\perp\omega + \omega\mathbf{u}^\perp,$$

we obtain

$$\int_{S_\tau} |\nabla\Phi|\, ds = \int_{S_\tau} \omega\mathbf{u}^\perp \cdot \mathbf{n}\, ds \le 2\sqrt{\tau} \int_{S_\tau} |\omega|\, ds \le 4\sqrt{2t}\varepsilon$$

(the integral of $\nabla^\perp\omega \cdot \mathbf{n} = \mathrm{curl}\omega \cdot \mathbf{n}$ over the close curve S_τ is equal to zero). This contradicts the first inequality in (4). Thus, if $F > 0$, then the assumption is false and $\mathbf{u}$ is uniformly bounded.

(b). Let $F < 0$. From

$$\int_{S_t} |\nabla \Phi| \, ds = \int_{S_\tau} |\nabla \Phi| \, ds + \int_{\Omega_{\tau,t}} \omega^2 dx + \frac{1}{\nu}(t - \tau)F,$$

we see that for large t the right-hand side becomes negative, while the left-hand side is positive for all t. We again obtain a contradiction. Thus, the proof for the case $F \neq 0$ is complete.

CASE II. The total flux is zero: $F = 0$. Then we have

$$\int\limits_{S_t} |\nabla\Phi|\, ds = \int\limits_{S_\tau} |\nabla\Phi|\, ds + \int\limits_{\Omega_{\tau,t}} \omega^2 dx.$$

From this it follows that $\int\limits_{S_t} |\nabla\Phi|\, ds$ is a bounded increasing function, i.e., it has a finite positive limit, in particular,

$$C_1 \leq \int\limits_{S_t} |\nabla\Phi|\, ds \leq C_2 \tag{5}$$

for sufficiently large t. Applying the Coarea formula, we obtain

$$\int\limits_t^{2t}\Big(\int\limits_{S_\tau} |\omega|\, ds\Big)\, d\tau = \int\limits_{E_t} |\omega|\cdot|\nabla\Phi| dx \leq \Big(\int\limits_{E_t} |\nabla\Phi|^2 dx\Big)^{\frac{1}{2}} \cdot \Big(\int\limits_{E_t} \omega^2 dx\Big)^{\frac{1}{2}}$$

$$= \Big(\int\limits_t^{2t}\Big(\int\limits_{S_\tau} |\nabla\Phi|\, ds\Big)\, d\tau\Big)^{\frac{1}{2}} \cdot \Big(\int\limits_{E_t} \omega^2 dx\Big)^{\frac{1}{2}} \leq \varepsilon\sqrt{t},$$

where $\varepsilon \to 0$ as $t \to \infty$.

From the mean value theorem there exists a value $\tau \in [t, 2t] \cap \mathcal{R}$ such that

$$\int\limits_{S_\tau} |\omega|\, ds \le \varepsilon \frac{2}{\sqrt{\tau}}.$$

As in the Case I we have $|\mathbf{u}| \sim \sqrt{2\tau}$ on S_τ. Therefore, integrating again the identity

$$\nabla \Phi = -\nu \nabla^\perp \omega + \omega \mathbf{u}^\perp,$$

we obtain

$$\int\limits_{S_\tau} |\nabla \Phi|\, ds = \int\limits_{S_\tau} \omega \mathbf{u}^\perp \cdot \mathbf{n}\, ds \le 2\sqrt{\tau} \int\limits_{S_\tau} |\omega| dx \le 4\varepsilon.$$

The last estimate is in contradiction with the first inequality in (5). Therefore, in the case $F = 0$ assumption (1) is again false and the solution $\mathbf{u}$ is uniformly bounded. Theorem is proved.

THEOREM 2. *Let* $\mathbf{u}$ *be a* D*-solution to the Navier-Stokes system (NS) in the exterior domain* $\Omega \subset \mathbb{R}^2$*. Then* $\mathbf{u}$ *converges uniformly at infinity, i.e.,*

$$\mathbf{u}(z) \to \mathbf{u}_0 \qquad \textit{uniformly as } |z| \to \infty,$$

where $\mathbf{u}_0 \in \mathbb{R}^2$ *is the constant from the G* $\&$ *W equality* $(*)$*.*

Proof. By G & W we know that there exist the constant $\mathbf{u_0}$ such that

$$\lim_{r \to +\infty} \int_0^{2\pi} |\mathbf{u}(r, \theta) - \mathbf{u}_0|^2 d\theta = 0. \qquad (*)$$

Moreover, if $\mathbf{u}_0 = 0$, then

$$\mathbf{u}(z) \to 0 \qquad \text{uniformly as } |z| \to \infty.$$

So, we need only to consider the case

$$\mathbf{u}_0 \neq 0.$$

Consider the vorticity $\omega = \partial_2 u_1 - \partial_1 u_2$. It satisfies the elliptic equation

$$\nu \Delta \omega = (\mathbf{u} \cdot \nabla)\omega.$$

In particular, ω satisfies two-sided maximum principle in $\mathbb{R}^2$; moreover (proved in G & W),

$$\int_{\Omega_0} r|\nabla \omega|^2 < \infty$$

LEMMA 1. (G & W) *Let* $\mathbf{u}$ *be a* D*-solution of (NS). Denoted by* $\bar{\mathbf{u}}(z,r)$ *the mean value of* $\mathbf{u}$ *over the circle* $S(z,r)$*:*

$$\bar{\mathbf{u}}(z,r) = \frac{1}{2\pi r} \int\limits_{|\xi - z| = r} \mathbf{u}(\xi)\, ds$$

and let $\varphi(z,r)$ *be the argument of the complex number associated to the vector* $\bar{\mathbf{u}}(z,r) = (\bar{u}_1(r), \bar{u}_2(r))$*, i.e.,* $\varphi(z,r) = \arg(\bar{u}_1(r) + i\bar{u}_2(r))$*. Suppose* $|z|$ *is large enough so that the disk* $D_z = \left\{\xi \in \mathbb{R}^2 : |\xi - z| \le \frac{4}{5}|z|\right\}$ *is contained in* Ω*. Assume also that*

$$|\bar{\mathbf{u}}(z,r)| \ge \sigma.$$

for some positive constant $\sigma > 0$ *and for all* $r \in \left(0, \frac{4}{5}|z|\right]$*. Then the estimate*

$$\sup_{0 < \rho_1 \le \rho_2 \le \frac{4}{5}|z|} |\varphi(z,\rho_2) - \varphi(z,\rho_1)| \le \frac{1}{4\pi\sigma^2} \int\limits_{D_z} \left(\frac{1}{r} |\nabla \omega| + |\nabla \mathbf{u}|^2 \right) d\xi$$

holds, where $r = |\xi - z|$*.*

We need also the following simple technical assertion.
LEMMA 2. *Let $\mathbf{u}$ be a D-solution to the Navier–Stokes system. For $z \in \Omega$ denote as above*

$$D_z = \big\{\xi \in \mathbb{R}^2 : |\xi - z| \leq \frac{4}{5}|z|\big\}.$$

Then

$$\int\limits_{D_z} \frac{1}{r}|\nabla\omega|\, d\xi \to 0$$

uniformly as $|z| \to \infty$ (here $r = |\xi - z|$).

We will use the following two criteria for the uniform convergence of $\mathbf{u}$:

LEMMA 3. *Let $\mathbf{u}$ be a D-solution. Suppose that at least one of the following two conditions is fulfilled:*

(i) $\omega(z) = o(|z|^{-1})$ *as* $|z| \to \infty$;

(ii) *the absolute value of the velocity has a uniform limit at infinity:*

$$|\mathbf{u}(z)| \to |\mathbf{u}_0| \qquad \text{uniformly as } |z| \to \infty.$$

Then $\mathbf{u}$ converges uniformly to $\mathbf{u}_0$.

Proof. (i) was proved by Amick. Recall, that his argument is based on the classical Cauchy-type representation formula of complex analysis:

$$w(z) = \frac{1}{2\pi i} \oint\limits_{|\xi - z_0| = r} \frac{w(\xi)\, d\xi}{\xi - z} + \frac{1}{2\pi i} \iint\limits_{|\xi - z_0| < r} \frac{\omega(\xi)}{\xi - z}\, dx\, dy,$$

where $w(\xi) = u_1(\xi) - i u_2(\xi)$ and $\xi = x + iy$.

PROOF OF THEOREM 2. For a point $z \in \Omega_0$ denote by $K(z)$ the connected component of the level set of the vorticity ω containing z, i.e., $K(z) \subset \{x \in \Omega_0 : \omega(x) = \omega(z)\}$.
We consider two possible cases:
Case I. Level sets of ω separate infinity from the origin:

$$\exists z_* \in \Omega_0 : \quad \omega(z_*) \neq 0 \quad \text{and} \quad K(z_*) \cap \partial\Omega_0 = \emptyset.$$

Case II. Level sets of ω do not separate infinity from the origin:

$$K(z) \cap \partial\Omega_0 \neq \emptyset \qquad \forall z \in \Omega_0,$$

In Case I, we shall show that

$$|z|\omega(z) \to 0 \qquad \text{uniformly as } |z| \to \infty$$

and we obtain the statement of Theorem applying Lemma 3 (i).
In Case II, we prove that

$$|\mathbf{u}(z)| \to |\mathbf{u}_0| \qquad \text{uniformly as } |z| \to \infty,$$

and the statement of Theorem follows from Lemma 3 (ii).

Case I. In this case the set $K(z_*)$ is compact.
Indeed, the set $K(z_*)$ is connected and if it is not compact, it should "reach" infinity. Since the vorticity tends to zero at infinity, $\omega(z)$ has to be zero on $K(z_*)$, but this contradicts the assumption.

Next, since ω is an analytic nonconstant function, $\omega(z) \neq \text{const}$ in any open neighborhood of z_*.

Moreover, there exists $\delta_0 > 0$ such that $K(z)$ is a compact set, and $K(z) \cap \partial\Omega_0 = \emptyset$, if $|z - z_*| < \delta_0$.

Recall, that a real number t is called *a regular value of* ω, if the set $\{z \in \Omega_0 : \omega(z) = t\}$ is nonempty and $\nabla\omega(z) \neq 0$ whenever $\omega(z) = t$. By the classical Morse–Sard theorem, almost all values of ω are regular.

Now take a point z_1 satisfying $|z_1 - z_*| < \delta_0$ with regular value $t_1 = \omega(z_1)$. The set $K(z_1)$ is a smooth compact curve homeomorphic to the circle. By maximum principle for ω, the curve $K(z_1)$ separates the boundary $\partial\Omega_0$ from infinity.

Denote $R_* = \max\{|z| : z \in K(z_1)\}$ and $\Omega_* = \{z \in \mathbb{R}^2 : |z| > R_*\}$. Then by construction we have

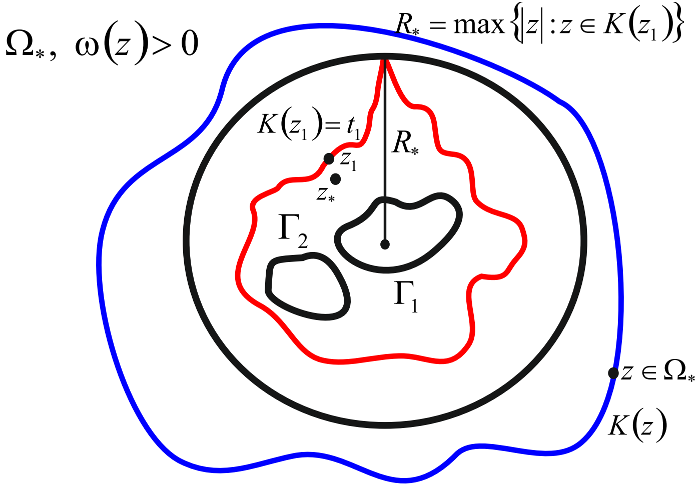
$$K(z) \cap \partial\Omega_0 = \emptyset \qquad \forall z \in \Omega_*.$$

Applying again the same Morse–Sard theorem, we obtain that for almost all $t \in \mathbb{R} \setminus \{0\}$ if $z \in \Omega_*$ and $\omega(z) = t$, then $K(z)$ is a smooth curve homeomorphic to the circle.

Since ω satisfies maximum principle, we conclude that this circle separates Ω_0 from infinity.
Moreover,

$$K(z_1) = K(z_2) \qquad \text{if } z_1, z_2 \in \Omega_* \quad \text{and } \omega(z_1) = \omega(z_2) \neq 0. \quad (Y_1)$$

Moreover, by maximum principle, $\omega(z)$ does not change sign in Ω_*.


$\Omega_*,\ \omega(z) > 0$
$R_* = \max\{|z| : z \in K(z_1)\}$
$K(z_1) = t_1$
z_1
z_*
R_*
Γ_2
Γ_1
$z \in \Omega_*$
$K(z)$

Thus we may suppose without loss of generality that

$$\omega(z) \geq 0 \qquad \text{in } \Omega_*.$$

By the maximum principle we have the strict inequality

$$\omega(z) > 0 \qquad \text{in } \Omega_*.$$

Moreover,

$$\omega(z) \to 0 \qquad \text{as } |z| \to \infty.$$

From Morse–Sard theorem we conclude that there exists a number $\delta > 0$ such that

> for almost all $t \in (0, \delta)$ the set $K_t := \{z \in \Omega_* : \omega(z) = t\}$ coincides with the smooth curve homeomorphic to the circle, $K_t \cap \partial\Omega_* = \emptyset$ and $\nabla\omega \neq 0$ on K_t.

Denote by $\mathcal{T}$ the set of full measure in the interval $(0, \delta)$ consisting of values t satisfying the last property. Denote also by Ω_t the unbounded connected component of the set $\mathbb{R}^2 \setminus K_t$. By maximum principle for ω, the sets K_t have the following monotonicity property:

$$\Omega_{t_1} \subset \Omega_{t_2} \qquad \text{if} \quad 0 < t_1 < t_2.$$

Moreover, from the uniform convergence to 0 of ω, it follows

$$\inf\{|z| : z \in \Omega_t\} \to \infty \qquad \text{as} \quad t \to 0+\,.$$

Our goal is to show that

$$|z|\omega(z) \to 0 \qquad \text{uniformly as} \quad |z| \to \infty.$$

The last condition is equivalent to

$$tg(t) \to 0 \qquad \text{as} \quad t \to 0+,$$

where $g(t) := \sup\{|z| : z \in K_t\}$. Obviously, $g(t) \leq \mathcal{H}^1(K_t)$, where $\mathcal{H}^1$ is the one dimensional Hausdorff measure (=length).

For $t \in \mathcal{T}$ and $R > R_*$ denote $\Omega_{t,R} = \Omega_t \cap B_R = \{z \in \Omega_t : |z| < R\}$. Then for sufficiently large R

$$\partial\Omega_{t,R} = K_t \cup S_R,$$

where $S_R = \{z \in \mathbb{R}^2 : |z| = R\}$ is the corresponding circle. Integrating the equation

$$\nu\Delta\omega = (\mathbf{u} \cdot \nabla)\omega.$$

over the domain $\Omega_{t,R}$ and taking into account that $(\mathbf{u} \cdot \nabla)\omega = \mathrm{div}(\mathbf{u}\omega)$, we obtain

$$\nu \int\limits_{K_t} |\nabla\omega|\, ds + \nu \int\limits_{S_R} \nabla\omega \cdot \mathbf{n}\, ds = t \int\limits_{K_t} \mathbf{u} \cdot \mathbf{n}\, ds + \int\limits_{S_R} \omega\, \mathbf{u} \cdot \mathbf{n}\, ds.$$

Here $\mathbf{n}$ is a unit vector of the outward with respect to $\Omega_{t,R}$ normal to $\partial\Omega_{t,R}$. Note also that the unit normal to the level set $K_t = \{z \in \Omega_* : \omega(z) = t\}$ is given by the formula $\mathbf{n} = \dfrac{\nabla\omega}{|\nabla\omega|}$.

Since $\operatorname{div}\mathbf{u} = 0$, we have $\int\limits_{K_t} \mathbf{u}\cdot\mathbf{n}\,ds = \int\limits_{\partial\Omega_*} \mathbf{u}\cdot\mathbf{n}\,ds = C_*$, i.e., this value does not depend on t. On the other hand, $\int\limits_{\Omega_0} \big(|\omega|^2 + |\nabla\omega|^2\big)\,d\mathcal{H}^2 < \infty$. Hence, there is a sequence $R_k \to +\infty$ such that

$$\int\limits_{S_{R_k}} \big(|\omega| + |\nabla\omega|\big)\,ds \to 0.$$

Taking $R = R_k$ and having in mind the uniform boundedness of the velocity, we deduce, passing $R_k \to +\infty$, that

$$\int\limits_{K_t} |\nabla\omega|\,ds = C_* t.$$

Further, for $t \in (0, \frac{1}{2}\delta)$ denote $E_t = \{z \in \Omega_* : \omega(z) \in (t, 2t)\}$. By construction,

$$\partial E_t = K_t \cup K_{2t}.$$

Applying the classical Coarea formula

$$\int_{E_t} f\, |\nabla \omega|\, d\mathcal{H}^2 = \int_t^{2t} \left(\int_{K_\tau} f\, ds \right) d\tau$$

for $f = |\nabla \omega|$ we obtain

$$\int_{E_t} |\nabla \omega|^2\, d\mathcal{H}^2 = \int_t^{2t} \left(\int_{K_\tau} |\nabla \omega|\, ds \right) d\tau = \int_t^{2t} C_* \tau\, d\tau = 3C_* t^2.$$

Applying now the same Coarea formula for $f = 1$ and using the Cauchy–Schwarz inequality, we get

$$\int_t^{2t} \mathcal{H}^1(K_\tau)\, d\tau = \int_{E_t} |\nabla\omega|\, d\mathcal{H}^2 \leq \left(\int_{E_t} |\nabla\omega|^2\, d\mathcal{H}^2 \right)^{\frac{1}{2}} \left(\text{meas} E_t \right)^{\frac{1}{2}}$$

$$= \sqrt{3C_*}\left(t^2 \text{meas}(E_t) \right)^{\frac{1}{2}} \leq \sqrt{\tfrac{3}{4}C_*}\left(\int_{E_t} \omega^2\, d\mathcal{H}^2 \right)^{\frac{1}{2}} \leq \varepsilon_t \to 0 \quad \text{as } t \to 0.$$

Here we have used also the fact that $t \leq |\omega(z)| \leq 2t$ in E_t. By virtue of the mean-value theorem, this implies that for any sufficiently small $t \in \mathcal{T}$ there exists a number $\tau \in [t, 2t]$ such that

$$t\mathcal{H}^1(K_\tau) \leq \varepsilon_t.$$

By construction, the closed curve K_τ surrounds K_{2t}. Therefore,

$$\sup\{|z| : z \in K_{2t}\} \leq \mathcal{H}^1(K_\tau) \leq \frac{\varepsilon_t}{t}$$

with $\varepsilon_t \to 0$ as $t \to 0$. From the last inequality it follow $|z|\omega(z) \to 0$ uniformly as $|z| \to \infty$. This finishes the proof in Case I.

Case II:

$$K(z) \cap \partial\Omega_0 \neq \emptyset \qquad \forall z \in \Omega_0.$$

Now we prove that

$$|\mathbf{u}(z)| \to |\mathbf{u}_0| \quad \text{uniformly as} \quad |z| \to \infty.$$

Ch. Amick proved this convergence under the assumption that $\mathbf{u}|_{\partial\Omega} = 0$. The assumption was used in order to define the stream function ψ in the neighborhood of infinity:

$$\nabla\psi = \mathbf{u}^{\perp} = (-v, u),$$

where $\mathbf{u} = (u, v)$. Using the stream function ψ, Amick introduced an auxiliary function $\gamma = \Phi - \omega\psi$, where $\Phi := p + \frac{1}{2}|\mathbf{u}|^2$ is the Bernoulli pressure. The gradient of this auxiliary function γ satisfies the identity

$$\nabla\gamma = -\nu\nabla^{\perp}\omega - \psi\nabla\omega.$$

Then $\nabla\gamma \cdot \nabla^{\perp}\omega = -\nu|\nabla^{\perp}\omega|^2$, and therefore, γ has the following monotonicity properties:

γ is monotone along level sets of the vorticity $\omega = c$,
vice versa — the vorticity ω is monotone along level sets of $\gamma = c$.

Obviously, the stream function ψ (and, consequently, the corresponding auxiliary function γ) is well defined in the neighborhood of infinity under the more general condition

$$\int_{\partial\Omega_0} \mathbf{u}\cdot\mathbf{n}\,ds = 0.$$

However, in general, the flow-rate of the velocity field is not zero,

$$\int_{\partial\Omega_0} \mathbf{u}\cdot\mathbf{n}\,ds \neq 0,$$

and, therefore, the stream function ψ can not be defined in the neighborhood of infinity.

We will overcome this difficulty using the assumption

$$K(z) \cap \partial\Omega_0 \neq \emptyset \qquad \forall z \in \Omega_0.$$

Take and fix a radius $R_* > R_0$ (R_* could be chosen arbitrary large) and consider the domain $\Omega_* = \{z \in \mathbb{R}^2 : |z| > R_*\}$. Denote by U_i the connected components of the open set $\{z \in \Omega_* : \omega(z) \neq 0\}$. Then there holds the following

Proposition.

(i) *There are only finitely many components* U_i, $i = 1, \dots, N$;

(ii) *Every* U_i *is a simply connected open set;*

(iii) *The vorticity* $\omega(z)$ *change sign in every neighborhood of infinity, i.e., there exist two sequences of points* z_n^+ *and* z_n^- *such that* $\omega(z_n^+) > 0$, $\omega(z_n^-) < 0$ *and* $\lim\limits_{n\to\infty} |z_n^+| = \lim\limits_{n\to\infty} |z_n^-| = \infty$.

Since U_i are simply connected, this allows us to define the stream function ψ in every component U_i. Moreover, since $\omega = 0$ on $\Omega_* \cap \partial U_i$, the auxiliary function $\gamma = \Phi - \omega\psi$ is well defined and continuous on the whole domain Ω_*. After the functions ψ and γ are defined, we can repeat Amick's arguments of and to prove the convergence of absolute value of the velocity at infinity. This proves the Case II.

Amick also used the components U_i, and he proved the same properties (i)–(iii) using the boundary condition $\mathbf{u}|_{\partial\Omega} = 0$. Here we get the properties (i)–(iii) because of the assumption

$$K(z) \cap \partial\Omega_0 \neq \emptyset \qquad \forall z \in \Omega_0.$$

Existence Results

Teorem 3 (KPR). *Let $\Omega \subset \mathbb{R}^2$ be an exterior domain with C^2-smooth boundary. Suppose that $\mathbf{h} \in W^{1/2,2}(\partial\Omega)$ and*

$$\int\limits_{\partial\Omega} \mathbf{h} \cdot \mathbf{n}\, ds = \sum_{i=1}^{N} \int\limits_{\Gamma_i} \mathbf{h} \cdot \mathbf{n}\, ds = \sum_{i=1}^{N} F_i = F = 0,$$

i.e., the total flux F is zero. Then there exists a D-solution $\mathbf{u}$ to the Navier–Stokes problem (NS).

The existence for arbitrary total flux F is known only in the symmetric case. Let Ω be a symmetric domain:

$$(x_1, x_2) \in \Omega \Rightarrow (x_1, -x_2) \in \Omega.$$

Theorem 4 (KPR). *Let $\Omega \subset \mathbb{R}^2$ be a symmetric exterior domain. Assume that $\mathbf{h}$ is a symmetric field in $W^{1/2,2}(\partial\Omega)$, i.e.,*

$$h_1(x_1, x_2) = h_1(x_1, -x_2), \quad h_2(x_1, x_2) = -h_2(x_1, -x_2)$$

Then problem (NS) admits at least one symmetric D-solution $\mathbf{u}$ for an arbitrary total flux F.

Existence Results

The existence of the solution to (NS) satisfying also (CI) condition is known only in the case when the domain and the data are symmetric with respect to the both coordinate axes:

$$x \in \Omega \Rightarrow (x_1, -x_2) \in \Omega, \quad (-x_1, x_2) \in \Omega.$$

Theorem 5 (KR). *Let $\Omega \subset \mathbb{R}^2$ be a symmetric exterior domain. Assume that $\mathbf{h}$ is a symmetric (with respect to the both coordinate axes) vector field in $W^{1/2,2}(\partial\Omega)$, i.e.,*

$$h_1(x_1, x_2) = h_1(x_1, -x_2) = -h_1(-x_1, x_2),$$
$$h_2(x_1, x_2) = -h_2(x_1, -x_2) = h_2(-x_1, x_2).$$

Then problem (NS) admits at least one symmetric (with respect to the both coordinate axes) D-solution $\mathbf{u}$ for an arbitrary total flux F. This solution satisfies the condition at infinity (CI).