Divergence-free positive symmetric tensors

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Applications to Fluid dynamics

and other models of Mathematical Physics

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The goal of these lectures is to present and justify new basic estimates for compressible fluids.

For instance, an inviscid gas in space dimension n obeys an inequality

$$\int_0^\infty dt \int_{\mathbb{R}^n} (\rho^{\frac{1}{n}} p)(t,y)\, dy \leqslant d_n M_0^{\frac{1}{n}} \sqrt{D_0},$$

where M_0 is the total mass, ρ the mass density, p the pressure, and

$$D_0 := \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \rho_0(y)\rho_0(y') \left(\frac{1}{4}|v_0(y') - v_0(y)|^2 + e_0(y)\right) dy\, dy'.$$

1st part

Divergence-free symmetric tensors in Mathematical Physics

An academic example

Burgers equation

$$\partial_t u + \partial_y \frac{u^2}{2} = 0.$$

Classical solutions satisfy also an entropy *equality*

$$\partial_t \frac{u^2}{2} + \partial_y \frac{u^3}{3} = 0.$$

Both identites write together as

$$\begin{pmatrix} \partial_t & \partial_y \end{pmatrix} \begin{pmatrix} u & \frac{u^2}{2} \\ \frac{u^2}{2} & \frac{u^3}{3} \end{pmatrix} = 0.$$

We condensate in

$$\mathrm{Div}_{t,y} T = 0$$

where $T(t,y) \in \mathbf{Sym}_2$. Remark that $T \geq 0_2$ iff $u \geq 0$.

This formalism extends (homework) to an arbitrary scalar conservation law

$$\partial_t u + \sum_{j=1}^{n} \partial_{y_j} f^j(u) = 0.$$

The Wave equation

Space-time variable (t, y) :

$$\partial_t^2 u = \Delta_y u.$$

Conservation of energy:

$$\partial_t \frac{1}{2}(u_t^2 + |\nabla_y u|^2) = \mathrm{div}_y(u_t \nabla_y u).$$

This writes $\mathrm{div}_{t,y}\vec{Q} = 0$.

We also have

$$\partial_t (u_t \partial_j u) = \mathrm{div}_y(\partial_j u \nabla_y u) + \partial_j \frac{1}{2}(u_t^2 - |\nabla_y u|^2).$$

Again $\mathrm{div}_{t,y}\vec{q}_j = 0 \ldots$

Assemble these equations into

$$\begin{pmatrix}\partial_t & \partial_1 & \dots & \partial_n\end{pmatrix}\begin{pmatrix}\frac{1}{2}(u_t^2+|\nabla_y u|^2) & -u_t\partial_1 u & \cdots & \cdots \\ -u_t\partial_1 u & (\partial_1 u)^2+\frac{1}{2}(u_t^2-|\nabla_y u|^2) & \partial_1 u\partial_j u & \cdots \\ \vdots & \partial_i u\partial_1 u & \ddots & \\ -u_t\partial_n u & \vdots & & \end{pmatrix}=0.$$

In other words

$$\mathrm{Div}_{t,y}T=0,$$

where

$$T=\begin{pmatrix}\frac{1}{2}(u_t^2+|\nabla_y u|^2) & -u_t\nabla_y u \\ -u_t\nabla_y u & \nabla_y u\otimes\nabla_y u+\frac{1}{2}(u_t^2-|\nabla_y u|^2)I_n\end{pmatrix}$$

is **symmetric**.

Examples in continuum mechanics

Fluids or deformable media.

Mass density ρ, velocity v, Cauchy stress Σ (symmetric).

Conservation laws of mass and momentum:

$$\partial_t \rho + \mathrm{div}_y(\rho v) = 0, \qquad \partial_t(\rho v) + \mathrm{Div}_y(\rho v \otimes v) = \mathrm{Div}_y \Sigma.$$

This rewrites as

$$\mathrm{Div}_{t,y} T = 0,$$

where

$$T = \begin{pmatrix} \rho & \rho v \\ \rho v & \rho v \otimes v - \Sigma \end{pmatrix}.$$

Applies to Euler, Navier–Stokes, Boltzman, ... and even to relativistic Euler !

Plasmas and galaxies

The density $f(t,y,v)$ depends also on the velocity variable.

No collision, but a self-induced force $F(t,y)$, either Coulomb or gravity. Given in terms of the local density

$$\rho(t,y) := \int_{\mathbb{R}^n} f(t,y,v)\,dv$$

by a Poisson equation

$$F = -\nabla_y \phi, \qquad \Delta_y \phi = \beta\rho.$$

The constant β is either positive (Coulomb) or negative (gravitation).

The density f evolves according to a *Vlasov equation* :

$$\underbrace{\partial_t f + v \cdot \nabla_y f}_{\text{pure transport}} \quad + \quad \underbrace{F \cdot \nabla_v f}_{\text{force}} = 0.$$

The conservation of mass and momentum writes as

$$\mathrm{Div}_{t,y} T = 0,$$

where

$$T(t,y) = \underbrace{\int_{\mathbb{R}^n} f(t,y,v) \begin{pmatrix} 1 \\ v \end{pmatrix} \otimes \begin{pmatrix} 1 \\ v \end{pmatrix} dv}_{\text{kinetic part}} + \begin{pmatrix} 0 & 0 \\ 0 & S \end{pmatrix}$$

and

$$S := \frac{1}{\beta} (F \otimes F - \frac{1}{2} |F|^2 I_n).$$

Once again, T is symmetric.

Maxwell's equations

The electromagnetic field is a closed 2-form Ω in the Minkowski space $\mathbb{R}^{1+3}$:

$$\mathrm{d}\Omega = 0.$$

The choice of an inertial frame allows us to define an electric field E and a magnetic induction B :

$$\Omega = (E_1 dy_1 + \cdots) \wedge dt + B_1 dy_2 \wedge dy_3 + \cdots .$$

The closeness writes (Maxwell–Faraday equations)

$$\partial_t B + \mathrm{curl}_y E = 0, \qquad \mathrm{div}_y B = 0. \tag{1}$$

The remaining equations come from a *variational principle*. One starts with a Lagrangian $\mathcal{L}(\Omega)$. Notice that it must not depend upon the choice of inertial frame:

$$\mathcal{L}(R^*\Omega) = \mathcal{L}(\Omega)$$

for every Lorentz transformation R. This yields the restriction

$$\mathcal{L} = L(\frac{1}{2}(|E|^2 - |B|^2), E \cdot B). \qquad (2)$$

One then declares (*Fiat Lux !*)

$$\delta\mathcal{L} = 0.$$

This yields (Gauss–Ampère)

$$\partial_t D - \mathrm{curl}_y H = 0, \qquad \mathrm{div}_y D = 0$$

for the magnetic field and the electric induction defined by

$$D := \frac{\partial \mathcal{L}}{\partial E}, \qquad H := -\frac{\partial \mathcal{L}}{\partial B}.$$

Energy density $W(B,D) = E \cdot D - L$ is conserved:

$$\partial_t W + \mathrm{div}_y (E \times H) = 0.$$

The energy flux $E \times H$ is the *Poynting vector* (Abraham's form).

There was a long-time controversy of whether the Poynting vector is merely (Minkowski's form)

$$D \times B.$$

In vacuum, both forms are valid:

$$D \times B \equiv E \times H, \qquad \text{(denoted } P \text{ afterwards)}$$

because of the frame-invariance of $\mathcal{L}$.

Important ! Because of the conservation of momentum

$$\partial_t (D \times B) + \mathrm{Div}_y(\cdots) = 0.$$

This yields

$$\mathrm{Div}_{t,y} T = 0,$$

where

$$T = \begin{pmatrix} W & P \\ P & g(E \otimes E + B \otimes B) + (L + B \cdot H) I_3 \end{pmatrix}$$

is symmetric.

Hereabove,

$$g = \frac{\partial}{\partial \sigma} L(\sigma, \pi), \qquad \sigma = \frac{1}{2}(|E|^2 - |B|^2), \quad \pi = E \cdot B.$$

Schrödinger equation

A quantum state $\psi(t,y) \in \mathbb{C}$ obeys to

$$i\hbar\partial_t\psi = -\frac{1}{2m}\Delta_y\psi + V(|\psi|^2)\psi.$$

The density $\rho = m|\psi|^2$ and momentum $\vec{q} := \frac{1}{\hbar}\Im(\bar{\psi}\nabla\psi)$ satisfy

$$\begin{aligned} \partial_t\rho + \mathrm{div}_y\vec{q} &= 0, \\ \hbar^2\partial_t\vec{q} + \frac{1}{2m}\mathrm{Div}_y\mathrm{Re}\left(\nabla_y\bar{\psi}\otimes\nabla_y\psi - \bar{\psi}\nabla_y^2\psi\right) + \nabla_y F(|\psi|^2) &= 0, \end{aligned}$$

where $F'(s) = sV'(s)$.

$\cdots \longrightarrow$

This is again

$$\mathrm{Div}_{t,y} T = 0,$$

where

$$T = \begin{pmatrix} \rho & \vec{q} \\ \vec{q} & \frac{1}{\hbar^2}\left(\frac{1}{2m}\mathrm{Re}\left(\nabla_y\bar{\psi}\otimes\nabla_y\psi - \bar{\psi}\nabla_y^2\psi\right) + F\left(|\psi|^2\right)I_3\right) \end{pmatrix}$$

is symmetric.

Whence a

Meta-question

Since Divergence-free Symmetric Tensors (DST) occur everywhere in physics, is there something to be said about them in general ?

Do they have any non-trivial mathematical property ?

General setting

$$\mathrm{Div}_x T = 0, \qquad x \in \mathbb{R}^d,\, T(x) \in \mathbf{Sym}_d.$$

In applications, $d = 1 + n$ and $x = (t, y)$.

Proposition 1 *The quantity* $\det T$ *has a well-defined physical dimension.*

Proof:

ℓ_j the dimension of x_j (time, length)

d_{ij} that of T_{ij} (mass density, momentum density, . . .).

From $(\mathrm{Div}\, T)_i = 0$, the terms $\partial_j T_{ij}$ have a common dimension m_i. One infers

$$d_{ij} = m_i \ell_j$$

for some dimension m_i.

Symmetry tells us

$$\vec{m} \otimes \vec{\ell} = \vec{\ell} \otimes \vec{m},$$

that is $\vec{m} = \mu \vec{\ell}$ for some dimension μ.

Then $d_{ij} = \mu \ell_i \ell_j$ and all the products in $\det T$ have the same dimension

$$\mu^d \left(\prod_{i=1}^{d} \ell_i \right)^2 .$$

Q.E.D.

Proposition 2 $\det T$ *is invariant under changes of inertial frame.*

Proof:

If $\tilde{x} = Rx$, then

$$\tilde{T}(\tilde{x}) = RT(R^{-1}\tilde{x})R^T$$

is symmetric and divergence-free.

Q.E.D.

We focus therefore upon $\det T$. Enhanced regularity ? integrability ?

Back to the wave equation

$$u_{tt} = \Delta_y u, \qquad y \in \mathbb{R}^n.$$

Cauchy problem:

$$u(0,y) = u_0(y), \qquad u_t(0,y) = u_1(y).$$

Energy:

$$E(t) = \int_{\mathbb{R}^n} \frac{1}{2}(u_t^2 + |\nabla_y u|^2)\, dy \equiv E_0.$$

Definition 1 EW *is the energy space of solutions, with norm* $\sqrt{2E_0}$.

Recall our DST

$$T = \begin{pmatrix} \frac{1}{2}(u_t^2 + |\nabla_y u|^2) & -u_t \nabla_y u \\ -u_t \nabla_y u & \nabla_y u \otimes \nabla_y u + \frac{1}{2}(u_t^2 - |\nabla_y u|^2) I_n \end{pmatrix}$$

Homework:

$$\det T = 2^{-d}(u_t^2 - |\nabla_y u|^2)^d, \qquad d = 1 + n.$$

We thus look for a (qualitative, quantitative) property of

$$u_t^2 - |\nabla_y u|^2.$$

Remark 1 (L. Tartar) *By* div-curl *Lemma, this is the only nontrivial* $F(u_t, \nabla_y u)$ *that is weakly continuous over* EW*.*

Actually, $\mathrm{div}_{t,y}(u_t, -\nabla_y u) = 0$, $\mathrm{curl}_{t,y} \nabla_{t,y} u = 0$ and

$$u_t^2 - |\nabla_y u|^2 = (u_t, -\nabla_y u) \bullet \nabla_{t,y} u.$$

Question: What is the integrability of $u_t^2 - |\nabla_y u|^2$ for $u \in EW$? Obviously $L_t^\infty L_y^1$, but this does not involve the DST structure.

At least (Coifman & al. 1989),

$$f := u_t^2 - |\nabla_y u|^2 \in \mathcal{H}^1 \qquad \text{(Hardy space)}.$$

If it has constant sign (unlikely) then $f \log(1+f) \in L^1$.

But this uses only $\nabla_{t,y} u \in L^1_{\rm loc}$, not the stronger

$$\nabla_{t,y} u \in L_t^\infty(L_y^1).$$

Dimensional analysis: the following quantities have the same dimension

$$\int_{\mathbb{R}} dt \int_{\mathbb{R}^n} |u_t^2 - |\nabla_y u|^2|^{1+\frac{1}{n}} dy \qquad \text{and} \qquad E_0^{1+\frac{1}{n}}.$$

Is there a functional inequality

$$\int_{\mathbb{R}\times\mathbb{R}^n} |\det T|^{\frac{1}{n}} dy\, dt \leq c_n \cdot E_0^{1+\frac{1}{n}} \quad ?$$

True if $n = 1$, by Fubini.

True for *radial* solutions in any dimension, but . . .

False otherwise:

Theorem 1 (Klainerman & Machedon, 1983.) *If $n = 2$ or 3, there exist finite energy solutions of the wave equation, for which*

$$\int_0^1 dt \int_{\mathbb{R}^n} |u_t^2 - |\nabla_y u|^2|^{1+\frac{1}{n}} dy = +\infty.$$

$\Longrightarrow$ a general theory of Divergence-free symmetric tensors needs another ingredient.

As suggested by Coifman & al.'s result, the sign of $\det T$ is important ...

2nd part

Theoretical analysis

Which exponent for $\det T$?

Observation: In practice, $T \in L^\infty_t(L^1_y)$, because of conservation of energy, hence $T \in L^1_{t,y}$, at least locally.

We infer $|\det T|^{\frac{1}{d}} \in L^1_{\rm loc}$. But this does not uses the Divergence-free structure ...

$\Longrightarrow$ we look for a non-trivial exponent $s > \frac{1}{d}$.

For the wave equation, $s = \frac{1}{d-1}$ **does not** work.

Not much room in $\left(\frac{1}{d}\,, \frac{1}{d-1}\right)$.

All that stuff, ... plus **Positivity**

Claim 1 *The missing ingredient is positivity:*

Definition 2 *A* Divergence-free Positive Symmetric Tensor *is a DST that takes positive semi-definite values:*

$$T(x) \in \mathbf{Sym}_d^+, \qquad \mathrm{Div}\, T = 0.$$

Acronym : *DPT.*

For the wave equation, T is **not** a DPT.

Recall

Proposition 3 • *$\mathbf{Sym}_d^+$ is a convex cone,*

- *The map $T \mapsto (\det T)^{\frac{1}{d}}$ is concave over $\mathbf{Sym}_d^+$.*

The exponent $\frac{1}{d}$ is optimal for this property.

Averages: With the help of Jensen Inequality, we infer

$$\fint_\Omega (\det T)^{\frac{1}{d}} dx \le \left(\det \fint_\Omega T \, dx \right)^{\frac{1}{d}}.$$

But again, this does not uses the Divergence-free structure.

$\cdots \quad \Longrightarrow$ We *really* look for a **non-trivial** exponent

$$s > \frac{1}{d}\,.$$

Question: For which s do we have

$$\fint_\Omega (\det T)^s dx \leq \left(\det \fint_\Omega T\,dx\right)^s$$

for every DPT ?

Searching for clues: the diagonal case

Suppose that T is diagonal:

$$T = \mathrm{diag}(a_1, \dots, a_d), \qquad a_j \geqslant 0, \qquad \det T = \prod_{j=1}^{d} a_j.$$

Divergence-freeness writes $\partial_j a_j = 0$ for every j. That is

$$a_j = a_j(\widehat{x_j}), \qquad \widehat{x_j} := (\dots, x_{j-1}, x_{j+1}, \dots).$$

If $d = 2$, then

$$\int \det T \, dx = \int a_1(x_2) a_2(x_1) \, dx \stackrel{\text{Fubini}}{=\!=} \int a_1 \cdot \int a_2 = \det \int T \, dx.$$

More generally

Theorem 2 (Gagliardo 1958.) *Let* $f_1, \ldots, f_d \in L^{d-1}(\mathbb{R}^{d-1})$ *be given. Define*

$$f(x) := \prod_{j=1}^{d} f_j(\widehat{x_j}), \qquad x \in \mathbb{R}^d.$$

Then $f \in L^1(\mathbb{R}^d)$ *and*

$$\|f\|_{L^1(\mathbb{R}^d)} \leqslant \prod_{j=1}^{d} \|f_j\|_{L^{d-1}(\mathbb{R}^{d-1})}.$$

yields . . .

$$\int \left(\prod_{j=1}^{d} a_j(\widehat{x_j}) \right)^{\frac{1}{d-1}} dx \quad \leqslant \quad \left(\prod_{j=1}^{d} \int a_j(z)\, dz \right)^{\frac{1}{d-1}},$$

that is

$$\int (\det T(x))^{\frac{1}{d-1}} dx \leq \left(\det \int T(x)\, dx \right)^{\frac{1}{d-1}}.$$

This is the limit exponent $\frac{1}{d-1}$ which failed for the wave equation.

Second clue: a sub-class among DPTs

Easy case: If $d = 2$, a DPT satisfies

$$\partial_1 t_{11} + \partial_2 t_{12} = 0, \qquad \partial_1 t_{21} + \partial_2 t_{22} = 0.$$

There exist potentials ϕ, ψ such that

$$t_{11} = \partial_2 \phi, \quad t_{12} = -\partial_1 \phi, \quad t_{21} = \partial_2 \psi, \quad t_{22} = -\partial_1 \psi.$$

The symmetry writes

$$\partial_1 \phi + \partial_2 \psi = 0,$$

whence a higher level potential θ such that

$$\phi = \partial_2 \theta, \qquad \psi = -\partial_1 \theta.$$

Eventually ($\widehat{M}$ the cofactor matrix) :

$$T = \begin{pmatrix} \partial_2^2\theta & -\partial_1\partial_2\theta \\ -\partial_1\partial_2\theta & \partial_1^2\theta \end{pmatrix} = \widehat{\mathrm{D}^2\theta}.$$

$T \geq 0_2$ means that θ is convex.

When $d \geqslant 3$, not all DPTs are of the form $\widehat{\mathrm{D}^2\theta}$. But

Proposition 4 ($d \geqslant 3$.) *Let* $\theta \in W^{2,d-1}_{\mathrm{loc}}(\mathbb{R}^d)$ *be a convex function, Then* $\widehat{\mathrm{D}^2\theta}$ *is a DPT.*

Proof : differential forms + exterior calculus.

Suppose now that $\mathrm{D}^2\theta$ is periodic. Equivalently,

$$\theta(x) = \underbrace{\frac{1}{2}x^T S x}_{\text{quadratic}} + \text{linear} + \underbrace{\rho(x)}_{\text{periodic}} \qquad \text{that is} \qquad \mathrm{D}^2\theta = S + \mathrm{D}^2\rho.$$

The t_{ij} are null-Lagrangians,

$$t_{ij} = (\widehat{S})_{ij} + \mathrm{div}(\text{ smthg periodic }),$$

and likewise

$$(\det T)^{\frac{1}{d-1}} = \det \mathrm{D}^2\theta = \det S + \mathrm{div}(\text{ smthg periodic }).$$

Therefore

$$\fint_{\mathbb{R}^d/\Gamma} T(x)\,dx = \widehat{S} \qquad \text{and} \qquad \fint_{\mathbb{R}^d/\Gamma} (\det T(x))^{\frac{1}{d-1}} dx = \det S.$$

With $\det S = \left(\det \widehat{S}\right)^{\frac{1}{d-1}}$, there comes

$$\fint_{\mathbb{R}^d/\Gamma} (\det T(x))^{\frac{1}{d-1}} dx = \left(\det \fint_{\mathbb{R}^d/\Gamma} T(x)\, dx\right)^{\frac{1}{d-1}}.$$

This is the same as in the diagonal case, but with ...

... **equality** !

If $T = \widehat{\mathrm{D}^2\theta}$ is defined instead on a bounded domain D, then

$$\int_D (\det T)^{\frac{1}{d-1}} dx = \int_D \det \mathrm{D}^2\theta \, dx = \mathrm{vol}(\nabla\theta(D)).$$

Notice that by convexity, $\nabla\theta$ is one-to-one.

By the isoperimetric inequality, this yields

$$\int_D (\det T)^{\frac{1}{d-1}} dx \leq \left(\frac{\mathrm{area}(\partial\nabla\theta(D))}{|S^{d-1}|}\right)^{\frac{d}{d-1}} |B_d|.$$

Using the formula

$$\mathrm{area}(\partial\nabla\theta(D)) = \int_{\partial D} \widehat{\mathrm{D}^2\theta} : \vec{n} \otimes \vec{n} \, ds(x)$$

we infer

$$\int_D (\det T)^{\frac{1}{d-1}} dx \;\; \leq \;\; c_d \int_{\partial D} \widehat{\mathrm{D}^2\theta} : \vec{n} \otimes \vec{n} \, ds(x) = \int_{\partial D} T : \vec{n} \otimes \vec{n} \, ds(x).$$

Whence

$$\int_D (\det T)^{\frac{1}{d-1}} dx \le c_d \int_{\partial D} |T\vec{n}|\, ds(x).$$

Special case : if moreover $\nabla\theta(D)$ is a ball, then

$$\int_D (\det T)^{\frac{1}{d-1}} dx = c_d \int_{\partial D} |T\vec{n}|\, ds(x) \quad !$$

These various special DPTs suggest more general statements

$\cdots \longrightarrow$

Abstract results

Theorem 3 *Let Γ be a lattice of $\mathbb{R}^d$, and T be a Γ-periodic DPT, with $T \in L^1_{\mathrm{loc}}$. Then $(\det T(x))^{\frac{1}{d-1}} \in L^1_{\mathrm{loc}}$, and there holds*

$$\fint_{\mathbb{R}^d/\Gamma} (\det T(x))^{\frac{1}{d-1}} dx \leqslant \left(\det \fint_{\mathbb{R}^d/\Gamma} T(x)\, dx\right)^{\frac{1}{d-1}}.$$

If T is smooth and uniformly positive, then the equality occurs if, and only if, there exists a convex function θ such that

$$T = \widehat{\mathrm{D}^2\theta}.$$

Mind that we do not expect $(T_{11}\cdots T_{dd})^{\frac{1}{d-1}}$ to be integrable. Only $\det T$ gains integrability.

Bounded domain

The fundamental inequality will involve a *boundary integral*.

Recall that when $\vec{q}$ and $\operatorname{div}\vec{q} \in L^p(\Omega)$, then $\vec{q}$ admits a normal trace $\gamma_n\vec{q} \sim \vec{q}\cdot\vec{n} \in W^{-\frac{1}{p},p}(\partial\Omega)$. Here, a DPT has a normal trace $T\vec{n}$ in $(Lip')^d$.

Here we consider fields such that $\operatorname{div}\vec{q}$ is a bounded measure. Then the extension

$$\vec{Q} = \begin{cases} \vec{q} & \text{in} \quad \Omega, \\ \vec{0} & \text{in} \quad \overline{\Omega}^c \end{cases}$$

has the property that $\operatorname{div}\vec{Q}$ is a bounded measure **iff** $\vec{q}\cdot\vec{n}$ is a bounded measure over $\partial\Omega$.

Theorem 4 *Let Ω be a bounded domain, and $T \in L^1(\Omega)$ be a DPT. Assume that the normal trace $T\vec{n}$ is a bounded measure over $\partial\Omega$. Then $(\det T)^{\frac{1}{d-1}} \in L^1(\Omega)$, and there holds*

$$\int_\Omega (\det T(x))^{\frac{1}{d-1}} dx \leqslant \frac{1}{d|S^{d-1}|^{\frac{1}{d-1}}} \|T\vec{n}\|_{\mathcal{M}(\partial\Omega)}^{\frac{d}{d-1}}. \qquad (3)$$

Remark: Applying (3) to $T(x) \equiv I_d$, we recover the **isoperimetric inequality**,

$$\frac{|\Omega|}{|B^d|} \leqslant \left(\frac{|\partial\Omega|}{|S^{d-1}|}\right)^{\frac{d}{d-1}}.$$

Amazingly, the constant in inequality (3) is sharp:

Proposition 5 (Equality case.) *Suppose that T is smooth and uniformly positive. Then the equality occurs in (3) if, and only if there exists a convex function $\theta : \Omega \to \mathbb{R}$ such that*

- $T = \widehat{\mathrm{D}^2\theta}$,
- *the image of Ω under $\nabla\theta$ is a ball B_r.*

Theorem 4 extends to the case where $\mathrm{Div}\,T \neq 0$ is a bounded measure. We have

$$\int_\Omega (\det T(x))^{\frac{1}{d-1}} dx \leqslant \frac{1}{d|S^{d-1}|^{\frac{1}{d-1}}} \left(\|T\vec{n}\|_{\mathcal{M}(\partial\Omega)} + \|\mathrm{Div}\,T\|_{\mathcal{M}(\Omega)} \right)^{\frac{d}{d-1}}. \qquad (4)$$

By applying (4) to ϕT where ϕ has compact support in Ω, and using the formula

$$\mathrm{Div}(\phi T) = \phi\,\mathrm{Div}\,T + T\nabla\phi,$$

we also have

Theorem 5 *Let Ω be a bounded domain, and $T \in L^1(\Omega)$ be a DPT. Then $(\det T)^{\frac{1}{d-1}} \in L^1_{\mathrm{loc}}(\Omega)$, and there holds*

$$\int_\Omega (\det T(x))^{\frac{1}{d-1}} \mathrm{dist}(x;\partial\Omega)^{\frac{d}{d-1}} dx \leqslant \frac{1}{d|S^{d-1}|^{\frac{1}{d-1}}} \|T\|_{L^1(\Omega)}^{\frac{d}{d-1}}. \qquad (5)$$

Suggested terminology: Theorems 3 and 4 state

Compensated integrability

+

non-diagonal Gagliardo inequality.

Semi-infinite domain

Let us consider a *slab*, that is $Q_\tau = (0,\tau)\times\mathbb{R}^n$ for some positive time τ, where $x=(t,y)$ is a space-time variable ($d=1+n$). Mind that $\frac{1}{d-1}=\frac{1}{n}$.

Theorem 6 *The same statement as Theorem 4 holds true in a slab:*

If $T\in L^1(Q_\tau)$ is a DPT and the normal trace $T\vec{e}_t$ is a bounded measure over $\partial Q_\tau=\{0,\tau\}\times\mathbb{R}^n$, then $(\det T)^{\frac{1}{n}}\in L^1(Q_\tau)$, and there holds

$$\int_0^\tau dt\int_{\mathbb{R}^n}(\det T(t,y))^{\frac{1}{n}}dy \leqslant \frac{1}{d|S^n|^{\frac{1}{n}}}\left(\|T\vec{e}_t\|_{\mathcal{M}(\{0\}\times\mathbb{R}^n)} + \|T\vec{e}_t\|_{\mathcal{M}(\{\tau\}\times\mathbb{R}^n)}\right)^{1+\frac{1}{n}}. \tag{6}$$

Mind that the constant does not depend upon τ.

Cylindrical domain

Let now $O \subset \mathbb{R}^n$ be an open set, and $Q_\tau = (0,\tau) \times O$.

Theorem 7 *The same statement as Theorem 5 holds true in a cylinder:*

If $T \in L^1(Q_\tau)$ is a DPT and the normal trace $T\vec{e}_t$ is a bounded measure over $\partial Q_\tau = \{0,\tau\} \times O$, then $(\det T)^{\frac{1}{n}} \in L^1_{\rm loc}(Q_\tau)$, and there holds

$$\int_0^\tau dt \int_O (\det T(t,y))^{\frac{1}{n}} \mathrm{dist}(y;\partial O)^{1+\frac{1}{n}} dy \leqslant \frac{1}{d|S^n|^{\frac{1}{n}}} \Big(\|T\vec{e}_t\|_{\mathcal{M}(\{0\}\times O)} + \|T\vec{e}_t\|_{\mathcal{M}(\{\tau\}\times O)} + \|T\|_{L^1(O)} \Big)^{1+\frac{1}{n}}.$$

Proofs

Preliminaries: the Monge–Ampère equation

Somehow, the constraint $\mathrm{Div}\, T = 0$ is dual to the Monge–Ampère equation (**MAE**)

$$\det \mathrm{D}^2\theta = f, \tag{7}$$

where f is given and θ is the unknown.

A fully nonlinear PDE !

The MAE can be read

$$\mathrm{Jac}(\nabla\theta) = f.$$

Like the Laplace equation $\Delta\theta = f$, the MAE bears the form

$$F(\mathrm{D}^2\theta) = f.$$

Ellipticity

$F : \mathbf{Sym}_d^+ \to \mathbb{R}$ is monotonous increasing (but not over the full space $\mathbf{Sym}_d$).

The Monge–Ampère equation is elliptic, when focusing on convex unknowns θ.

In particular, the data f must be > 0.

For $S_\pm \in \mathbf{SPD}_d$, define

$$\mathbb{A}(S_+, S_-) := \int_0^1 \mathrm{D}F(\lambda S_- + (1-\lambda) S_+)\, d\lambda \quad \in \mathbf{SPD}_d.$$

Maximum principle

Consider sub/super-solutions $\theta_\pm$:

$$F(\mathrm{D}^2\theta_+) \leqslant f \leqslant F(\mathrm{D}^2\theta_-).$$

Taylor formula yields

$$\mathrm{Tr}\,(A(x)\mathrm{D}^2(\theta_+ - \theta_-)) \leqslant 0, \tag{8}$$

where $A(x) = \mathbb{A}(\mathrm{D}^2\theta_+(x), \mathrm{D}^2\theta_-(x))$, In particular:

$$\min_{\Omega}(\theta_+ - \theta_-) = \min_{\partial\Omega}(\theta_+ - \theta_-),$$

which implies uniqueness for the Dirichlet BVP (Rellich's Theorem).

Well-posedness results for Monge–Ampère BVPs

The constraint (convexity of the solutions) suggests that the domain itself be convex. At least three interesting boundary-value problems ...

Dirichlet. $\theta|_{\partial\Omega} = \phi$ given.

The uniform convexity of Ω ensures that every $\phi \in C(\partial\Omega)$ is the trace of some convex function.

The Dirichlet problem in a uniformly convex domain is uniquely solvable in the set of convex functions (see Gilbarg–Trudinger).

We shall not use the Dirichlet BVP.

Periodic. Let $f > 0$ be Γ-periodic. One looks for a convex solution $\theta = \frac{1}{2}x^T Sx + \rho(x)$ where $S \in \mathbf{SPD}_d$ and ρ is Γ-periodic. One has $D^2\theta = S + D^2\rho$.

Recall that

$$\det(S + D^2\rho) = \det S + \text{ null-Lagrangian.}$$

This implies

$$⨍_{\mathbb{R}^d/\Gamma} \det D^2\theta(x)\,dx = \det S.$$

Whence the *necessary condition*

$$\det S = ⨍_{\mathbb{R}^d/\Gamma} f(x)\,dx.$$

This NC turns out to be sufficient:

Theorem 8 (Yan Yan Li, 1990) *Let f be Γ-periodic, smooth and such that $\inf_x f > 0$. Let $S \in \mathbf{SPD}_d$ satisfy*

$$\det S = \fint_{\mathbb{R}^d/\Gamma} f(x)\,dx.$$

Then there exists a periodic function ρ, such that $S + D^2\rho > 0_d$ and $\det(S + D^2\rho) = f$.

This solution is unique, up to an additive constant.

“Second” BVP. Suppose Ω convex and $0 < \inf_x f \leqslant \sup_x f < \infty$. Let D be another convex domain.

One considers maps $\phi : \Omega \longrightarrow D$ that *push forward* the measure $(f(x)\,dx, \Omega)$ onto the Lebesgue measure (dx, D), that is

$$\int_{\phi^{-1}(A)} f(x)\,dx = \mathrm{vol}(A), \qquad \forall A \subset D.$$

Compatibility condition

$$\mathrm{vol}\, D = \int_\Omega f(x)\,dx.$$

A classical problem is the $\cdots \longrightarrow$

Monge–Kantorovich problem: Among all these maps, one looks for the one which minimizes the cost

$$\int_\Omega |\phi(x) - x|^2 dx.$$

The solution is given by the **Brenier's transport** :

Theorem 9 *The Monge–Kantorovich problem admits a unique solution.*

The optimal transport is a gradient, $\phi = \nabla\theta$ *where*

- θ *is strongly convex (hence* ϕ *is one-to-one),*
- $\det \mathrm{D}^2\theta = f$*,*
- $\phi(\Omega) = D$ *(plays the role of a boundary condition).*

Proofs (I) ; the periodic case

Let $f \in C^\infty_{\text{per}}$, $f > 0$ be given, and $S \in \mathbf{SPD}_d$ be such that

$$\det S = \fint_{\mathbb{R}^d/\Gamma} f(x)\,dx, \tag{9}$$

Solve the periodic Monge–Ampère equation (Y.-Y. Li)

$$\det(\underbrace{S + \mathrm{D}^2\rho}_{\mathrm{D}^2\theta}) = f.$$

Let us form

$$(f \det T)^{\frac{1}{d}} = (\det(T\mathrm{D}^2\theta))^{\frac{1}{d}},$$

the *geometric mean* of the spectrum of $M := T\mathrm{D}^2\theta = T(S + \mathrm{D}^2\rho)$.

M is not symmetric, but ...

... as a product of an SPD matrix with a another in $\mathbf{Sym}_d^+$,

it is diagonalisable with non-negative real eigenvalues.

Whence

$$
\begin{aligned}
(\det(T\mathrm{D}^2\theta))^{\frac{1}{d}} = GM(\mathrm{Sp}\, M) \quad \overset{\mathrm{AGI}}{\leqslant} \quad & AM(\mathrm{Sp}\, M) = \frac{1}{d}\,\mathsf{Tr}\, M \\
& = \frac{1}{d}\left(\mathsf{Tr}\,(TS) + \mathsf{Tr}\,(T\mathrm{D}^2\rho)\right) \\
& \overset{\mathrm{Div}\, T = 0}{=\!=} \frac{1}{d}\left(\mathsf{Tr}\,(TS) + \mathrm{div}(T\nabla\rho)\right).
\end{aligned}
$$

Integrating,

$$\fint_{\mathbb{R}^d/\Gamma} (f \det T)^{\frac{1}{d}}\, dx \leqslant \frac{1}{d} \operatorname{Tr}\,(S \underbrace{\fint_{\mathbb{R}^d/\Gamma} T(x)\, dx}_{T_+}).$$

Next step: minimize the rhs $\operatorname{Tr}\,(ST_+)$ over all the $S \in \mathbf{SPD}_d$ satisfying the constraint (9). This amounts to choosing

$$S = \lambda T_+^{-1}, \qquad \lambda^d = \fint_{\mathbb{R}^d/\Gamma} f(x)\, dx \cdot \det T_+.$$

One obtains

$$\fint_{\mathbb{R}^d/\Gamma} (f \det T)^{\frac{1}{d}} dx \leqslant \lambda = (\det T_+)^{\frac{1}{d}} \left(\fint_{\mathbb{R}^d/\Gamma} f(x)\, dx \right)^{\frac{1}{d}}.$$

Working with $\phi = f^{\frac{1}{d}}$ instead, this reads

$$\fint_{\mathbb{R}^d/\Gamma} (\det T)^{\frac{1}{d}} \phi \, dx \leqslant (\det T_+)^{\frac{1}{d}} \|\phi\|_{L^d_{\rm per}}, \qquad \forall \phi \in \mathcal{C}^\infty_{\rm per}, \, \phi > 0.$$

This proves $(\det T)^{\frac{1}{d}} \in L^{d'}_{\rm per}$, with

$$\|(\det T)^{\frac{1}{d}}\|_{L^{d'}_{\rm per}} \leqslant (\det T_+)^{\frac{1}{d}}.$$

Because $d' = \frac{d}{d-1}$, this is $(\det T)^{\frac{1}{d-1}} \in L^1_{\rm per}$ and

$$\fint_{\mathbb{R}^d/\Gamma} (\det T(x))^{\frac{1}{d-1}} dx \leqslant (\det T_+)^{\frac{1}{d-1}}.$$

Q.E.D.

Proofs (II) : the equality case

Equality means that when

$$f = (\det T)^{\frac{1}{d-1}}, \qquad S = \lambda T_+^{-1} \quad \text{and} \quad M := T\mathrm{D}^2\theta = T(S + \mathrm{D}^2\rho),$$

then

$$GM(\mathrm{Sp}\,M) = AM(\mathrm{Sp}\,M).$$

Hence M has only one eigenvalue $\alpha > 0$, with $\# d$.

Because M is diagonalisable, $M = \alpha I_d$, that is $T = \alpha(\mathrm{D}^2\theta)^{-1}$. In other words

$$T = \widehat{\mathrm{D}^2\theta'}.$$

Proofs (III) : bounded convex domain

Once again, start with an $f \in \mathcal{C}^\infty(\overline{\Omega})$, $f > 0$. Consider Brenier's transport

$$(f(x)\,dx, \Omega) \longrightarrow (dx, B_r).$$

The solvability condition imposes the radius r :

$$\frac{r^d}{d}\,|S^{d-1}| = \int_\Omega f(x)\,dx.$$

Recall that the optimal transport map is $\nabla\theta = \Omega \to B_r$ where

$$\det \mathrm{D}^2\theta = f.$$

The same arguments as above yield

$$(f \det T)^{\frac{1}{d}} \leqslant \frac{1}{d}\,\mathrm{Tr}\,(T\mathrm{D}^2\theta) = \frac{1}{d}\,\mathrm{div}(T\nabla\theta).$$

Integrating:

$$\int_\Omega (f \det T)^{\frac{1}{d}} dx \leqslant \frac{1}{d} \int_{\partial\Omega} \underbrace{(T\nabla\theta)\cdot\vec{n}}_{=(T\vec{n})\cdot\nabla\theta} ds(x).$$

With $|\nabla\theta| \leqslant r$, there comes

$$\int_\Omega (f \det T)^{\frac{1}{d}} dx \leqslant \frac{r}{d} \int_{\partial\Omega} |T\vec{n}|\, ds(x).$$

Combining with the expression for the value of r, we obtain

$$\int_\Omega (f \det T)^{\frac{1}{d}} dx \leqslant \frac{1}{d^{1/d'}|S^{d-1}|^{1/d}} \left(\int_\Omega f(x)\, dx\right)^{\frac{1}{d}} \int_{\partial\Omega} |T\vec{n}|\, ds(x),$$

and we conclude as in the periodic case.

Q.E.D.

The equality case is treated the same way.

- The proof works the same way when $\operatorname{Div} T$ is a non-zero bounded measure, but yields an extra term

$$\|\operatorname{Div} T\|_{\mathcal{M}(\Omega)}.$$

- Proof for a general bounded domain:
  - choose a ball B that contains $\overline{\Omega}$,
  - extend T by 0_d to $B \setminus \Omega$. Notice that $\operatorname{Div} T$ involves the bounded measure

$$T\vec{n} \otimes \delta_{\partial\Omega}.$$

  - Apply the convex case to this extension.

Lecture #3 : Applications to gas dynamics

I – Euler equations

Recall the context:

- n the space dimension, $y \in R^n$ the space variable, t the time variable.
- $d = 1 + n$, $x = (t, y)$.
- ρ, v, p are the mass density, the fluid velocity, the pressure.
- The DPT is

$$T = \begin{pmatrix} \rho & \rho v^T \\ \rho v & \rho v \otimes v + pI_n \end{pmatrix}, \qquad \det T = \rho p^n.$$

Assume a finite mass. Then

$$\forall t > 0, \qquad \int_{\mathbb{R}^n} \rho(t,y)\, dy = M_0.$$

Assume a finite energy at initial time. Then

$$\forall t > 0, \qquad \int_{\mathbb{R}^n} \left(\frac{1}{2} \rho |v|^2 + \rho e \right) (t,y)\, dy \leqslant E_0. \tag{10}$$

For most reasonable gases, $p = O(\rho e)$ and (10) implies

$$\sup_{t \geq 0} \int_{\mathbb{R}^n} p(t,y)\, dy < \infty,$$

that is $T \in L^\infty(0,\tau; L^1(\mathbb{R}^n))$.

Apply Theorem 6 in the slab $Q_\tau = (0,\tau) \times \mathbb{R}^n$:

$$\int_0^\tau dt \int_{\mathbb{R}^n} (\rho^{\frac{1}{n}} p)(t,y)\, dy \leqslant c_n \left(\int_{t=0} + \int_{t=\tau} |T\vec{e}_t|\, dy \right)^{1+\frac{1}{n}}.$$

Top & bottom:

$$T\vec{e}_t = \begin{pmatrix} \rho \\ \rho v \end{pmatrix}.$$

We have

$$\int_{t=0 \text{ or } \tau} |T\vec{e}_t|\, dy = \int_{\mathbb{R}^n} \rho \sqrt{1+|v|^2} dy \leqslant \int_{\mathbb{R}^n} \rho \left(1 + \frac{1}{2} |v|^2 \right) dy \leqslant M_0 + E_0.$$

Summary:

$$\int_0^\tau dt \int_{\mathbb{R}^n} (\rho^{\frac{1}{n}} p)(t,y)\, dy \leqslant c'_n (M_0 + E_0)^{1+\frac{1}{n}}. \tag{11}$$

Dimensional analysis:

$$M^{1+\frac{1}{n}} L T^{-1} \quad vs \quad (M + ML^2 T^{-2})^{1+\frac{1}{n}}.$$

Unbalanced from the physical point of view !

This suggests $\cdots \longrightarrow$

Exploiting invariance groups

Observations: – The Euler equations are invariant upon rescalings and Galilean transformations. – Our inequality does not depend upon the time interval $(0, \tau)$.

Strategy:

- Choose a one-parameter group,
- Rescale accordingly the (in)dependent variables,
- Apply inequality (11) to the rescaled DPT,
- Optimize the parameter.

(I) Scaling group

For $\lambda > 0$, define

$$t' = \lambda t, \qquad y' = y, \qquad \rho' = \lambda^2 \rho, \qquad u' = \frac{1}{\lambda} u, \qquad p' = p, \qquad e' = \lambda^{-2} e.$$

For instance

$$M_0' = \int_{\mathbb{R}^n} \rho' dy' = \lambda^2 M_0, \qquad E_0' = E_0.$$

Then (11) becomes

$$\lambda^{1+\frac{2}{n}} \int_0^\tau dt \int_{\mathbb{R}^n} (\rho^{\frac{1}{n}} p)(t,y)\, dy \leqslant c_n' \left(\lambda^2 M_0 + E_0\right)^{1+\frac{1}{n}}.$$

Choose

$$\lambda = \sqrt{\frac{E_0}{M_0}}\,.$$

One obtains a well-balanced (equal physic units) inequality

$$\int_0^\tau dt \int_{\mathbb{R}^n} (\rho^{\frac{1}{n}} p)(t,y)\, dy \leqslant c_n'' M_0^{\frac{1}{n}} \sqrt{M_0 E_0}\,. \tag{12}$$

But this is not the end of the story ...

(II) Galilean group

The left-hand side of (12) is Galilean-invariant / the right-hand side is not.

A Galilean change of variables turns the velocity v into $v-\bar{v}$ for some constant $\bar{v}$, and the energy into

$$\overline{E_0} = \int_{\mathbb{R}^n} (\frac{1}{2}\rho_0|v_0-\bar{v}|^2+\rho_0 e_0)\,dy = E_0 - \bar{v}\cdot\int_{\mathbb{R}^n}\rho_0 v_0 dy + \frac{1}{2}|\bar{v}|^2 M_0.$$

The modified energy $\overline{E_0}$ is minimal when

$$\bar{v} = \frac{1}{M_0}\int_{\mathbb{R}^n}\rho_0 v_0 dy$$

is the *mean velocity*.

This yields our final result:

Theorem 10 *Let a flow in* $(0,\tau)\times\mathbb{R}^n$ *have finite mass* M_0 *and energy. Then it obeys the estimate*

$$\int_0^\tau dt\int_{\mathbb{R}^n}(\rho^{\frac{1}{n}}p)(t,y)\,dy\leqslant d_n M_0^{\frac{1}{n}}\sqrt{D_0}\,, \qquad (13)$$

where

$$D_0:=\frac{1}{4}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\rho_0(y)\rho_0(y')|v_0(y')-v_0(y)|^2dy\,dy'+M_0\int_{\mathbb{R}^n}\rho_0 e_0 dy.$$

This improves the known integrability.

Mind that the bound is independent of τ. If the flow is globally defined, then

$$\int_0^\infty dt\cdots\leqslant\cdots$$

Example: Polytropic gas.

The equation of state is

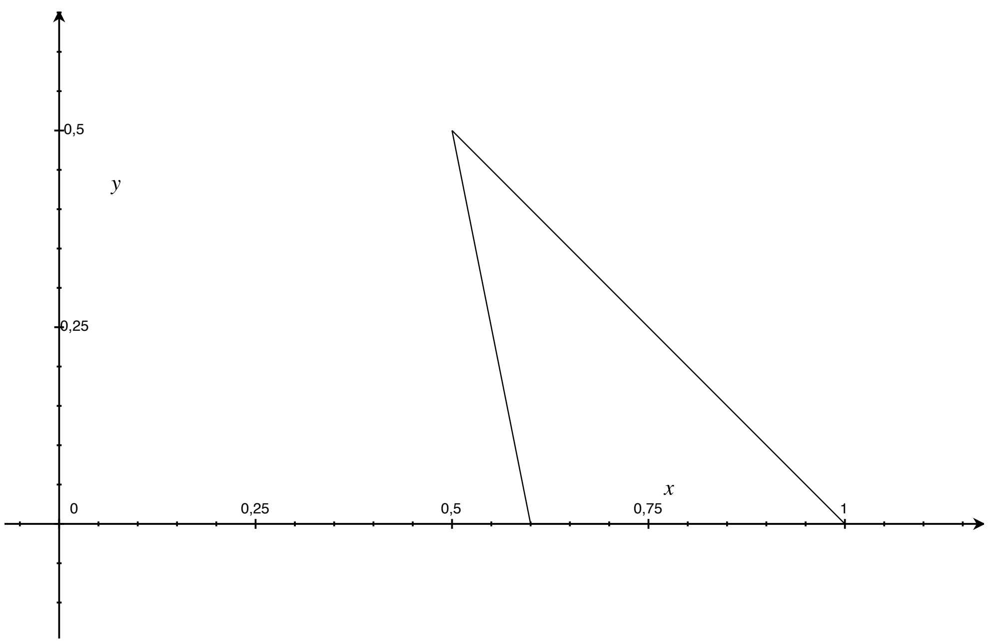
$$p = \rho^{\gamma}, \qquad \gamma > 1.$$

It satisfies the requirement $p = O(\rho e)$; actually $p = (\gamma - 1)\rho e$.

- Conservation of mass: $\rho \in L_t^{\infty}(L_y^1)$.
- Energy estimate: $\rho \in L_t^{\infty}(L_y^{\gamma})$.
- Theorem 10 above: $\rho \in L_{t,y}^{\gamma+\frac{1}{n}}$.

By interpolation, a triangle of *a priori* estimates $\rho \in L^q_t(L^r_y)$:

$$(\gamma - 1 + \frac{1}{n})\frac{1}{q} + \frac{1}{r} \leqslant 1, \qquad \frac{1}{nq} + \frac{\gamma}{r} \geqslant 1.$$



Coordinates are

$$\frac{1}{r} \text{ and } \frac{1}{q}.$$

In this example,

$$n = 3 \text{ and } \gamma = \frac{5}{3}$$

(mono-atomic gas).

II – Flow in a bounded domain

Consider a flow in a bounded physical domain O, with an impermeable wall. Boundary condition

$$v \cdot \vec{n} = 0 \qquad \text{over } \partial O.$$

Then the mass is conserved

$$M \equiv M_0$$

and the energy decays

$$E(t) \leqslant E_0.$$

We do not control the boundary trace $T\vec{n}$. We apply instead the generalized inequality to $T^\dagger(t,y) := \phi(y)T(t,y)$ where $\phi(y) = \mathrm{dist}(y;\partial O)$. We use

$$\mathrm{Div}_{t,y} T^\dagger = T\nabla_{t,y}\phi = T \begin{pmatrix} 0 \\ \nabla_y \phi \end{pmatrix},$$

whence

$$|\mathrm{Div}_{t,y}T^{\dagger}| \leq \|(\rho v, \rho v\otimes v + pI_n)\|.$$

After exploiting the freedom degree of scaling, we obtain

Proposition 6 *Consider an admissible flow in the bounded domain* O *with impermeable wall. Assume a finite mass and energy at initial time. Then*

$$\int_0^\tau\int_O \mathrm{dist}(y;\partial O)^{1+\frac{1}{n}}\rho^{\frac{1}{n}}p\,dx\,dt \leqslant c_n' M_0^{\frac{1}{2n}}\sqrt{E_0}\left(\tau\sqrt{E_0}+\sqrt{M_0}\,\mathrm{diam}(O)\right)^{1+\frac{1}{n}},$$

for every $\tau > 0$.

Notice that this estimate is homogeneous from the physical point of view.

III – Kinetic models

Unknown: $f(t,y,\xi)$ the density of particles at time t, position y, with velocity ξ.

Governing equation:

$$\underbrace{(\partial_t + \xi \cdot \nabla_y)}_{\text{pure transport}} \; f = \underbrace{Q[f(t,y,\cdot)]}_{\text{interaction}}. \tag{14}$$

Elastic assumption: the mass, momentum and energy are conserved. That is

$$\int_{\mathbb{R}^n} Q[g]\,d\xi = 0, \qquad \int_{\mathbb{R}^n} Q[g]\xi\,d\xi = 0, \qquad \int_{\mathbb{R}^n} Q[g]\,|\xi|^2\,d\xi = 0.$$

$\Longrightarrow$ Physically relevant solutions satisfy hydrodynamic conservation laws

$\cdots \longrightarrow$

Mass Formal integration of (14) yields

$$\partial_t \rho + \operatorname{div} m = 0,$$

for

$$\rho := \int_{\mathbb{R}^n} f \, d\xi \quad \text{and} \quad m := \int_{\mathbb{R}^n} f \xi \, d\xi.$$

Momentum Multiply (14) by ξ, then integrate, to obtain

$$\partial_t m + \operatorname{Div} S = 0,$$

for

$$S := \int_{\mathbb{R}^n} f \xi \otimes \xi \, d\xi.$$

The two first conservation laws write

$$\operatorname{Div}_{t,y} T = 0, \qquad T := \begin{pmatrix} \rho & m^T \\ m & S \end{pmatrix} \geqslant 0_d.$$

Energy Multiply by $\frac{1}{2}|\xi|^2$, then integrate to obtain formally

$$\partial_t E + \operatorname{div} F = 0,$$

for

$$E := \int_{\mathbb{R}^n} f \frac{1}{2}|\xi|^2 \, d\xi \quad \text{and} \quad F := \int_{\mathbb{R}^n} f \frac{1}{2}|\xi|^2 \xi \, d\xi.$$

So-called *renormalized solutions* (R. DiPerna & P.-L. Lions, 1989) satisfy

$$\int_{\mathbb{R}^n} \rho(t,y)\, dy \equiv M_0, \qquad \int_{\mathbb{R}^n} E(t,y)\, dy \leqslant E_0.$$

We infer $T \in L^\infty_t(L^1_y)$. We may apply Theorem 6.

Theorem 11 *Renormalized solutions satisfy an estimate*

$$\int_0^\tau dt \int_{\mathbb{R}^n} dy \left(\int_{\mathbb{R}^n}^{\otimes n+1} f(\xi^0)\cdots f(\xi^n)\Delta^2 d\xi^0 \cdots d\xi^n \right)^{\frac{1}{n}} \leqslant d_n M_0^{\frac{1}{n}} D_0^{\frac{1}{2}}$$

where

$$\Delta = \Delta(\xi^0,\ldots,\xi^n) = \begin{vmatrix} 1 & \ldots & 1 \\ \xi^0 & \ldots & \xi^n \end{vmatrix}$$

is the volume of the simplex with vertices $\xi^0,\ldots,\xi^n$*, and*

$$D_0 := \int_{\mathbb{R}^n}^{\otimes 4} f_0(y,\xi) f_0(y',\xi') |\xi' - \xi|^2 d\xi\, d\xi'\, dy\, dy'.$$

Remarks.

- (Gain of integrability.) One estimates something like $\rho^{1+\frac{1}{n}}$, up to the factor Δ^2. "Better" than the conservation of mass and energy, which are linear in f.

- This estimate was known in space dimension $n = 1$ (J.-M. Bony 1987).

  Bony used it to prove the global well-posedness of discrete kinetic models.

- Still for $n = 1$, Cercignani (2005) used it to prove that *renormalized solutions* of the Boltzmann equation are actually distributional solutions.

- For $n \geqslant 2$, the existence of distributional solutions is an open question.

About renormalized solutions

R. DiPerna & P.-L. Lions (1989) proved the existence of *renormalized solutions* to the Cauchy problem for Boltzmann equation. These are weaker than distributional ; the right-hand side $Q[f]$ does not make sense.

RenSols do satisfy the conservation of mass,

$$\partial_t \rho + \mathrm{div}_y m = 0.$$

Whether these satisfy the conservation of momentum is unknown. We only know

$$\partial_t m + \mathrm{Div}_y (S + \Sigma) = 0$$

where

$$S := \int_{\mathbb{R}^n} f(\xi)\, \xi \otimes \xi \, d\xi \geqslant \frac{1}{\rho} m \otimes m$$

and Σ is a *defect measure* with values in $\mathbf{Sym}_d^+$.

This is recast as

$$\mathrm{Div}_{t,y} T^* = 0, \qquad T^* := \begin{pmatrix} \rho & m^T \\ m & S + \Sigma \end{pmatrix}.$$

Our estimate applies to T^*.

- Because of $0_d \leqslant T \leqslant T^*$, we have $\det T \leqslant \det T^*$.

  Hence the same estimate as in Theorem 11.

- Because of $\det T^* = \rho \det(\Sigma + S - \rho^{-1} m \otimes m) \geqslant \rho \det \Sigma$, we also have

$$\int_0^\tau dt \int_{\mathbb{R}^n} \rho^{\frac{1}{n}} (\det \Sigma(t,\cdot))^{\frac{1}{n}} \leqslant d_n M_0^{\frac{1}{n}} D_0^{\frac{1}{2}}.$$

  Remark that $(\det \Sigma(t,\cdot))^{\frac{1}{n}}$ is a non-negative measure.

Open questions

- Do the new estimates help in the theory of the Cauchy problem for either the Euler equation, the Boltzmann equation or the Vlasov–Poisson equation ?

- The Div-quasiconcavity of $(\det)^{\frac{1}{d-1}}$ over $\mathbf{Sym}_d^+$ is a necessary condition for its weak$*$-upper semi-continuity on the space
$$\left\{T \in L^\infty(\Omega;\mathbf{Sym}_d^+) \,|\, \mathrm{Div}\, T \in L^\infty(\Omega;\mathbb{R}^d)\right\}.$$
Is this condition also sufficient ?

  In other words, if $T^\varepsilon(x) \in \mathbf{Sym}_d^+$, $T^\varepsilon \overset{*}{\rightharpoonup} T$ and $\|\mathrm{Div}\, T^\varepsilon\|_\infty$ remains bounded, do we have
$$w*\lim(\det T^\varepsilon)^{\frac{1}{d-1}} \leqslant \left(\det(w*\lim T^\varepsilon)\right)^{\frac{1}{d-1}} \qquad ?$$

Other kinetic equations: Vlasov models

A ionized gas (plasma) or a gravitational cluster (galaxy) obeys the Vlasov–Poisson equation

$$(\partial_t + v \cdot \nabla_y) f + F(t, y) \cdot \nabla_v f = 0$$

which expresses at the continuum level Newton's dynamics

$$\frac{dX}{dt} = v(X, t), \qquad \frac{dv}{dt} = F(X, t).$$

Coupling: the *force* F is determined by the density of the fluid (self-induction):

$$F = -\nabla_y \phi, \qquad \Delta_y \phi = \beta \rho, \qquad \beta \neq 0 \quad \text{a constant.}$$

More general Vlasov couplings:

$$\phi = \chi *_y \rho = \int_{\mathbb{R}^n} \chi(y - z) \rho(z) \, dz$$

where $\chi = \chi(r)$ is radial.

With the same notations as above, we have

$$\partial_t \rho + \mathrm{div}_y m = 0, \qquad \partial_t m + \mathrm{Div}_y S - \rho F = 0,$$

For the Vlasov–Poisson model, we may write $-\rho F = \frac{1}{\beta} \Delta\phi \, \nabla\phi = \mathrm{Div}_y \Sigma$ where

$$\Sigma = \frac{1}{\beta} (\nabla\phi \otimes \nabla\phi - \frac{1}{2} |\nabla\phi|^2 I_n)$$

but this tensor is not positive $\longrightarrow$ *our theory doesn't seem to apply !*

An alternate choice, valid for every kernel χ : one has $-\rho F = \mathrm{Div}_y \Sigma^\dagger$ for

$$\Sigma^\dagger := -\frac{1}{2}\int_{\mathbb{R}^n} \frac{\chi'(|z|)}{|z|} z \otimes z \, dz \int_{-\frac{1}{2}}^{\frac{1}{2}} \rho(y + (t - \frac{1}{2})z)\rho(y + (t + \frac{1}{2})z)\, dt.$$

Under the assumption that $\chi' \leqslant 0$ (repulsive force, OK for a plasma), we have

$$\Sigma^\dagger \in \mathbf{Sym}_n^+.$$

Then our theory applies to

$$T = \begin{pmatrix} \rho & m^T \\ m & S + \Sigma^\dagger \end{pmatrix} = T^B + \begin{pmatrix} 0 & \\ 0 & \Sigma^\dagger \end{pmatrix}.$$

Notice

$$\det T \geqslant \det T^{B},$$

whence the same estimate as for the Boltzmann equation.

We also have

$$\det T \geqslant \rho \det \Sigma^{\dagger},$$

whence the estimate

$$\int_0^\infty dt \int_{\mathbb{R}^n} (\rho \det \Sigma^{\dagger})^{\frac{1}{n}} dy \leqslant d_n M_0^{\frac{1}{n}} D_0^{\frac{1}{2}}.$$

Lecture #4 : Application to scalar conservation laws

Collaboration with Luis SILVESTRE, Univ. of Chicago

Multi-D Burgers equations

This is

$$u_t + \mathrm{div}_y f(u) = 0, \qquad f(u) = (\frac{u^2}{2}, \dots, \frac{u^{n+1}}{n+1}).$$

The most elementary flux f such that $f([a,b])$ is never contained in a hyperplane (genuine nonlinearity).

If $n = 1$ (original Burgers), $u_t + (\frac{u^2}{2})_y = 0$. Tartar (compensated compactness) and Golse (regularity) used compensated compactness and the symmetric tensor

$$T = \begin{pmatrix} u & \frac{1}{2}u^2 \\ \frac{1}{2}u^2 & \frac{1}{3}u^3 \end{pmatrix}.$$

$$\mathrm{Div}\, T = \begin{pmatrix} 0 & \mu \end{pmatrix}, \qquad \mu = -\text{entropy production}.$$

For $n \geqslant 2$, we use Compensated integrability with

$$T := \left(\frac{u^{i+j+p}}{i+j+p} \right)_{0 \leqslant i,j \leqslant n} \in \mathbf{Sym}_d.$$

If $u_0 \geqslant 0$ (hence $u \geqslant 0$), then $T(t,y) \in \mathbf{Sym}_d^+$. One has

$$\det T = H u^{d(d-1+p)}, \qquad H := \text{ Hilbert-like determinant.}$$

If $u_0 \in L^1 \cap L^d(\mathbb{R}^n)$, then $\operatorname{Div} T \in \mathcal{M}(\mathbb{R}_+ \times \mathbb{R}^n)$.

Compensated integrability gives

$$\int_0^{+\infty} dt \int_{\mathbb{R}^n} u^{p^*} dy \leqslant c_d \left(\sum_{j=0}^{n} \int_{\mathbb{R}^n} u_0^{j+p} dy \right)^{\frac{d}{d-1}}, \qquad p^* = d(1 + \frac{p}{d-1}).$$

As in gas dynamics, we exploit scale invariance, to infer

$$\int_0^{+\infty} dt \int_{\mathbb{R}^n} u^{p^*} dy \leqslant c_d \left(\int_{\mathbb{R}^n} u_0^p dy \right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^n} u_0^{p+n} dy \right)^{\frac{1}{2}}. \tag{15}$$

Application : Shift the initial time to $\tau \geqslant 0$. Use Hölder

$$\|u(\tau)\|_{p+n} \leqslant \|u(\tau)\|_p^{1-\theta} \|u(\tau)\|_{p^*}^{\theta},$$

then the decay of every L^p-norm:

$$\|u(\tau)\|_{p+n} \leqslant \|u_0\|_p^{1-\theta} \|u(\tau)\|_{p^*}^{\theta}.$$

Then (15) becomes a differential inequality. Gronwall argument yields the decay

$$\|u(t)\|_{p^*} \leqslant c_d t^{-\delta} \|u_0\|_p^{\gamma}.$$

Using an iteration *à la* De Giorgi, one obtains a dispersion estimate

$$\|u(t)\|_\infty \leqslant c_d t^{-\beta} \|u_0\|_p^{\alpha}$$

where $\alpha, \beta > 0$ depend only upon p and n.

This yields a new well-posedness result:

Theorem 12 *If $u_0 \in L^p(\mathbb{R}^n)$, the multi-dimensional Burgers equation admits an entropy solution.*

For every $t > 0$,

$$S_t L^p(\mathbb{R}^n) \subset \bigcap_{p \leqslant q \leqslant \infty} L^q(\mathbb{R}^n).$$

Thank you for your attention !

Merci !

Děkuji !