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Gravity wave propagation in inhomogeneous media : wave scattering and interference process

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Abstract

This lecture aims to give an overview of water waves and their propagation in inhomogeneous media. Effects of varying bathymetries, varying currents, or structures including porous media are then considered. After some generalities on 1st and 2nd order Stokes waves, governing equations for regular plane waves for the study of wave scattering due to either varying bathymetry and currents or structures are presented for both 2D and 3D cases. Analytical and numerical solutions are then presented and compared to experiments. For 2-D cases, examples are given for wave reflection through interference process including Bragg resonance. For 3-D cases, various examples including wave scattering due to a shoal, a structure, periodic structures or varying currents are given. Applications to both shore protection solutions and wave energy device are given.

Keywords: Gravity wave, interference process, Bragg scattering, wave propagation equations, multi-scale expansions, integral matching methods, 2nd order Stokes effects, Linear and non linear wave damping.

1 Introduction

This lecture aims to give an overview of water waves and their behavior in the presence of varying bathymetries, structures or currents. For 2-D cases, examples are given for wave reflection through interference process including Bragg resonance. For 3-D cases, examples are given for wave focusing due to a shoal, a structure, periodic structures arrays or in the presence of current. Both analytical and experimental approaches including a discussion on scale effects are presented. Applications to both shore protection solutions and wave energy device are given.

After some generalities on the Stokes waves in Section 2, governing equations for regular plane waves for the study of wave scattering due to either varying bathymetry or structures are presented in Section 3 for 2D cases and in Section 4 for 3D cases. Examples on wave scattering, and wave focusing due to inhomogeneous media are given in the PDF support of the presentation.

2 Water waves : Stokes' wave solutions

2.1 Velocity potential

The Stokes' solutions assume a small-amplitude, monochromatic, irrotational motion of an ideal fluid. The complex velocity potential $\Phi(x, y, z, t)$, defined by $\vec{v} = \vec{\nabla}\Phi$, satifies the Laplace's equation:

$$\nabla^2\Phi = 0 \tag{1}$$

where $\nabla^2 \left(\frac{\partial^2}{\partial x^2}, \frac{\partial^2}{\partial y^2}, \frac{\partial^2}{\partial z^2}\right)$. The departure of the water from its mean level $z = 0$ (z oriented upwards) is taken as $\eta(x, y, t)$ where t is the time, and x, y the horizontal coordinates. Neglecting the surface tension effects, the kinematic and dynamic conditions at the free surface $z = \eta(x, y, t)$ are respectively:

$$\frac{\partial\eta}{\partial t} + \frac{\partial\Phi}{\partial x}\frac{\partial\eta}{\partial x} - \frac{\partial\Phi}{\partial z} = 0 \tag{2}$$

and

$$g\eta + \frac{1}{2}\vec{v}^2 + \frac{\partial\Phi}{\partial t} = 0 \tag{3}$$

Writting $\vec{v}.\vec{\nabla}\left(\frac{\partial\Phi}{\partial t}\right) = \frac{1}{2}\frac{\partial \vec{v}^2}{\partial t}$, and assuming the pressure to be constant at the air-sea interface, the total derivative of the dynamic condition (3) on $z = \eta$, combined to the kinematic condition (2) gives for the potential Φ:

$$g\frac{\partial\Phi}{\partial z} + \frac{\partial \vec{v}^2}{\partial t} + \frac{\partial^2\Phi}{\partial t^2} + \frac{1}{2}\vec{v}.\vec{\nabla}(\vec{v}^2) = 0 \quad (4)$$

In waters of finite depth, the bottom condition is:

$$\frac{\partial\Phi}{\partial n} = 0 \quad (5)$$

where $\vec{n}$ is normal to the bottom.

2.2 1st order Stokes waves on uneven bottoms

1st ordre Stokes or Airy waves correspond to the solutions of the linearized problem. For simplicity, we consider here a wave propagating in the x-axis direction. The linearized surface conditions for the potential $\Phi(x, z, t)$, written at $z = 0$ becomes after Taylor expansion at first order,

$$g\frac{\partial\Phi}{\partial z} + \frac{\partial^2\Phi}{\partial t^2} = 0 \quad (6)$$

The bottom condition, for $z = -h$,

$$\frac{\partial\Phi}{\partial z} = 0 \quad (7)$$

The (complex) velocity potential is assumed to have time dependance

$$\Phi(x, z, t) = \phi(x, z)e^{i\omega t}$$

where $\omega = \frac{2\pi}{T}$ is the wave frequency. The solutions for $\phi(x, z)$, satisfying the boundary conditions (6) and (7) are

$$\phi(x, z) = \left[A^{-}e^{-ikx} + A^{+}e^{+ikx}\right]\cosh k(z + h) \quad (8)$$

where k, wavenumber of the surface wave is given by the dispersion relation

$$\omega^2 = gk\tanh(kh) \quad (9)$$

The phase velocity C is given by

$$C = \frac{\omega}{k} = \sqrt{\frac{g}{k}\tanh(kh)} \tag{10}$$

Excepted in shallow water conditions ($kh \ll 1$), the gravity wave, which depends on the wave frequency, is dispersive. For a progressive wave, surface deformation and velocity potential are then as follows:

$$\eta(x,t) = a\sin(\omega t - kx) \tag{11}$$

and

$$\Phi(x,z,t) = \frac{a\omega}{k}\frac{\cosh[k(z+h)]}{\sinh(kh)}\cos(\omega t - kx) = \frac{ag}{\omega}\frac{\cosh[k(z+h)]}{\cosh(kh)}\cos(\omega t - kx) \tag{12}$$

where the amplitude $a = H/2$ is half the wave height. It is sometime convenient for calculations for linear equations to use the complex expressions of the free surface η and the potential Φ:

$$\tilde{\eta}(x,t) = -iae^{i(\omega t - kx)} \tag{13}$$

and

$$\tilde{\Phi}(x,z,t) = \frac{a\omega}{k}\frac{\cosh[k(z+h)]}{\sinh(kh)}e^{i(\omega t - kx)} = i\frac{\omega}{k}\frac{\cosh[k(z+h)]}{\sinh(kh)}\eta(x,t) \tag{14}$$

The solution is then the real part of the complex expression, $\eta = Re(\tilde{\eta})$, $\Phi = Re(\tilde{\Phi})$. For non-linear calculations, the complex expression of both η and Φ must be written with the complex conjugate:

$$\eta(x,t) = -\frac{1}{2}iae^{i(\omega t - kx)} - \frac{1}{2}iae^{-i(\omega t - kx)} \tag{15}$$

and

$$\Phi(x,z,t) = \frac{1}{2}\frac{a\omega}{k}\frac{\cosh[k(z+h)]}{\sinh(kh)}e^{i(\omega t - kx)} + \frac{1}{2}\frac{a\omega}{k}\frac{\cosh[k(z+h)]}{\sinh(kh)}e^{-i(\omega t - kx)} \tag{16}$$

The averaged energy flux across a section of width dy normal to the direction of propagation, is given by

$$dE_t = \int_t^{t+T}\int_{x=-h}^{\eta} p\vec{v}.\vec{n}dydzdt \tag{17}$$

where

$$p = -\rho \frac{\partial \Phi}{\partial t} \tag{18}$$

is the dynamic pressure and $\vec{n}$ is normal to $y0z$. The expression (17) becomes for a Airy wave and per unit width along $y-$axis:

$$E_t = \frac{\rho g a^2}{4} C \left[1 + \frac{2kh}{\sinh 2kh} \right] = \frac{\rho g a^2}{2} C_g \tag{19}$$

This energy propagates at the group velocity C_g:

$$C_g = \frac{\partial \omega}{\partial k} = \frac{1}{2} C \left[1 + \frac{2kh}{\sinh 2kh} \right] \tag{20}$$

The limitations of this solution are $ak, \frac{a}{h}, ak^{-2}h^{-3} \ll 1$ (see also next section).

2.3 2nd order Stokes waves

In the calculations for Airy waves, non-linear terms like $\frac{\partial \Phi}{\partial x}\frac{\partial \eta}{\partial x}$ are neglected. When replacing η et Φ by their expressions (11) and (12), the term $\frac{\partial \Phi}{\partial x}\frac{\partial \eta}{\partial x}$ becomes proportional to $\sin[2(\omega t - kx)]$ at leading order. Both surface deformation and velocity potential may be developped in Fourier series:

$$\eta = a \sin[\omega t - kx] + A_2 a \sin[2(\omega t - kx)] + ... \tag{21}$$

$$\Phi = a \cosh k(z+h) \cos[\omega t - kx] + B_2 a^2 \cosh[2k(z+h)] \cos[2(\omega t - kx)] + ... \tag{22}$$

Note that with that form, Φ verifies the Laplace's equation (1) and the bottom condition (7). If "ka" (or "a") is small, η et Φ can be expressed by:

$$\eta = \varepsilon \eta_1 + \varepsilon^2 \eta_2 + O(\epsilon^2) \tag{23}$$

$$\Phi = \varepsilon \Phi_1 + \varepsilon^2 \Phi_2 + O(\epsilon^2) \tag{24}$$

where ε is a small parameter, and $O(\varepsilon^n)$ denotes the terms negligible at order n.

The surface condition (4) expressed at $z = \eta$, can be written at $z = 0$ by using the Taylor expansion:

$$f(x,\eta,t) = f(x,0,t) + \eta \left[\frac{\partial f}{\partial z}\right]_{z=0} + \frac{\eta^2}{2}\left[\frac{\partial^2 f}{\partial z^2}\right]_{z=0} + ... \quad (25)$$

The boundary conditions at $z = 0$ and $z = -h$, and the Laplace's equation are then expressed at orders $O(\varepsilon)$ and $O(\varepsilon^2)$ by replacing η and Φ in the relations (1), (4) and (7) by the expressions (23) and (24), and by using the Taylor expansion (25):

*Laplace's condition$(0 \geq z \geq -h)$:

At each order i, $i = 1, 2$:

$$\frac{\partial^2 \Phi_i}{\partial x^2} + \frac{\partial^2 \Phi_i}{\partial z^2} = 0 \quad (26)$$

*Bottom condition $(z = -h)$:

At each order i, $i = 1, 2$:

$$\frac{\partial \Phi_i}{\partial z} = 0 \quad (27)$$

*Surface condition $(z = 0)$:

at order ϵ :

$$\frac{\partial^2 \Phi_1}{\partial t^2} + g\frac{\partial \Phi_1}{\partial z} = 0 \quad (28)$$

at order ϵ^2 :

$$\frac{\partial^2 \Phi_2}{\partial t^2} + g\frac{\partial \Phi_2}{\partial z} + \eta_1\left(\frac{\partial^3 \Phi_1}{\partial t^2 \partial z} + g\frac{\partial^2 \Phi_1}{\partial z^2}\right) + \frac{\partial}{\partial t}\left[\left(\frac{\partial \Phi_1}{\partial x}\right)^2 + \left(\frac{\partial \Phi_1}{\partial z}\right)^2\right] = 0 \quad (29)$$

The free surface deformation (3) becomes :

at order ε :

$$-g\eta_1 = \frac{\partial \Phi_1}{\partial t} \tag{30}$$

at order ε^2 :

$$-g\eta_2 = \frac{\partial \Phi_2}{\partial t} + \eta_1 \frac{\partial^2 \Phi_1}{\partial z \partial t} + \frac{1}{2}\left[\left(\frac{\partial \Phi_1}{\partial x}\right)^2 + \left(\frac{\partial \Phi_1}{\partial z}\right)^2\right] \tag{31}$$

Solution at order ε:
The solution is the Airy wave:

$$\Phi_1 = \frac{a\omega}{k}\frac{\cosh[k(z+h)]}{\sinh(kh)}\cos[\omega t - kx] \tag{32}$$

$$\eta_1 = a\sin[\omega t - kx] \tag{33}$$

with the dispersion relation $\omega^2 = gk\tanh(kh)$, which is still valid at order ε^2.

Solution at order ε^2:

After use of the expressions (32) and (33), the surface condition (29) becomes:

$$\frac{\partial^2 \Phi_2}{\partial t^2} + g\frac{\partial \Phi_2}{\partial z} = -\frac{3a^2\omega^3}{2\sinh^2(kh)}\sin 2(\omega t - kx) \tag{34}$$

which solution may be of the general form $\Phi_2(x,z,t) = A\cosh[2k(z+h)]\sin 2(\omega t - kx)$. We find after simplifications:

$$\Phi_2 = \frac{3}{8}\frac{a^2\omega}{\sinh^4 kh}\cosh[2k(z+h)]\sin 2(\omega t - kx) \tag{35}$$

and after use of (31), the solution for η_2 is

$$\eta_2 = \frac{a^2 k}{2\sinh(2kh)} + \frac{a^2 k}{4}\frac{(3-\tanh^2 kh)}{\tanh^3 kh}\cos[2(\omega t - kx)] \tag{36}$$

Let us note for example for η_2 that $k\eta_2$ is of order $a^2k^2 = O(\epsilon^2)$. The "ε" and "ε^2" in the expressions (23) and (24) were useful for the indication of the order of the different terms. For the final result, "ε" is "eliminated" and the solutions for the potential and the surface deformation are then respectively

$$\Phi(x,z,t) = \frac{a\omega}{k}\frac{\cosh[k(z+h)]}{\sinh(kh)}\cos[\omega t - kx] + \frac{3}{8}\frac{a^2\omega}{\sinh^4 kh}\cosh[2k(z+h)]\sin 2(\omega t - kx) \tag{37}$$

and

$$\eta = a\sin[\omega t - kx] + \frac{a^2k}{2\sinh(2kh)} + \frac{a^2k}{4}\frac{(3-\tanh^2 kh)}{\tanh^3 kh}\cos[2(\omega t - kx)] \tag{38}$$

Writting $|\Phi_2| \ll |\Phi_1|$, since the $2nd$ order solution is a perturbation,

$$\left|\frac{\Phi_2}{\Phi_1}\right| = \left|ak\frac{\cosh[2k(z+h)]}{\cosh[k(z+h)]}\frac{1}{\sinh^3 kh}\right| \ll 1 \tag{39}$$

Remarks:

Deep water conditions ($kh \gg 1$):

For progressive waves, $\left|\frac{\Phi_2}{\Phi_1}\right| \simeq \frac{ak}{e^{2kh}} \to 0$ when $h \to \infty$. We find $\Phi_2 = 0$ in deep water conditons. Particle trajectories are still circular for non-linear waves, as assumed by Gerstner in 1802 in its Lagrangien appoach.

For partially standing waves, 1st order velocity potential and surface deformation are of the form, with an adequate choice of the origins,

$$\Phi_1 = \frac{a^-\omega}{k}e^{kz}\cos[\omega t - kx] + \frac{a^+\omega}{k}e^{kz}\cos[\omega t + kx] \tag{40}$$

$$\eta_1 = a^-\sin[\omega t - kx] + a^+\sin[\omega t + kx] \tag{41}$$

The second order solution for the potential is no more null

$$\Phi_2 = -a^-a^+\sin[2\omega t] \tag{42}$$

If we remember that $p = -\rho\frac{\partial\Phi}{\partial t}$, we can observe that Φ_2 is at the origin of pressure oscillations at frequency $2f$ which do not depend on the water depth, as shown by Longuet-Higgins (1950).

In shallow water conditions, ($kh \ll 1$), $\left|\frac{\Phi_2}{\Phi_1}\right| \simeq \frac{a}{k^2h^3} = \frac{a\lambda^2}{4\pi h^3} = \frac{1}{4\pi}Ur$ where $U_r = \frac{a\lambda^2}{h^3}$ is the Ursell number. It must remain small. The non-linear Stokes solution is then no more valid when $h \to 0$. Some other solutions, formulated for swallow water conditions, and extended to finite depth conditions, as the non-linear Boussinesq equations, assume U_r of order 1.

3 Water scattering by varying depth : 2D case

For water waves, the phase celerity depends on water depth h (see expression (10)). We can then define for a given depth h a parameter $n(h) = \frac{C_0}{C(h)} \geqslant 1$, where index 0 refers to deep water conditions. Generally, for monotonous mild slopes (corresponding to slowly varying index n), wave reflection is negligible and the bathymetric influence on the progressive wave amplitude is known as the "shoaling effect":

$$\left(\frac{a}{a_0}\right)^2 = \left(\frac{H}{H_0}\right)^2 = \frac{C_{g0}}{C_g} = \frac{1}{\tanh kh + \frac{kh}{\cosh^2 kh}} \tag{43}$$

After a weak decrease (minimum of amplitude $\frac{a}{a_0} = 0.9129$ for $h/\lambda = 0.191$), the wave amplitude increases up to breaking. For undulating beds, even for weakly reflecting beds, interference process may lead to significant reflection for particular shapes. Sinusoidal beds are then considered in the following to evidence the interference process leading to Bragg resonance. Theoretical approches for the calculation of wave scattering by abrut bathymetries or steps are also presented.

3.1 Case of smooth bathymetry : sinusoidal beds

Let us consider a mean water depth h, and a bottom modulation δ, defined as

$$\delta = \frac{1}{2}De^{iKx} + * \tag{44}$$

Throughout this paragraph, the symbol $*$ denotes the complex conjugate. Assuming small amplitude modulations, of order $\varepsilon = O(D/h)$, we will note $\varepsilon\delta$ the bed modulation. At leading order, assuming a flat bed, the velocity potential $\Phi(x,z,t)$ is of the form (16) for a progressive wave. The bottom boundary impermeability condition at $z=-h+\varepsilon\delta$ is in the present case $\frac{\partial\Phi}{\partial n}=0$, where $\vec{n}$ is normal to the bottom. The bottom condition can be written at $z=-h$ after a Taylor expansion:

$$\frac{\partial\Phi}{\partial z}=-\frac{\partial\delta}{\partial x}.\frac{\partial\Phi}{\partial x}=\varepsilon\frac{\partial}{\partial x}\left(\delta\frac{\partial\Phi}{\partial x}\right)+.. \tag{45}$$

Using the expression (44) for δ, we will observe that some of the terms of $\frac{\partial\Phi}{\partial z}$ are proportional to $e^{i(\omega t+kx)}$ when $K=2k$ (see section 3.1.1). The bottom condition then forces a reflected wave. For a periodic bed of large number of oscillations, reflection tends to 1. The order of the reflected wave amplitude is then of order 1, of same order as the incident wave.

Calculations either based on perturbation methods, numerical resolution based on vertically integrated equation (mild-slope and "modified" mild-slope equations), or integral matching at vertical boundaries between successive domains of constant depth can be used.

3.1.1 Perturbation method with multiscale expansion for sinusoidal beds of finite extend

We assume the small waves amplitudes and the linearized free surface condition (6). In the fluid $(-h+\varepsilon\delta<z<0)$, the velocity potential satisfies the Laplace equation where h denotes the constant mean depth and $\epsilon\delta(x)$ is the bar height above the mean bottom with $\varepsilon=O(D/h)$ being the small parameter measuring the ratio of bar amplitude to depth. Within the domain of $0<x<L$, where $kL\gg 1$, the bed profile is given by (44). If the bed is weakly modulated, we can assume that the wave amplitude and phase are slowly modulated over the sinusoidal bed (Mei, 1983). By introducing the slow variables $x_1=\varepsilon x$ and $t_1=\varepsilon t$, and the multiple-scale expansion, the velocity potential is as follows:

$$\Phi(x,z,t)=\varepsilon\Phi_1(x,z,t,x_1,t_1)+\varepsilon^2\Phi_2(x,z,t,x_1,t_1)+... \tag{46}$$

From the Laplace's equation (1), the linearized free surface condition (6) and the bottom condition (45), a sequence of perturbation problems are obtained at successive orders of ε. The partial derivative functions for variables x and t then become $\frac{\partial}{\partial x}+\varepsilon\frac{\partial}{\partial x_1}$ and $\frac{\partial}{\partial t}+\varepsilon\frac{\partial}{\partial t_1}$.

At order ε, the equations are homogeneous for the velocity potential; the solution is a linear combination of incident and reflected waves

$$\Phi_1 = \psi^- e^{i(\omega t - kx)} + * + \psi^+ e^{i(\omega t + kx)} + *. \tag{47}$$

The superscripts $-$ and $+$ refer, respectively, to incident and reflected waves. The vertical profiles ψ^- and ψ^+ are governed by the homogeneous ordinary differential equations and boundary conditions at $z = 0, h$, which can be solved by (see 12)

$$\psi^{\pm} = i\frac{g}{2\omega}\frac{\cosh\left[k\left(z+h\right)\right]}{\cosh\left(kh\right)}a^{\pm}, \tag{48}$$

where a^- and a^+ are the complex wave amplitudes. By anticipating the wave reflection by the modulated bottom, we assume a^- and a^+ to be of same order. Let us note that Davies (1982) and Davies and Heathershaw (1984) had developed a model based on the technique of regular perturbations. The results are discussed at the end of this the section.

The wavenumber k satisfies the dispersion relation (9).

At order ε^2, the equations are as follows:

$*$Surface condition $(z = 0)$:

$$\frac{\partial^2 \Phi_2}{\partial t^2} + g\frac{\partial \Phi_2}{\partial z} = -2\frac{\partial^2 \Phi_1}{\partial t \partial t_1} \tag{49}$$

$*$Laplace's condition $(0 \geq z \geq -h)$:

$$\frac{\partial^2 \Phi_2}{\partial x^2} + \frac{\partial^2 \Phi_2}{\partial z^2} = -2\frac{\partial^2 \Phi_1}{\partial x \partial x_1} \tag{50}$$

$*$Bottom condition $(z = -h)$:

$$\frac{\partial \Phi_2}{\partial z} = \frac{\partial}{\partial x}\left(\delta\frac{\partial \Phi_1}{\partial x}\right) \tag{51}$$

If $K = 2k$, that is at resonance,

$$\delta\frac{\partial \Phi_1}{\partial x} = -\frac{1}{2}ikD\psi^- e^{i(\omega t + kx)} + \frac{1}{2}ikD\psi^+ e^{i(\omega t + 3kx)} - \frac{1}{2}ikD\psi^- e^{i(\omega t - kx)} + \frac{1}{2}ikD\psi^+ e^{i(\omega t - 3kx)} + * \tag{52}$$

If we only consider the resonating terms, then equation (51) becomes:

$$\frac{\partial \Phi_2}{\partial z} = \frac{1}{2}k^2 D\psi^+ e^{i(\omega t - kx)} + \frac{1}{2}k^2 D\psi^- e^{i(\omega t + kx)} + * \tag{53}$$

The second rigth-hand term is a resonating term for the reflected wave. If Φ_2 is expressed as:

$$\Phi_2 = i\gamma^- e^{i(\omega t - kx)} + * + i\gamma^+ e^{i(\omega t + kx)} + * \tag{54}$$

Equations (49),(50) and (53) give for $\gamma^\pm$:

$$\begin{aligned}
\frac{\partial \gamma^\pm}{\partial z} - \frac{\omega^2}{g}\gamma^\pm &= i\frac{\partial a^\pm}{\partial t_1} \text{ for } z = 0 \\
\frac{\partial^2 \gamma^\pm}{\partial z^2} - k^2\gamma^\pm &= \frac{igk}{\omega}\frac{\cosh\left[k\left(z+h\right)\right]}{\cosh\left(kh\right)}\frac{\partial a^\pm}{\partial x_1} \\
\frac{\partial \gamma^\pm}{\partial z} &= -\frac{1}{4}\frac{gk^2 D}{\omega}\frac{1}{\cosh\left(kh\right)}a^\mp \text{ for } z = -h
\end{aligned} \tag{55}$$

By using the function $F(z) = \cosh\left[k\left(z+h\right)\right]$, solution of the homogeneous equations, and writing

$$\int_{-h}^{0} F\left(\frac{\partial^2 \gamma^\pm}{\partial z^2} - k^2\gamma^\pm\right)dz = \left[F\frac{\partial \gamma^\pm}{\partial z} - \gamma^\pm\frac{\partial F}{\partial z}\right]_{-h}^{0} \tag{56}$$

, the solvability of $\gamma^\pm$ yields over the rippled bed:

$$\left(\frac{\partial}{\partial t_1} \pm C_g\frac{\partial}{\partial x_1}\right)a^\pm = -i\Omega_0 a^\mp \tag{57}$$

where

$$\Omega_0 = \frac{gk^2 D}{4\omega\cosh\left(kh\right)} = \frac{\omega k D}{2\sinh\left(2kh\right)}$$

has the dimension of a frequency. For $x_1 < 0$ and $x_1 > L_1$, a^- and a^+ are uncoupled:

$$\left(\frac{\partial}{\partial t_1} \pm C_g\frac{\partial}{\partial x_1}\right)a^\mp = 0 \tag{58}$$

We impose the condition that there is no reflected wave for $x > L_1$. Continuity of a^+ and a^- at $x_1 = 0$ and $x_1 = L_1$ gives four matching conditions.

The leading-order wave potential is given on the incidence side, $x_1 < 0$, by Eq. (47). The complex amplitude of the incident and reflected waves may be expressed respectively by:

$$a^- = a_0 e^{i(\Omega t_1 - \kappa x_1)}, \tag{59}$$

and

$$a^+ = a_0 R(0) e^{i(\Omega t_1 + \kappa x_1)}, \tag{60}$$

where $\epsilon\kappa$ corresponds to a wavenumber detuning (equivalent to a frequency detuning of $\varepsilon\Omega = \varepsilon C_g \kappa$) and on the transmission side, $x_1 > L_1$, by

$$a^- = a_0 T(L_1) e^{i(\Omega t_1 - \kappa x_1)} \tag{61}$$

$$a^+ = 0 \tag{62}$$

Over the patch of bars in the region $0 < x_1 < L_1$, the complex amplitudes are given respectively by:

$$a^- = a_0 T(x_1) e^{i(\Omega t_1 - \kappa x_1)}, \tag{63}$$

and

$$a^+ = a_0 R(x_1) e^{i(\Omega t_1 + \kappa x_1)}, \tag{64}$$

$T(L_1)$ and $R(0)$ are respectively the reflection and transmission coefficients. Four cases may be distinguished with respect to the cutoff frequency Ω_0 for the solutions $T(x_1)$ and $R(x_1)$:

Case (i): $\Omega > \Omega_0$

$$T(x_1) = \frac{P C_g \cos\left[P\left(L_1 - x_1\right)\right] - i\Omega \sin\left[P\left(L_1 - x_1\right)\right]}{P C_g \cos\left(P L_1\right) - i\Omega \sin\left(P L_1\right)} \tag{65}$$

$$R(x_1) = \frac{-i\Omega_0 \sin\left[P\left(L_1 - x_1\right)\right]}{P C_g \cos\left(P L_1\right) - i\Omega \sin\left(P L_1\right)} \tag{66}$$

where

$$P = \frac{(\Omega^2 - \Omega_0^2)^{\frac{1}{2}}}{C_g}. \tag{67}$$

Case (ii): $0 < \Omega < \Omega_0$

$$T(x_1) = \frac{iQC_g \cosh\left[Q\left(L_1 - x_1\right)\right] + \Omega \sinh\left[Q\left(L_1 - x_1\right)\right]}{iQC_g \cosh\left(QL_1\right) + \Omega \sinh\left(QL_1\right)} \tag{68}$$

$$R(x_1) = \frac{\Omega_0 \sinh\left[Q\left(L_1 - x_1\right)\right]}{iQC_g \cosh\left(QL_1\right) + \Omega \sinh\left(QL_1\right)} \tag{69}$$

where

$$Q = iP = \frac{\left(\Omega_0^2 - \Omega^2\right)^{\frac{1}{2}}}{C_g} \tag{70}$$

Now dependences on x_1 and L_1 are monotonic in this case of subcritical detuning.

Case (iii): $\Omega = 0$

$$T(x_1) = \frac{\cosh\left[\frac{\Omega_0}{C_g}\left(L_1 - x_1\right)\right]}{\cosh\left[\frac{\Omega_0}{C_g} L_1\right]}, R(x_1) = \frac{-i \sinh\left[\frac{\Omega_0}{C_g}\left(L_1 - x_1\right)\right]}{\cosh\left[\frac{\Omega_0}{C_g} L_1\right]} \tag{71}$$

Case (iv): $\Omega = \Omega_0$ At the cutoff frequency, we take $Q \to 0$ in (69) to get the solutions for $0 < x_1 < L_1$

$$T(x_1) = \frac{1 - i\left[\frac{\Omega_0}{C_g}\left(L_1 - x_1\right)\right]}{1 - i\frac{\Omega_0}{C_g} L_1}, R(x_1) = \frac{-i\frac{\Omega_0}{C_g}\left(L_1 - x_1\right)}{1 - i\frac{\Omega_0}{C_g} L_1} \tag{72}$$

Energy balance and reflection power: The wave energy is conserved,

$$R(0)^2 + T(L_1)^2 = 1 \tag{73}$$

For a perfect tuning,

$$|R(0)| = \tanh\left[\frac{\Omega_0}{C_g} L_1\right]. \tag{74}$$

As the zone of bars widens, $|R(0)| \to 1$ and there is complete reflection. The reflection increase with the number of bars under the cutoff frequency Ω_0, while an oscilatory behaviour is observed above Ω_0.

Remarks:

* The perturbation method with multiscale expansions of Mei (1985) anticipates the strong reflection by the sinusoidal patch by assuming the reflected wave to be of same order of the incident wave. If the sinusoidal bed extend remains small (several periods), the reflection is quite small, $R(0) = O(\epsilon)$, even near the resonance, and then $A^- \ll A^+$. Then at first order, (69) over the patch can be approximated by:

$$\left(\frac{\partial}{\partial t_1} \pm C_g \frac{\partial}{\partial x_1}\right) A^+ = 0 \tag{75}$$

and

$$\left(\frac{\partial}{\partial t_1} \pm C_g \frac{\partial}{\partial x_1}\right) A^- = -i\epsilon\Omega_0 A^+ \tag{76}$$

The reflection coefficient is for a frequency detuning of $\varepsilon\Omega = \varepsilon C_g \kappa$:

$$R(0) = \frac{\varepsilon\Omega_0}{2\Omega}\left[1 - e^{2i\kappa L_1}\right] \tag{77}$$

This result is valid only when $\kappa = O(1)$ is not small. However, when tuning is nearly perfect, i.e., $\kappa \ll 1$, we get in the limit $\Omega = C_g\kappa = 0$,

$$|R(0)| = \frac{\varepsilon\Omega_0 L_1}{C_g}. \tag{78}$$

which increases with increasing L_1. We can notice that the energy balance is only verified at order ε. For $\varepsilon L_1 = O(1)$, (78) is no more valid and the method anticipating the reflected wave may be used.

Let us note that the coefficient ε only indicates in the equations the order of the different terms, it may be omitted for the numerical applications. In a same way, the length $L_1 = L$ is the length of the modulated bed.

* For a sinusoidal bed such that

$$\delta = D \sin Kx \tag{79}$$

over the patch $0 < x < L = \frac{2m\pi}{K}$, where m is the number of periods, equation (78) becomes

$$|R(0)| = \frac{2kD}{2kh + \sinh(2kh)} \frac{m\pi}{2} \tag{80}$$

which is the result found by Davies (1982) and Davies and Heathershaw (1984) by using a regular perturbation method where the potential Φ is expressed $\Phi(x,z,t) = \varepsilon\Phi_1(x,z,t) + \varepsilon^2\Phi_2(x,z,t) + ...$ In this calculation, equations for the potential and its derivative are obtained at each order. The solution for Φ_1 is homogeneous, the bottom boundary condition for Φ_2 has a non-zero right-hand term over the rippled patch. Assuming that Φ_2 is bounded and also that Φ_2, and its first and second derivatives, tend to zero as $|x| \to \infty$, Fourier transform exists in x, and the mathematical problem can be solved by intruducing into the formulation a small amount of friction proportional to the relative velocity.

* We only considered here the case of a sinusoidal bed in waters of constant mean water depth, and two dimensional problems. Similar perturbation methods with multi-scales expansions where carried out in the presence of currents [17] or for other topographies as sinusoidal beds over sloping beds [22, 23] , doubly sinusoidal beds [28], pseudo-sinusoidal beds of modulated amplitude [24]. In these studies, first and higher order Bragg resonances were evidenced. It was shown [28] that even second order Bragg resonance can lead to strong reflection, since they can occur at low frequencies, for swallow water conditions where the bottom has a strong influence on the waves.

3.1.2 Mild slope and modified mild slope equations

Berkhoff (1976) derived a linear, elliptic equation taking into account the combined effects of reflection, diffraction and refraction of water waves on varying bathymetries. Details of the derivation are presented in Section 4.2 for 3D cases. For waves of normal incidence on one-dimensional bottom $h(x)$ of gentle slope ($\mu = \frac{dh/dx}{kh} \ll 1$), the velocity potential for a periodic wave may be expressed by the approximate form (considering the solution for a flat bottom):

$$\Phi(x,y,z,t) = \phi(x,z)e^{i\omega t} = F(h,z)\varphi(x)e^{i\omega t} = \frac{ig}{\omega}\frac{\cosh[k(z+h)]}{\sinh(kh)}\varphi(x)e^{i\omega t} \tag{81}$$

The reduced velocity potential $\phi(x,z)$ satisfies:

$$
\begin{aligned}
\frac{\partial^2 \phi}{\partial z^2} &= -\nabla_h^2 \phi \text{ for } 0 \geqslant z \geqslant -h \\
\frac{\partial \phi}{\partial z} - \frac{\omega}{g}\phi &= 0 \text{ for } z = 0 \\
\frac{\partial \Phi}{\partial z} &= \frac{dh}{dx}\frac{\partial \phi}{\partial x} \text{ for } z = -h(x)
\end{aligned} \tag{82}
$$

A weak formulation of the Laplace's equation is used, by integrating between $z = -h$ and $z = 0$ the Laplace's equation multiplied by the Green function $f = \cosh[k(z+h)]$. After applying the surface and bottom boundary conditions, and using the dispersion relation (9) and restricting to the terms of $O(\mu)$, we obtain the one-dimensional form of the Berkhoff or mild-slope equation

$$
\frac{d}{dx}\left(CC_g \frac{d\varphi}{dx}\right) + k^2 CC_g \varphi = 0 \tag{83}
$$

Booij (1983) numerically examined the validity of the mild-slope equation, and concluded that it is applicable with quite good accuracy for slopes up to $\tan\alpha = \frac{1}{3}$. This solution which assumes a gentle slope was extended by Kirby (1986) to rapid undulations. For modulated bottom shapes $\delta(x)$ of constant mean depth, the right-hand term of the bottom condition takes the form (51), and CC_g remains constant at the order considered. The "modified" mild-slope equation takes then the form[16, 20]:

$$
\frac{d}{dx}\left(\frac{d\varphi}{dx}\right) + k^2\varphi - \frac{g}{CC_g \cosh^2 kh}\frac{d}{dx}\left(\delta \frac{d\varphi}{dx}\right) = 0 \tag{84}
$$

Let us note that the expression for the velocity potential (81) does not take into account the evanescent modes which were found to be negligible in the mild-slope equation (Smith et al, 1975). The role of the evanescent modes will be introcuced in the next section.

Equation (84) is then solved numerically with the boundary conditions at both ends of the undulting bed:

$$
\frac{d\varphi}{dx} = ik(\varphi - 2\varphi_I) \tag{85}
$$

upwave where $\varphi_I = e^{-ikx}$ is the incident wave of unit amplitude and, assuming only a progressive wave on the down-wave side

$$\frac{d\varphi}{dx} = -ik\varphi \tag{86}$$

The reflection coefficient R may be evaluated by writting the reflected wave $\varphi_R = Re^{+ikx}$ and using the expression $\varphi = \varphi_I + \varphi_R = e^{-ikx} + Re^{+ikx}$ upwave the computational grid.

3.2 Case of abrupt bathymetries

For steep slopes or in the presence of coastal structures with vertical boundaries, the mild slope approaches are no more valid. In the presence of vertical boundaries, a classical method is based on a general formulation of the velocity potential in each domain of constant depth. Solutions for domains of finite or semi-infinite extend along the wave propagation direction include "evanescent modes" which exist whatever the wave direction in addition to the well-known evanescent waves due to oblique wave incidence which can appear for oblique incidence at the interface of two successive media of different indices (or wave celerities). The solution of the problem is solved by use of integral matching method for the boundaries conditions between successive domains. This method can be applied for smooths beds by discretizing the bed into a series of narrow shelves (see for instance Guazzelli et al, 1992; Gouaud et al, 2010) even if the solution is not valid near the bed as pointed out by Athanassoulis and Belibassakis(1999) who proposed an additional term in the velocity potential expression to take into account the local slope. Even if only 2D cases are considered in the present part, the method presented in Rey (1995), which is valid for obliquely incident waves in the horizontal $x0y$ plane in the presence of bathymetric shape of the form $h = h(x)$, is summerized in the following section.

3.2.1 General expression of the velocity potentials

The propagation of obliquely incident waves on one-dimensional bottom topographies or in presence of cylindrical obstacles can be studied at first order by using the linearized potential theory. Details on the method and detailed bibliography can be found in Rey (1992, 1995). We consider a varying bottom made of $(N + 1)$ regions labelled m $(m = 0, ..., N)$ of constant depths.

The coordinate system (O, x, y, z) is chosen so that $z = 0$ at the free surface (z positive upwards), $x = 0$ for the first step, and $x = x_m$ for each step m. We consider a plane surface wave whose direction of propagation forms an angle θ_0 with the x-axis. The bottom is so as $h = h(x)$ and the solid obstacles are cylindrical, with a boundary surface $S = S(x)$.

For each rectangular domain of constant depth h, with a free surface condition, the (complex) velocity potential $\Phi(x, y, z, t)$ can be written [Kirby and Dalrymple, 1983]:

$$\begin{aligned} \Phi(x,y,z,t) &= \phi(x,z) e^{i(\omega t - k_y y)} \\ &= \left[\left[A^- e^{-ik_x x} + A^+ e^{+k_x x}\right] \chi(z) + \underbrace{\sum_{n=1}^{\infty} \left[B_n^- e^{-k_{xn} x} + B_n^+ e^{+k_{xn} x}\right] \psi_n(z)}_{evanescent} \right] e^{i(\omega t - k_y y)} \end{aligned} \quad (87)$$

where n a priori infinite, but troncated at some order $n = P$, and

$$\chi(z) = \cosh k(h+z) \quad (88)$$

$$\psi_n(z) = \cos k_n(h+z) \quad (89)$$

k and k_n $(n = 1, ..., P)$ are respectively given by:

$$\frac{\omega^2}{g} = K = k \tanh(kh), \quad (90)$$

$$\frac{\omega^2}{g} = K = -k_n \tan(k_n h) \quad (91)$$

The evanescent modes have a physical signification for finite or semi-infinite domains of constant depth. For the numerical procedure, it is convenient to write $\chi(z) = \cosh k(h+z) = \psi_0(z) = \cos k_0(h+z)$, with $k_0 = ik$. Since $\nabla^2 \Phi = 0$, the components of the wavenumbers along the x-axis verify:

$$\begin{aligned} k_x &= \left(k^2 - k_y^2\right)^{\frac{1}{2}} \\ k_{nx} &= \left(k_n^2 + k_y^2\right)^{\frac{1}{2}} \end{aligned} \quad (92)$$

The k_y value is a constant, which depends only on the incident surface wave characteristics: $k_y = k_0 \sin\theta_0$, where the index 0 relates to the incident wave (see section 4.1).

The general expression of the potential $\Phi(x, y, z, t)$ between two solid horizontal boundaries at $y = h_1$ and $y = h(h > h_1)$ satisfying the phase matching condition along the z-axis is given, for a oblique incidence ($k_y \neq 0$), by:

$$\Phi(x,y,z,t) = \left[A^{\mp} e^{\mp k_y x}\chi(z) + \sum_{n=1}^{\infty}\left[B_n^- e^{-k_{xn}x}\psi_n(z) + B_n^+ e^{+k_{xn}x}\psi_n(z)\right]\right] e^{i(\omega t - k_y y)} \tag{93}$$

with

$$\chi(z) = 1 \tag{94}$$

$$\psi_n(z) = \cos k_n(h + z) \tag{95}$$

where k_n, for the evanescent modes are determined from $k_n = \frac{n\pi}{h-h_1}$, and $k_{xn} = \left(k_n^2 + k_y^2\right)^{\frac{1}{2}}$. For normal incidence, ($k_y = 0$), $\Phi(x, y, z, t)$ becomes [Takano, 1960]:

$$\Phi(x,y,z,t) = \left[(A^- x + A^+)\chi(z) + \sum_{n=1}^{\infty} B_n^- e^{-k_n x}\psi_n(z) + B_n^+ e^{+k_n x}\psi_n(z)\right] e^{i\omega t} \tag{96}$$

The functions χ, ψ_n and the wavenumber k_n are similar to those found for oblique incidence.

Note that expressions (87), (93) and (96) have similar forms and can be written as $\Phi(x, y, z, t) = \phi(x, z)e^{i(\omega t - k_y y)}$, which is of practical interest in the numerical processing.

3.2.2 Integral matching conditions method

We use the method based on an integral matching along vertical boundaries between successive rectangular domains for the numerical resolution of the continuity equations at the vertical boundaries for both fluid velocity and pressure. The complex expressions allow not to distinguish propagative and evanescent modes in the equations. If the wavenumber k for the semi-infinite down-wave region is such that $k < k_y$, the incident wave is totally reflected. For the opposite case, both reflection and transmission coefficients along the x-axis are defined as for a normal incidence case since the matching conditions are applied to the reduced form $\phi(x, z)$.

Let us consider the particular case of two domains labelled 1 and 2 of respective water depths h_1 and h_2 ($h_1 > h_2$) separated by the surface $x = x_m$. The dynamic pressure is given by (18), and the horizontal velocities are given by $\frac{\partial \phi_i}{\partial x}$. Matching conditions to ensure continuity of both fluid velocity and surface elevation between the successive domains are as follows for respectively pressures and velocities:

$$\int_{-h_2}^{0} \phi_1.\psi_{2,n} dz = \int_{-h_2}^{0} \phi_2.\psi_{2,n} dz \tag{97}$$

and

$$\int_{-h_1}^{0} \frac{\partial \phi_1}{\partial x}.\psi_{1,n} dz = \int_{-h_2}^{0} \frac{\partial \phi_2}{\partial x}.\psi_{1,n} dz \tag{98}$$

for $n = 0, .., P$.

For N steps, or $(N+1)$ domains labelled d, $d = 0, .., N$, $2N(P+1)$ equations of the forms (97) and (98) are written. For the semi-infinite domains, the non-divergency when $x \to \infty$ imposes $B_{0,n}^{-}$ and $B_{N,n}^{+} = 0$, whatever $n > 0$. A_0^{-} is the coefficient for the incident wave, and $A_N^{+} = 0$ if we assume no beach reflection. Let us note that the beach reflection may have a significant impact on the reflected wave (see for instance Rey and Touboul, 2011). The $2N(P+1)$ unknown (complex) coefficients are then obtained by solving the $2N(P+1)$ complex linear equations. Reflection and transmission coefficient R et T are given by :

$$R = \frac{\left|A_0^{+}\right|}{\left|A_0^{-}\right|}, T = \frac{\left|A_N^{+}\right|}{\left|A_0^{-}\right|} \frac{\chi_N(0)}{\chi_0(0)} \tag{99}$$

Energy flux conservation can be used as a numerical test:

$$R^2 + T^2 \left[\frac{n_N k_0^2 k_{Nx}}{n_0 k_N^2 k_{0x}} \right] = 1 \tag{100}$$

with

$$n_i = \frac{1}{2} \left[1 + \frac{2 k_i h_i}{\sinh(2 k_i h_i)} \right] \tag{101}$$

$i = 0, N$. One can verify the rapid convergency of the coefficients as the number P of evanescent modes increases.

4 Wave scattering and focusing : 3D case

In 3D cases, the wave direction of propagation is modified when the celerity is no more constant along the wave crest. As shown in Section 2, wave celerity depends on the water depth h. Bathymetric changes are then at the origin of wave refraction or diffraction (in this latter case, plane wave approximation no more valid). Let us note that the wave celerity C may also change in the presence of currents or in the presence of porous structures.

In the presence of a constant current of the form $\vec{u} = (U, 0, 0)$, the dispersion relation for a linear surface wave propagating in the direction $0x$ is given by:

$$(\omega - Uk)^2 = \sigma^2 = gk \tanh(kh) \tag{102}$$

where ω is the wave frequency in a fixed frame, and σ is the wave frequency in a moving frame at the velocity U. For wave-following current conditions, the wave wavelength is increased. In wave-opposing current conditions, the wavelength decreases down to a limit when there is no more wave propagation. In deep water conditions, the linearized theory shows that this limit is reached for $|U| = C_0/4$, where C_0 is the wave celerity in the absence of current. In fact the wave amplitude increases up to breaking before this theoretical limit assuming small amplitude waves, due to the increase of wave steepness for increasing wave-opposing current conditions.

In the presence of porous on a whole water column, the dispersion relation can be approximated by (Yu & Chwang, 1994):

$$Z\omega^2 = igk \tanh(kh) \tag{103}$$

where $Z = f_R + iS$ is the dimensionless impedance of the porous medium, f_R is a linearised resistance coefficient and S is a reactance,

$$S_r = 1 + C_M \frac{(1-\gamma)}{\gamma} \tag{104}$$

S_r depends on the porosity γ, and C_M, an added mass coefficient of the medium grains. The solution of Eq. (103) has a complex form $k = k_r + ik_i$ and the wave attenuation is related to its wavenumber through the dispersion relation. If damping is neglected, dispersion relation takes the form

$$S_r\omega^2 = gk \tanh(kh) \tag{105}$$

Since $S_r > 1$, wavelength (and celerity) decrease within the porous media.

4.1 Refraction - Snell - Descartes' Law

In the nearshore, when both time and space scales are small compared to the media variations, the local properties of a wave train can be expressed as follows:

$$\eta = a\exp(iS)$$

where a is the local wave amplitude and $S(\vec{x}, t, \vec{k}, \omega)$ its phase. For varying bathymetries, or non-uniform currents, S depends slowly on time and space coordinates. In the plane wave approximation, one can define the local wavenumber $\vec{k}$ and the angular frequency ω:

$$\vec{k} = -\vec{\nabla}_h S \text{ and } \omega = \frac{\partial S}{\partial t} \tag{106}$$

where $\vec{\nabla}_h \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right)$ is the horizontal gradient. Eliminating S, one obtains

$$\frac{\partial \vec{k}}{\partial t} + \vec{\nabla}_h \omega = 0 \tag{107}$$

In the $x0y$-plane, a wave ray is a line which is tangent to the local wave number $\vec{k}$ at any point. The velocity potential $\Phi(x, y, z, t)$, for a linear wave of local wavenumber $\vec{k}(k_1 = k\cos\theta; k_2 = k\sin\theta)$, where θ is the wave angle with respect to the x-axis, is given by:

$$\Phi(x, y, z, t) = \frac{a\omega}{k}\frac{\cosh[k(z+h)]}{\sinh(kh)}\cos S \tag{108}$$

with $S = \omega t - k_1 x - k_2 y$. $k = \left\|\vec{k}\right\|$ verifies the dispersion relation (9). From Eq. (106), one obtains the eikonal equation

$$\left(\frac{\partial S}{\partial x}\right)^2 + \left(\frac{\partial S}{\partial y}\right)^2 = \left(\vec{\nabla}_h S\right)^2 = k^2 \tag{109}$$

and then

$$\frac{\partial(k\sin\theta)}{\partial x} - \frac{\partial(k\cos\theta)}{\partial y} = 0 \tag{110}$$

If water depth $h = h(x)$, the y-component of $\vec{k}$ is constant, and Eq. (110) reduces to

$$\frac{d(k\sin\theta)}{dx} = 0$$

Hence, $k_2 = k\sin\theta = k_0\sin\theta_0 = const$, where index 0 refers to an initial condition. We then obtain the Descartes-Snell's law

$$\frac{\sin\theta}{C} = \frac{\sin\theta_0}{C_0} \tag{111}$$

which shows that θ decreases as water depth h decreases. The wave evolution from deep water up to the shoreline can be calculated step by step along wave rays. If wave propagates from shallow waters to deeper waters, total reflection occurs above a critical value of the incident angle (see Kirby and Dalrymple, 1983). If bathymetric changes are too rapid, wave rays may cross leading to "caustics". The present "ray theory" is no more valid and diffraction effects must be taken into account for the wave propagation.

4.2 Refraction-Diffraction

The impermeability bottom condition for varying topograhies with water depth $h(x, y)$, writes

$$\frac{\partial\Phi}{\partial z} = \vec{\nabla}_h h \vec{\nabla}_h \Phi \tag{112}$$

For gentle slope variations defined by $\mu = \frac{\nabla h}{kh} \ll 1$, the velocity potential for a periodic wave may be expressed by the approximate form (considering the solution for a flat bottom):

$$\Phi(x, y, z, t) = \phi(x, y, z)e^{i\omega t} = \frac{ig}{\omega}\frac{\cosh[k(z+h)]}{\cosh(kh)}\eta(x, y)e^{i\omega t} \tag{113}$$

The velocity potential ϕ satisfies:

$$\begin{aligned} \frac{\partial^2\phi}{\partial z^2} &= -\nabla_h^2\phi \text{ for } 0 \geqslant z \geqslant -h(x, y) \\ \frac{\partial\phi}{\partial z} - \frac{\omega}{g}\phi &= 0 \text{ for } z = 0 \\ \frac{\partial\Phi}{\partial z} &= \vec{\nabla}_h h \vec{\nabla}_h \phi \text{ for } z = -h(x, y) \end{aligned} \tag{114}$$

A weak formulation of the Laplace's equation by integrating between $z = -h$ and $z = 0$ the Laplace's equation multiplied by the Green function $f = \cosh[k(z + h)$ gives, after applying the surface and bottom boundary conditions and using the dispersion relation:

$$\int_{-h}^{0} \left(k^2 f\phi + \nabla_h^2 \phi f\right) dz = \left[(\nabla_h h \nabla_h \phi)\, f\right]_{z=-h} \tag{115}$$

writting

$$\phi(x, y, z) = \frac{ig}{\omega} \frac{\cosh[k(z + h)]}{\cosh(kh)} \eta(x, y) \tag{116}$$

and using the expressions (see Mei, 1983)

$$\vec{\nabla}_h \phi = \frac{ig}{\omega} \left[f \vec{\nabla}_h \eta + \eta \frac{\partial f}{\partial h} \vec{\nabla}_h h \right] \tag{117}$$

$$\nabla_h^2 \phi = \frac{ig}{\omega} \left[f \vec{\nabla}_h^2 \eta + 2 \frac{\partial f}{\partial h} \vec{\nabla}_h \eta \vec{\nabla}_h h + \eta \frac{\partial^2 f}{\partial h^2} \left[\vec{\nabla}_h h\right]^2 + \eta \frac{\partial f}{\partial h} \vec{\nabla}_h^2 h \right] \tag{118}$$

Eq. (115) becomes, after multiplication by $\frac{ig}{\omega}$:

$$\int_{-h}^{0} \left(f^2 \vec{\nabla}_h^2 \eta + 2f \frac{\partial f}{\partial h} \vec{\nabla}_h \eta \vec{\nabla}_h h + \eta f \frac{\partial^2 f}{\partial h^2} \left[\vec{\nabla}_h h\right]^2 + \eta f \frac{\partial f}{\partial h} \vec{\nabla}_h^2 h + k^2 \eta f^2 \right) dz$$
$$= -\left[f^2 \vec{\nabla}_h h \vec{\nabla}_h \eta + \eta f \frac{\partial f}{\partial h} \left(\vec{\nabla}_h h\right)^2 \right]_{z=-h} \tag{119}$$

After use of the Leibnitz formula

$$\frac{\partial}{\partial x} \int_{a(x)}^{b(x)} f(x, y) dy = \int_{a(x)}^{b(x)} \frac{\partial f}{\partial x} dy + f(x, b(x)) \frac{\partial b}{\partial x} - f(x, a(x)) \frac{\partial a}{\partial x} \tag{120}$$

one obtains

$$\vec{\nabla}_h . \int_{-h}^{0} f^2 \vec{\nabla}_h \eta dz + \int_{-h}^{0} k^2 f^2 \eta dz \tag{121}$$
$$= -\left[\eta f \frac{\partial f}{\partial h} \left(\vec{\nabla}_h h\right)^2 \right]_{z=-h} - \int_{-h}^{0} \eta f \frac{\partial^2 f}{\partial h^2} \left[\vec{\nabla}_h h\right]^2 dz - \int_{-h}^{0} \eta f \frac{\partial f}{\partial h} \vec{\nabla}_h^2 h dz$$

Restricting to the terms of $O(\mu)$, since $\frac{\vec{\nabla}_h h}{kh} = O(\mu)$ and $O(\vec{\nabla}_h) = O(k)$, right-hand terms of Eq. (121), of order $O(\mu^2)$ can be neglected. After integration with respect to z, one obtains the Berkhoff or mild-slope equation (see Mei, 1983):

$$\vec{\nabla}_h.\left(a_1\vec{\nabla}_h\eta\right) + \omega^2 a_2\eta = 0 \tag{122}$$

with

$$a_1 = gh\frac{\tanh(kh)}{kh}\frac{1}{2}\left[1+\frac{2kh}{\sinh 2kh}\right] \tag{123}$$

and

$$a_2 = \frac{1}{2}\left[1+\frac{2kh}{\sinh 2kh}\right] \tag{124}$$

since $C = \frac{\omega}{k} = \sqrt{\frac{g}{k}\tanh(kh)}$ and $C_g = \frac{\partial\omega}{\partial k} = \frac{1}{2}C\left[1+\frac{2kh}{\sinh 2kh}\right]$, Eq. (122) becomes,

$$\vec{\nabla}_h.\left(CC_g\vec{\nabla}_h\eta\right) + k^2 CC_g\eta = 0 \tag{125}$$

In eq. (113), $\varphi(x, y)$ represents the (complex) local amplitude of the wave. For a progressive wave, one can define $H(x, y) = 2\eta(x, y)$, the complex representation of the crest-to-trough height of the surface elevation. H can be written (see Jarry et al, 2011)

$$H = \hat{H}e^{iS} \tag{126}$$

where $\hat{H}(x, y)$ is the height envelope and $S(x, y)$ its phase. Berkhoff equation (125) can be rewritten:

$$\vec{\nabla}.\left(CC_g\vec{\nabla}_h H\right) + k^2 CC_g H = 0 \tag{127}$$

Taking the mild-slope equation, following the height envelope and phase, one obtains after the separation of real and imaginary parts:

$$\left(\vec{\nabla}_h S\right)^2 = k^2 + \frac{\vec{\nabla}_h.\left(CC_g\vec{\nabla}_h\hat{H}\right)}{CC_g\hat{H}} \tag{128}$$

and

$$\vec{\nabla}_h.\left(CC_g\hat{H}^2\vec{\nabla}_h S\right) = 0 \tag{129}$$

In the pure refraction case, where the amplitude variation is considered as negligible, Eq. (128) leads to the eikonal equation $k^2 = \left(\vec{\nabla}S\right)^2$ (Eq. 109). When diffraction effect becomes preponderant, the second term of the right-hand-side of Eq. (128) can not be neglected since it cannot be directly assimilated to the wave number of a progressive wave. Writing

$$k' = \vec{\nabla}S$$

and introducing this relation in Eqs (128) and (129), one obtains the following diffraction parameter (Holtuijsen and al., 2003):

$$\delta_H = \frac{\vec{\nabla}_h . \left(CC_g\hat{H}^2\vec{\nabla}_h S\right)}{k^2 CC_g\hat{H}} \tag{130}$$

This parameter, which can be either positive or negative, indicates that in the presence of diffraction effect, the wave number, the wave phase and group velocities are modified as follows:

$$\begin{aligned} k' &= k\sqrt{1+\delta_H} \\ C' &= C/\sqrt{1+\delta_H} \\ C'_g &= C_g/\sqrt{1+\delta_H} \end{aligned} \tag{131}$$

where k', C and C_g respectively the modified wave number, phase and group velocities in the presence of diffraction.

Chamberlain and Porter (1995) extended Berkhoff's equation to include the effect of local modes. More recently, Athanassoulis and Belibassakis (1999) improved the equation by including a bottom mode in the local mode decomposition. In the framework of coastal wave propagation modelling, Booij(1981)and Liu (1983) were the first authors to extend Berkhoff's equation, allowing to take wave - current interactions into account in the presence of arbitrary bathymetric variations. In these formulations, the current was assumed to vary horizontally, presenting a uniform vertical structure. However, both of these equations neglected some terms describing the horizontal variation of the currents. A full formulation of the problem was finally introduced by Kirby (1984). An extension of this equation, taking into account the linear variation of the current with depth, which results in a constant horizontal vorticity, slowly varying horizontally, within the background current field was recently proposed by Touboul et al (2016).

4.3 Diffraction

For constant water depth h, wave diffraction occurs in the presence of walls (dykes, cliffs). For a small amplitude wave, the velocity potential $\Phi(x, y, z, t)$ is given by:

$$\Phi(x, y, z, t) = \frac{a\omega}{k}\frac{\cosh[k(z+h)]}{\sinh(kh)}F(x, y)e^{i\omega t} \tag{132}$$

where F verifies Helmotz's equation

$$\nabla_h^2 F + k^2 F = 0 \tag{133}$$

and k verifies the dispersion relation (9).

4.3.1 Analytic solution : Semi-infinite dike

For a semi-infinite obstacle along the $x-$axis, for $y > 0$, the analytic solution for an incident wave of propagation direction α with respect to $x-$axis, is given by (see Penney and Price, 1952; Horikawa, 1988)

$$F(r, \theta) = I\left(-\sqrt{\frac{4kr}{\pi}}\sin\frac{\alpha-\theta}{2}\right)e^{ikr\cos(\alpha-\theta)} + I\left(-\sqrt{\frac{4kr}{\pi}}\sin\frac{\alpha+\theta}{2}\right)e^{-ikr\cos(\alpha+\theta)} \tag{134}$$

in the polar coodinate, where r is the distance to point 0, and θ the angle with respect to $x-$axis. $I(u) = \frac{1}{2}(1+i)\int_{-\infty}^{u} e^{-i\frac{\pi u'}{2}}du'$.

4.3.2 Channels of finite width

In the case of closed basins or semi-closed basins, oscillations called "seiching" may be observed.

In the case of vertical boundaries, resonant conditions are fulfilled for standing waves if antinodes, which correspond to vertical fluid velocity, are present at the boundaries. In the particular case of a rectangular closed basin, $0 < x < a$ and $0 < y < b$, of constant water depth h, the general solutions for the free surface verifying the boundary conditions are of the form

$$\eta_1 = a_1 \cos \frac{n\pi x}{a} \tag{135}$$
$$\eta_2 = a_2 \cos \frac{m\pi y}{b}$$

for $\eta(x, y, t) = \eta_0(x, y)e^{i\omega t} = \eta_1(x)\eta_2(y)e^{i\omega t}$.The surface deformation is the sum of the particular solutions: Since $\eta_0(x, y)$ verifies the Helmotz equation (133), one obtains

$$k = k_{n,m} = \sqrt{\left(\frac{n\pi}{a}\right)^2 + \left(\frac{m\pi}{b}\right)^2} \tag{136}$$

Periods $T_{n,m}$ are then given by

$$T_{n,m} = \frac{2\pi}{\omega_{n,m}} \text{ with } \omega_{n,m}^2 = gk_{n,m} \tanh(k_{n,m}h) \tag{137}$$

If $a > b$, the harmonic $(n = 1, m = 0)$ is the fundamental mode.

If we consider an incoming wave in the $x-$direction in a channel of constant depth h of varying finite width d, a general form of the potential can be written for a domain of given constant width. The solution of the problem is solved by use of integral matching method for the boundaries conditions between successive domains.

General expressions for the velocity potential within domains of finite width .

In the linear problem, a general expression of the velocity potential in a cartesian frame $\Phi(x, y, z, t)$ for a regular wave of angular frequency ω propagating over a constant water depth h is of the form

$$\Phi(x, y, z, t) = \frac{\alpha\omega}{k} \frac{\cosh[k(z + h)]}{\sinh(kh)} F(x, y)e^{i\omega t} + cc. \tag{138}$$

where α is the amplitude of free surface elevation. Wavenumber k verifies the dispersion relation:

$$\omega^2 = gk\tanh(kh) \tag{139}$$

Considering the fluid domain to be bounded along y-axis in the interval $d_m < y < d_M$, with $d = d_M - d_m$ the width of the channel, velocity potential (138) takes the form

$$\Phi(x, y, z, t) = \cosh[k(z + h)]\phi(x, y)e^{i\omega t} + cc. \tag{140}$$

with the free surface potentiel $\phi(x, y)$ given by

$$\phi(x, y) = \sum_{n=0}^{\infty} \left[A_n^- e^{-ik_{xn}x} + A_n^+ e^{+ik_{xn}x}\right] \psi_n(y) \tag{141}$$

where

$$\psi_n(y) = \cos\left[k_{yn}(y - d_m)\right] \tag{142}$$

and

$$k_{yn} = \frac{n\pi}{d_M - d_m} = \frac{n\pi}{d} \tag{143}$$

since no flux conditions are required at $y = d_m$ and $y = d_M$.

Since $\Delta\Phi = 0$, where $\Delta = \left(\frac{\partial^2}{\partial x^2}, \frac{\partial^2}{\partial y^2}, \frac{\partial^2}{\partial z^2}\right)$,

$$k_{xn} = \left(k^2 - k_{yn}^2\right)^{\frac{1}{2}} \tag{144}$$

For $n = 0$, $k_{xn} = k$, and k_{xn} is purely imaginary for $k_{yn} > k$. The number of propagating modes n_{prop} is then given by

$$n_{prop} = 1 + Int\left[\frac{kd}{\pi}\right] \tag{145}$$

with Int denoting the integer part. For propagating modes n along x-axis, $k_{xn} = k\cos\theta_n$ and $k_{yn} = k\sin\theta_n$, where θ_n is the angle of propagation with respect to the x-axis at given mode n. For narrow channels or long waves ($k < \frac{\pi}{d_M - d_m}$), only the first mode propagates along x-axis, the other are evanescent along x-axis. For the numerical processing, general expression of the velocity potential is truncated at a given order $n = P$, the first $(P + 1)$ modes are hence considered (see Rey et al, 2018).

For two successive domains labelled 1 and 2, of respective widths $d_1 = d_{M1} - d_{m1}$ ($d_{m1} < y < d_{M1}$) and $d_2 = d_{M2} - d_{m2}$ ($d_{m2} < y < d_{M2}$), velocity and pressure matching conditions at their interface $x = x_0$ are the following :

$$\left\{ \begin{array}{c} \phi_1 = \phi_2 \, for \, \max(d_{m1}, d_{m2}) \leqslant y \leqslant \min(d_{M1}, d_{M2}) \\ \frac{\partial \phi_1}{\partial x} = \frac{\partial \phi_2}{\partial x} \, for \, \max(d_{m1}, d_{m2}) \leqslant y \leqslant \min(d_{M1}, d_{M2}) \\ \frac{\partial \phi_j}{\partial x} = 0 \, for \, d_{mj} \leqslant y \leqslant \max(d_{m1}, d_{m2}) \, and \, \min(d_{M1}, d_{M2}) \leqslant y \leqslant d_{Mj} \, , \, j = 1 \, , \, 2 \end{array} \right. \tag{146}$$

Functions $\psi_{j,m}, \psi_{j,n}$ form an orthogonal system for the scalar product

$$< f_j \mid g_j >= \int_{d_{mj}}^{d_{Mj}} f_j . g_j dy \tag{147}$$

Taking advantage of the orthogonal set of functions $\psi_{j,m}$, interface conditions are written in an integral form, for any n. In the case $d_{m1} \leqslant d_{m2}$ and $d_{M2} \leqslant d_{M1}$, conditions are given by

$$\int_{d_{m2}}^{d_{M2}} \phi_1 . \psi_{2,n} dy = \int_{d_{m2}}^{d_{M2}} \phi_2 . \psi_{2,n} dy \tag{148}$$

and

$$\int_{d_{m1}}^{d_{M1}} \frac{\partial \phi_1}{\partial x} . \psi_{1,n} dy = \int_{d_{m2}}^{d_{M2}} \frac{\partial \phi_2}{\partial x} . \psi_{1,n} dy \tag{149}$$

for $n = 0, .., P$.

In the following, we will only considerer domains of single channel thanks to the symetries along $y-$axis. For multiple channels, one may write in the integral form pressure continuity condition along the boundary (in the $y-$direction) between two successive domains. The velocity condition is written along the whole channel width, through integral form taking into account both the velocity condition between successive domains and the absence of velocity normal to the boundaries. The method is similar to the method used in the two-dimensional vertical case for a submerged obstacle (see for instance, Rey, 1995; Rey et al, 2011). Similar method has been used by Belibassakis et al, 2014 to study tridimensional diffraction in the vicinity of openings in coastal structures.

Numerical formulation : Construction of the matricial system .

Let us consider M domains, separated by $N = M-1$ discontinuities along the x-axis, x_j, $j = 0,..,N-1$. Domain D_0 is defined for $x < 0$ and $d_m^{(0)} \leqslant y \leqslant d_M^{(0)}$, domain D_M is for $x \geqslant 0$ and $d_m^{(M)} \leqslant y \leqslant d_M^{(M)}$ where superscript (j) refers to domain j. The discontinuities delimit $(N-1)$ segments S_j of length $L_j = x_{j+1} - x_j$. For each segment j, domains labelled D_j are defined, bounded along $y - axis$ by $y = d_m^{(j)}$ and $y = d_M^{(j)}$, $d_M^{(j)} > d_m^{(j)}$. For domain D_0, the absence of incident wave of oblique incidence or the absence of divergency of the evanescent modes when $x \rightarrow -\infty$ leads to the following general expression for the free surface potential:

$$\phi^{(0)}(x,y) = A_0^{(0)-} e^{-ik_{x0}^{(0)}x} \psi_0^{(0)}(y) + \sum_{n=0}^{P} \left[A_n^{(0)+} e^{+ik_{xn}^{(0)}(x)} \right] \psi_n^{(0)}(y) \; for \, x \leqslant 0 \tag{150}$$

where $A_0^{(0)-}$ is the coefficient for the incoming wave, $A_n^{(0)-}$, $n = 0,..,(n_{prop}-1)$ the coefficients for the scattered waves ($n = 0$ corresponds to the reflected wave, opposite to the incoming wave), and $A_n^{(0)-}$, $n > (n_{prop}-1)$ the coefficients for the evanescent waves. For domain D_M, the absence of reflected wave of any incidence or the absence of divergency of the evanescent modes when $x \rightarrow \infty$ leads to the following general expression

$$\phi^{(M)}(x,y) = \sum_{n=0}^{P} \left[A_n^{(M)-} e^{-ik_{xn}^{(M)}(x-x_N)} \right] \psi_n^{(M)}(y) \; for \, x \geqslant L \tag{151}$$

$A_n^{(M)-}$, $n = 0,..,(n_{prop}-1)$ are the coefficients for the scattered waves ($n = 0$ corresponds to the transmitted wave along the incoming wave direction), and $A_n^{(M)-}$, $n > (n_{prop}-1)$ the coefficients for the evanescent waves. $(P+1)$ complex coefficients are then to be calculated for each of these semi-infinite domains. General expressions for the free surface potentials are the following for domains D_j, $j = 1,..,N-1$:

$$\phi^{(j)}(x,y) = \sum_{n=0}^{P} \left[A_n^{(j)-} e^{-ik_{xjn}(x-x_j)} + A_n^{(j)+} e^{+ik_{xj}^{(j)}(x-x_{j+1})} \right] \psi_n^{(j)}(y) \; for \, x_j < x < x_{j+1} \tag{152}$$

where $A_n^{(j)\pm}$are the $2(P+1)$ unknown complex coefficients for each bounded domain D_j. The $2N(P+1)$ unknown coefficients are then numerically solved thanks to the $2N(P+1)$ matching conditions written at each abscissa $x_j, j = 1, ..., N$ derived from Eq. (150), (151) and (152).

Energy conservation .

The mean energy flux across $y0z$ plane is given by:

$$E_t = \frac{1}{T}\int_t^{t+T}\int_{y_m}^{y_M}\int_{x=-h}^{\eta} p\vec{v}.\vec{n}dtdydz \tag{153}$$

where $\vec{n}$ is a unit vector normal to $y0z$ and $p = -\rho\frac{\partial\Phi}{\partial t}$, $\vec{v}.\vec{n} = \frac{\partial\Phi}{\partial x}$. In the absence of dissipation, conservation of wave energy flux along the direction of propagation $0x$ takes the form:

$$\frac{k_{x0}^{(0)}}{k}\left|A_0^{(0)-}\right|^2 = \frac{k_{x0}^{(0)}}{k}\left|A_0^{(0)+}\right|^2 + \frac{k_{x0}^{(M)}}{k}\left|A_0^{(M)+}\right|^2 + \frac{1}{2}\sum_{n=1}^{n_{prop}-1}\left[\frac{k_{xn}^{(0)}}{k}\left|A_n^{(0)+}\right|^2 + \frac{k_{xn}^{(M)}}{k}\left|A_n^{(M)+}\right|^2\right] \tag{154}$$

Defining the following reflection and transmission coefficients for modes n

$$R_n = \left|\frac{A_n^{(0)+}}{A_0^{(0)-}}\right| \; and \, T_n = \left|\frac{A_n^{(M)+}}{A_0^{(0)-}}\right| \tag{155}$$

and assuming the same values for the boundaries y_m and y_M for both incident wave and transmitted wave domains, expression (154) becomes,

$$1 = R_0^2 + T_0^2 + \frac{1}{2}\sum_{n=1}^{n_{prop}-1}\frac{k_{xn}^{(0)}}{k}\left[R_n^2 + T_n^2\right] \tag{156}$$

which reduces to the classical energy conservation condition $1 = R_0^2 + T_0^2$ when only the first mode ($n = 0$) is propagative. Reflected energy coefficient for the first mode is defined as the ratio between the reflected and incident energy, it is given by

$$E_{R0} = R_0^2 \tag{157}$$

Reflected energy coefficient for mode $n > 1$ is

$$E_{Rn} = \frac{1}{2}R_n^2\frac{k_{xn}^{(0)}}{k}E_I \tag{158}$$

The reflected energy flux remains lower than the incident energy flux. Since $k_{xn}^{(0)} < k$, the reflected coefficient R_n defined in (155) can be higher than 1. The total reflected energy coefficient E_R is given by

$$E_R = \sum_{n=0}^{n_{prop}-1} E_{Rn} \tag{159}$$

Energy flux conservation (Eq. 154) is used as a control parameter for the numerical computations.

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