Waves and related processes in Geophysical Fluid Dynamics

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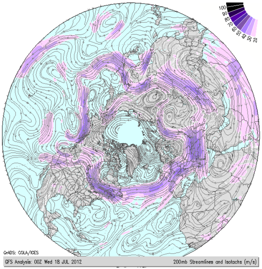
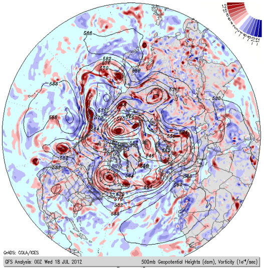
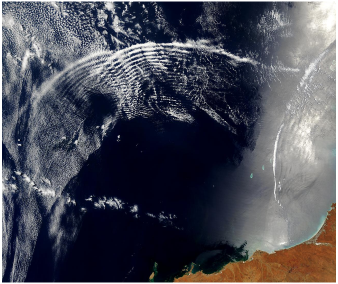
Plan

Atmospheric data : streamlines of the flow and velocity (colour) at the 200 mb level (left), and vorticity (colour)at the 500 mb level (right) of the atmosphere in the Northern hemisphere



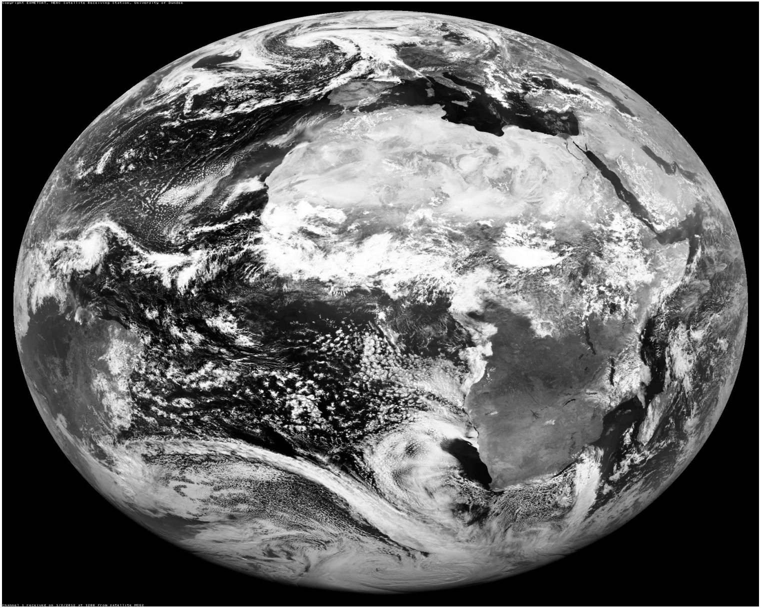


Internal waves in the atmosphere (left) and in the ocean (right), as seen from satellite.



Coast line and an island give an idea of spatial scale.

GFD : space view



GFD : what's that ?

Hydrodynamics in all its complexity plus :

- Rotating frame
- Thermal/stratification effects
- Spherical geometry (large- and meso-scales)
- Fluid in the complex domains (coasts, topography/bathymetry)
- Multi-component, multi- phase fluids (water vapour, salt, ice ...)

But !

These additional effects often allow to simplify the analysis

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Motion in the rotating frame

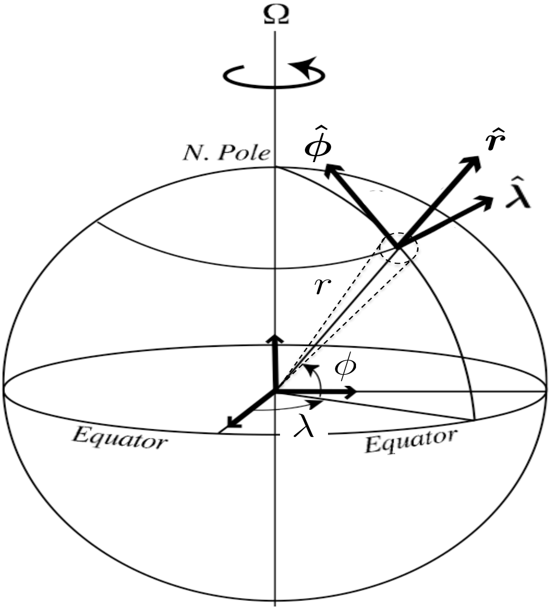
Euler equations in the rotating frame in the presence of gravity :

$$\frac{\partial \vec{v}}{\partial t} + \vec{v} \cdot \vec{\nabla} \vec{v} + 2\vec{\Omega} \wedge \vec{v} = -\frac{\vec{\nabla} P}{\rho} + \vec{g}^* \tag{1}$$

Effective gravity :

$$\vec{g}^* = \vec{g} + m\vec{\Omega} \wedge \left(\vec{\Omega} \wedge \vec{r} \right) \tag{2}$$

Spherical coordinates



Euler and continuity equations

$$\frac{dv_r}{dt} - \frac{v_\lambda^2 + v_\phi^2}{r} - 2\Omega\cos\phi v_\lambda + g^* = -\frac{1}{\rho}\partial_r P,$$

$$\frac{dv_\lambda}{dt} + \frac{v_r v_\lambda - v_\phi v_\lambda \tan\phi}{r} + 2\Omega\left(-\sin\phi v_\phi + \cos\phi v_r\right) = -\frac{1}{\rho r\cos\phi}\partial_\lambda P,$$

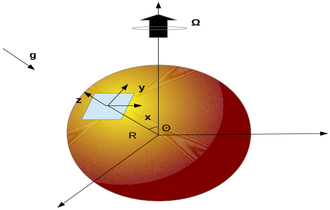
$$\frac{dv_\phi}{dt} + \frac{v_r v_\phi + v_\lambda^2\tan\phi}{r} + 2\Omega\sin\phi v_\lambda = -\frac{1}{\rho r}\partial_\phi P,$$

$$\frac{d\rho}{dt} + \rho\left[\frac{1}{r^2}\frac{\partial(r^2 v_r)}{\partial r} + \frac{1}{r\cos\phi}\left(\frac{\partial(\cos\phi v_\phi)}{\partial\phi} + \frac{\partial v_\lambda}{\partial\lambda}\right)\right],$$

$$\frac{d}{dt} = \frac{\partial}{\partial t} + v_r\partial_r + \frac{v_\phi}{r}\partial_\phi + \frac{v_\lambda}{r\cos\phi}\partial_\phi$$

Traditional approx. : green + red $\rightarrow$ out, $r \rightarrow R = \text{const}$
Non-traditional approx : green $\rightarrow$ out.

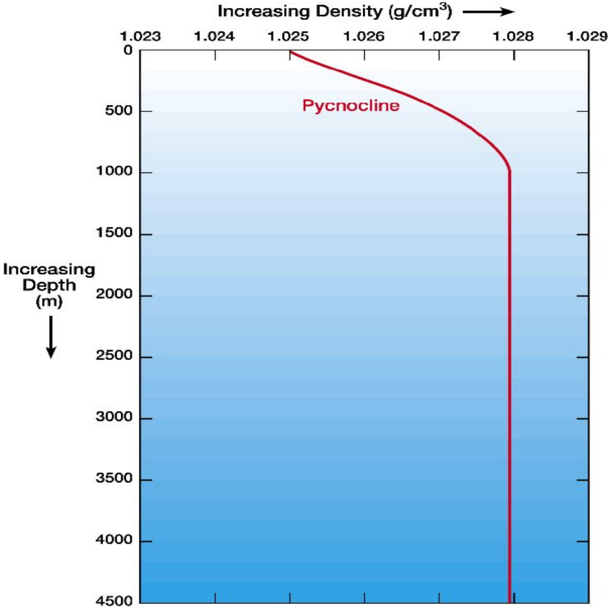
Tangent plane approximation



$$\frac{\partial \vec{v}}{\partial t} + \vec{v} \cdot \vec{\nabla} \vec{v} + f \hat{z} \wedge \vec{v} = -\frac{\vec{\nabla} P}{\rho} + \vec{g}$$

f - plane : $f = const$; β - plane : $f = f + \beta y$; f - Coriolis parameter : $f = 2\Omega \sin \phi$

Mean oceanic stratification



Primitive equations : ocean

Hydrostatics

$$g\rho + \partial_z P = 0, \qquad (3)$$

$$\begin{aligned} P &= P_0 + P_s(z) + \pi(x, y, z; t), \\ \rho &= \rho_0 + \rho_s(z) + \sigma(x, y, z; t), \quad \rho_0 \gg \rho_s \gg \sigma \end{aligned}$$

Incompressibility

$$\vec{\nabla} \cdot \vec{v} = 0, \quad \vec{v} = \vec{v}_h + \hat{z} w. \qquad (4)$$

Euler :

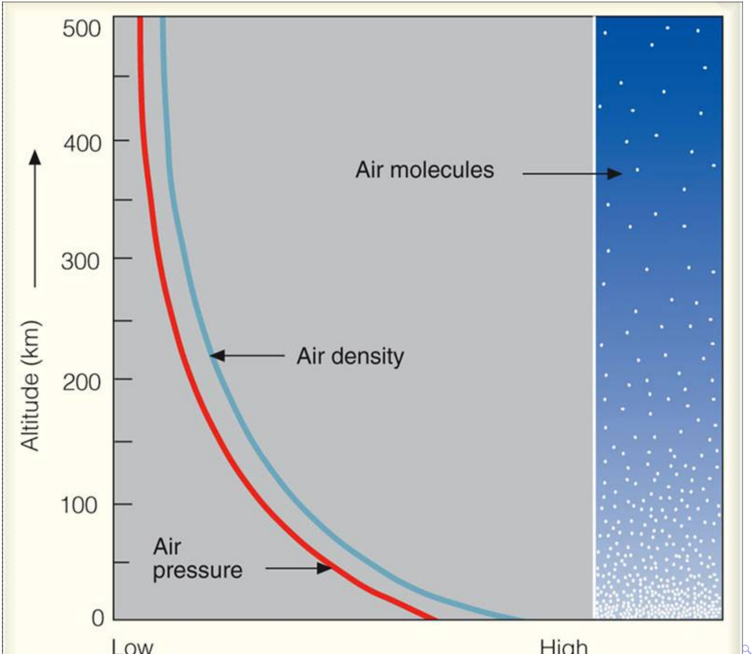
$$\frac{\partial \vec{v}_h}{\partial t} + \vec{v} \cdot \vec{\nabla} \vec{v}_h + f \hat{z} \wedge \vec{v}_h = -\vec{\nabla}_h \phi. \qquad (5)$$

$\phi = \frac{\pi}{\rho_0}$ - geopotential.

Continuity :

$$\partial_t \rho + \vec{v} \cdot \vec{\nabla} \rho = 0. \qquad (6)$$

Mean atmospheric stratification



Primitive equations : dry atmosphere, pseudo-height vertical coordinate

$$\frac{\partial \vec{v}_h}{\partial t} + \vec{v} \cdot \vec{\nabla} \vec{v}_h + f \hat{z} \wedge \vec{v}_h = -\vec{\nabla}_h \phi, \tag{7}$$

$$-g \frac{\theta}{\theta_0} + \frac{\partial \phi}{\partial z} = 0, \tag{8}$$

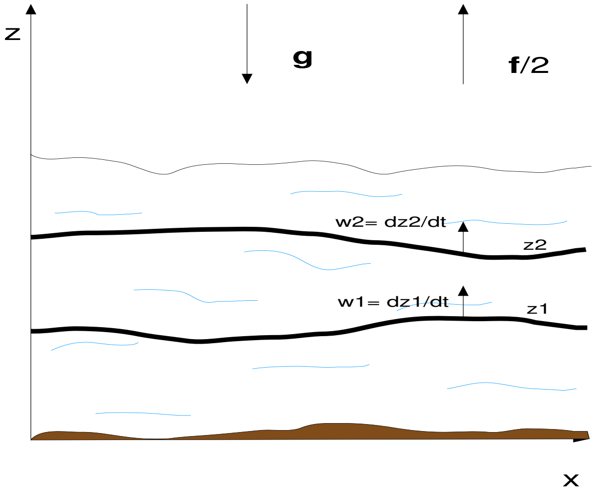
$$\frac{\partial \theta}{\partial t} + \vec{v} \cdot \vec{\nabla} \theta = 0; \quad \vec{\nabla} \cdot \vec{v} = 0. \tag{9}$$

Identical to oceanic ones with $\sigma \to -\theta$, potential temperature, directly related to entropy. Vertical coordinate : pseudo-height, P - pressure.

$$\bar{z} = z_0 \left(1 - \left(\frac{P}{P_s} \right)^{\frac{R}{c_p}} \right), \tag{10}$$

$R = c_p - c_v$, Mayer relation for ideal gas.

Material surfaces



Vertical averaging and RSW models

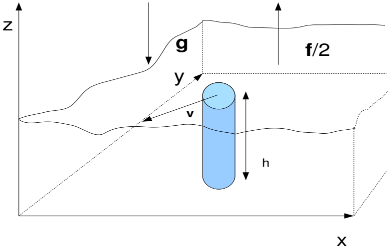
- Take horizontal momentum equation in conservative form : and integrate between a pair of material surfaces $z_{1,2}$,
- Use Leibniz formula and boundary conditions on material surfaces to eliminate vertical velocity
- Introduce the vertical (mass-) averages : and get averaged mass and horizontal momentum equations
- Use hydrostatics supposing mean constant mean density
- Use the mean-field (= columnar motion) approximation and get shallow water momentum equation for the fluid layer.
- Pile up layers, with lowermost boundary fixed by topography, and uppermost free or fixed.

1-layer RSW, $z_1 = 0$, $z_2 = h$

$$\partial_t \mathbf{v} + \mathbf{v} \cdot \nabla \mathbf{v} + f \hat{\mathbf{z}} \wedge \mathbf{v} + g \nabla h = 0 \,, \tag{11}$$

$$\partial_t h + \nabla \cdot (\mathbf{v} h) = 0 \,. \tag{12}$$

$\Rightarrow$ 2d barotropic gas dynamics + Coriolis force.
In the presence of nontrivial topography $b(x, y)$:
$h \to h - b$ in the second equation.



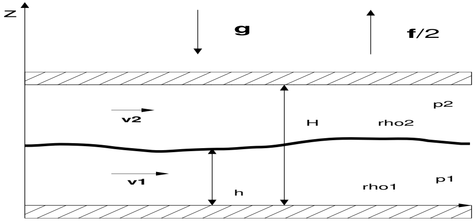
2-layer RSW, rigid lid : $z_1 = 0$, $z_2 = h$, $z_3 = H = \text{const}$

$$\partial_t \mathbf{v}_i + \mathbf{v}_i \cdot \nabla \mathbf{v}_i + f\hat{\mathbf{z}} \wedge \mathbf{v}_i + \frac{1}{\bar{\rho}_i} \nabla \pi_i = 0\,, i = 1, 2; \tag{13}$$

$$\partial_t h + \nabla \cdot (\mathbf{v}_1 h) = 0\,, \tag{14}$$

$$\partial_t (H - h) + \nabla \cdot (\mathbf{v}_2 (H - h)) = 0\,, \tag{15}$$

$$\pi_1 = (\bar{\rho}_1 - \bar{\rho}_2) g h + \pi_2\,. \tag{16}$$



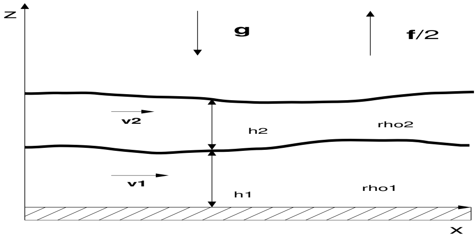
2-layer rotating shallow water model with a free surface : $z_1 = 0$, $z_2 = h_1$, $z_3 = h_1 + h_2$

$$\partial_t \mathbf{v}_2 + \mathbf{v}_2 \cdot \nabla \mathbf{v}_2 + f\hat{\mathbf{z}} \wedge \mathbf{v}_2 = -g\nabla(h_1 + h_2) \tag{17}$$

$$\partial_t \mathbf{v}_1 + \mathbf{v}_1 \cdot \nabla \mathbf{v}_1 + f\hat{\mathbf{z}} \wedge \mathbf{v}_1 = -g\nabla(rh_1 + h_2), \tag{18}$$

$$\partial_t h_{1,2} + \nabla \cdot \left(\mathbf{v}_{1,2} h_{1,2}\right) = 0\,, \tag{19}$$

where $r = \frac{\rho_1}{\rho_2} \leq 1$ - density ratio, and $h_{1,2}$ - thicknesses of the layers.



Equations of horizontal motion

$$\frac{\partial \vec{v}_h}{\partial t} + \vec{v} \cdot \vec{\nabla} \vec{v}_h + f \hat{z} \wedge \vec{v}_h = -\vec{\nabla}_h \Phi. \qquad (20)$$

$$f = f_0(1 + \beta y), \quad \Phi = \Phi_0 + \phi = g(H_0 + h) \qquad (21)$$

h - geopotential (perturbation) height.

Scaling for eddy motions

- Velocity $\vec{v}_h = (u, v), \ \ u, v \sim U, w \sim W << U$
- Unique horizontal spatial scale L,
- Vertical scale $H << L$,
- Time-scale : turn-over time $T \sim L/U$.

Characteristic parametres

Intrinsic scale of the system : deformation (Rossby) radius :

$$R_d = \frac{\sqrt{gH_0}}{f_0} \tag{22}$$

- Rossby number : $Ro = \frac{U}{f_0 L}$ - ratio of fast and slow time-scales,
- Burger number : $Bu = \frac{R_d^2}{L^2}$,
- Characteristic non-linearity : $\lambda = \Delta H / H_0$, where ΔH is the typical value of h,
- Dimensionless gradient of f : $\tilde{\beta} \sim \beta L$

Geostrophic balance

Non-dimensional equations of horizontal motion

$$Ro\,(\partial_t \mathbf{v}_h + \mathbf{v}\cdot\nabla\mathbf{v}_h) + (1+\tilde{\beta})\hat{\mathbf{z}}\wedge\mathbf{v}_h = -\frac{\lambda Bu}{Ro}\nabla_h h\,, \quad (23)$$

Geostrophic equilibrium

Balance between the Coriolis force and the pressure force $\rightarrow$ geostrophic wind :

$$\hat{\mathbf{z}}\wedge\mathbf{v}_g = -\nabla h \quad (24)$$

Conditions of realisation :

- $Ro \rightarrow 0$,
- $\lambda\, Bu \sim Ro$,
- $\tilde{\beta} \rightarrow 0$.

QG regime in RSW

Non-dimensional RSW equations

$$Ro\,(\partial_t \mathbf{v} + \mathbf{v}\cdot\nabla\mathbf{v}) + (1+\tilde{\beta}y)\hat{\mathbf{z}}\wedge\mathbf{v} = -\frac{\lambda Bu}{Ro}\nabla\eta\,, \tag{25}$$

$$\lambda\partial_t\eta + \nabla\cdot(\mathbf{v}(1+\lambda\eta)) = 0\,. \tag{26}$$

QG regime

$$\lambda \sim Ro, \Rightarrow Bu \sim 1, \Rightarrow L \sim R_d,\ \tilde{\beta} \sim Ro \ll 1. \tag{27}$$

Asymptotic expansions

$$\epsilon\left(\partial_t \mathbf{v} + \mathbf{v}\cdot\nabla\mathbf{v}\right) + (1+\epsilon y)\hat{\mathbf{z}}\wedge\mathbf{v} = -\nabla\eta\,, \tag{28}$$

$$\epsilon\partial_t\eta + \nabla\cdot(\mathbf{v}(1+\epsilon\eta)) = 0\,, \epsilon \equiv Ro \ll 1. \tag{29}$$

$$\mathbf{v} = \mathbf{v}^{(0)} + \epsilon\mathbf{v}^{(1)} + \epsilon^2\mathbf{v}^{(2)} + \ldots \tag{30}$$

Order ϵ^0

$$u^{(0)} = -\partial_y\eta,\quad v^{(0)} = \partial_x\eta \quad\Rightarrow\quad \partial_x u^{(0)} + \partial_y v^{(0)} = 0, \tag{31}$$

$$\frac{d^{(0)}}{dt}\cdots = \partial_t\ldots + u^{(0)}\partial_x\ldots + v^{(0)}\partial_y\ldots \equiv \partial_t\cdots + \mathcal{J}(\eta,\ldots). \tag{32}$$

$$\mathcal{J}(A,B) \equiv \partial_x A\partial_y B - \partial_y A\partial_x B. \tag{33}$$

Order ϵ^1

$$u^{(1)} = -\frac{d^{(0)}}{dt} v^{(0)} - y u^{(0)}, \quad v^{(1)} = \frac{d^{(0)}}{dt} u^{(0)} - y v^{(0)}, \Rightarrow \tag{34}$$

$$\partial_x u^{(1)} + \partial_y v^{(1)} = -\frac{d^{(0)}}{dt} \vec{\nabla}^2 \eta - v^{(0)}, \Rightarrow \tag{35}$$

$$\frac{d^{(0)}}{dt} \left(\eta - \vec{\nabla}^2 \eta \right) - \partial_x \eta = 0 \leftrightarrow \frac{d^{(0)}}{dt} \left(\eta - \vec{\nabla}^2 \eta - y \right) = 0. \tag{36}$$

Detailed writing with $\bar{\beta} = \tilde{beta}/Ro$ restored, for convenience.

$$\partial_t \eta - \vec{\nabla}^2 \partial_t \eta - \mathcal{J}(\eta, \vec{\nabla} \eta) - \bar{\beta} \partial_x \eta = 0. \tag{37}$$

Similar procedure and scaling → equations for the pressures in the layers

$$\frac{d_i^{(0)}}{dt}\left[\nabla^2\pi_i - (-1)^i D_i^{-1}\eta + y\right] = 0\,,\ i = 1, 2. \qquad (38)$$

where

$$\frac{d_i^{(0)}}{dt}(\ldots) := \partial_t(\ldots) + J(\pi_i, \ldots)\,,\ i = 1, 2\,, \qquad (39)$$

where $D_i = \frac{H_i}{H}$, non-dimensional heights of the layers, and

$$\pi_2 - \pi_1 + \frac{N}{2}(\pi_2 + \pi_1) = \eta. \qquad (40)$$

$N = 2\frac{\rho_2 - \rho_1}{\rho_2 + \rho_1}$. Standard limit : weak stratification → $\rho_2 \to \rho_1 \Rightarrow \eta = \pi_2 - \pi_1$

Non-dimensional PE under slow-motion scaling

$$\epsilon \frac{d}{dt}\mathbf{v}_h + (1+\tilde{\beta}y)\hat{z}\wedge\mathbf{v}_h = -\vec{\nabla}_h \pi. \quad (41)$$

$$\frac{d}{dt}\sigma + \rho_s' w = 0, \quad \partial_z \pi + \sigma = 0. \quad (42)$$

$$\vec{\nabla}_h \cdot \mathbf{v}_h + \lambda \partial_z w = 0; \quad (43)$$

where $\frac{d}{dt} = \partial_t + \mathbf{v}_h \cdot \nabla_h + \lambda w \partial_z$, λ - typical deviation of the isopycnals $\sigma =$ const Boundary conditions - rigid lid/flat bottom :

$$w|_{z=0,1} = 0. \quad (44)$$

QG regime : $\epsilon \sim \lambda \sim \tilde{\beta} \ll 1$.

QG equations

Asymptotic expansion in ϵ + elimination of w and σ $\rightarrow$

$$\frac{d^{(0)}}{dt}\left(-\nabla_h^2\pi - y + \partial_z\left(\frac{1}{\rho_s'(z)}\partial_z\pi\right)\right) = 0, \tag{45}$$

$$\text{c.l. : } w\big|_{z=0,1} = 0 \Rightarrow \left.\frac{d^{(0)}}{dt}\partial_z\pi\right|_{z=0,1} = 0. \tag{46}$$

Linearising Primitive Equations on the f-plane about the state of rest

$$\begin{cases} u_t - fv + \phi_x = 0, \\ v_t + fu + \phi_y = 0, \\ \phi_z + \frac{g}{\rho_0}\sigma = 0, \\ \sigma_t + w\rho_s' = 0, \\ u_x + v_y + w_z = 0. \end{cases} \quad (47)$$

u, v, w are three components of velocity perturbation, ϕ - geopotential perturbation, σ - perturbation of the profile of background density ρ_s. Successive elimination of σ and w :

$$\begin{cases} u_t - fv + \phi_x = 0, \\ v_t + fu + \phi_y = 0, \\ u_x + v_y - N^{-2}\phi_{zzt} = 0. \end{cases} \quad (48)$$

If Brunt - Väisälä frequency $N^2 = -\frac{g\rho_s'}{\rho_0}$ is constant, this is a system of linear equations with constant coefficients which can be treated by the method of Fourier.

Dispersion relation and spectral gap

Harmonic waves :

$$(u, v, \phi) = (u_0, v_0, \phi_0)e^{i(\omega t - \mathbf{k}\cdot\mathbf{x})} + c.c. \tag{49}$$

Dispersion relation

$$\omega\left(\omega^2 - \left(N^2\frac{k_x^2 + k_y^2}{k_z^2} + f^2\right)\right) = 0. \tag{50}$$

Three roots : two different kinds of solutions :

- Propagative inertia-gravity waves (IGW) with dispersion relation :

$$\omega = \pm\sqrt{N^2\frac{k_x^2 + k_y^2}{k_z^2} + f^2}, \tag{51}$$

- Stationary solutions with $\omega = 0$ $\leftrightarrow$ linearised conservation of Potential Vorticity, vortices.

Spectral gap : $\omega \geq f$ for IGW.

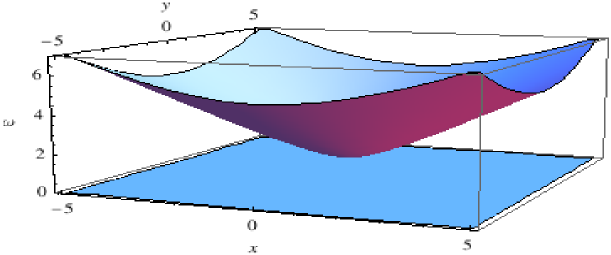
Linearising 1-layer RSW over the state of rest

Small perturbations u, v, η about the state of rest with $\mathbf{v} = 0$, $h = H_0 = const$ in the f-plane $\rightarrow$

$$\begin{cases} u_t - fv + g\eta_x = 0, \\ v_t + fu + g\eta_y = 0, \\ \eta_t + H_0(u_x + v_y) = 0, \end{cases} \tag{52}$$

Dispersion relation :

$$\omega\left(\omega^2 - gH_0\mathbf{k}^2 - f^2\right) = 0. \tag{53}$$



Artist's view of shallow vortex dynamics : vortices, waves, and topography (to appear in the next lecture)in shallow water

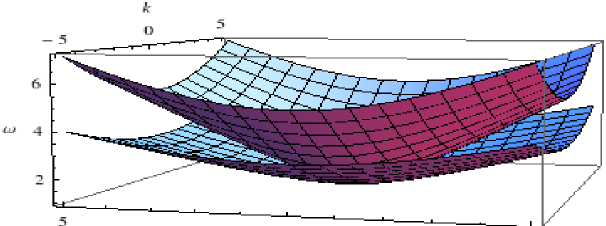
Linearising 2-layer RSW over the state of rest

$\eta_i,\ i = 1, 2$ - perturbations of free surface and interface :

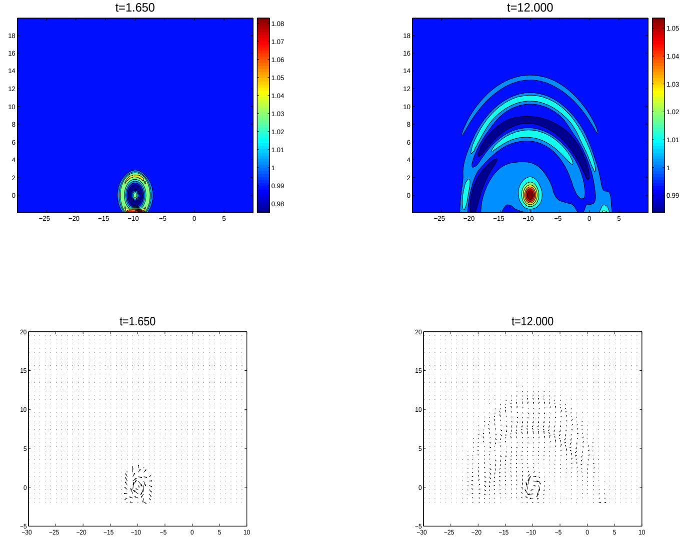
$$\begin{cases} \partial_t \vec{v}_i + f\hat{\mathbf{z}} \wedge \vec{v}_i + g\vec{\nabla}\left(r^{i-1}\eta_1 + \eta_2\right) = 0, \\ \partial_t \eta_i + H_i \vec{\nabla} \cdot \vec{v}_i = 0. \end{cases} \tag{54}$$

Simplest case : $H_1 = H_2 = \frac{H}{2}$. Barotropic-baroclinic decomposition $\vec{v}^{\pm} = \sqrt{r}\vec{v}_1 \pm \vec{v}_2, \quad \eta^{\pm} = 2\left(\sqrt{r}\eta_1 \pm \eta_2\right)$.
Dispersion relation (zero roots - out) :

$$\omega_{\pm}^2 = c_{\pm}^2 \mathbf{k}^2 + f^2, \quad c_{\pm} = \sqrt{gH\frac{1 \pm \sqrt{r}}{2}}. \tag{55}$$



Ubiquity of IGW : relaxation of pressure anomaly



1-layer (barotropic) QG model

f-plane

QG equation

$$\partial_t \vec{\nabla}^2 \eta - \partial_t \eta + \mathcal{J}(\eta, \vec{\nabla}^2 \eta) = 0.$$

Linearisation : $\partial_t \vec{\nabla}^2 \eta = 0$ - no waves

β- plane

$$\partial_t \vec{\nabla}^2 \eta - \partial_t \eta + \mathcal{J}(\eta, \vec{\nabla}^2 \eta) + \partial_x \eta = 0.$$

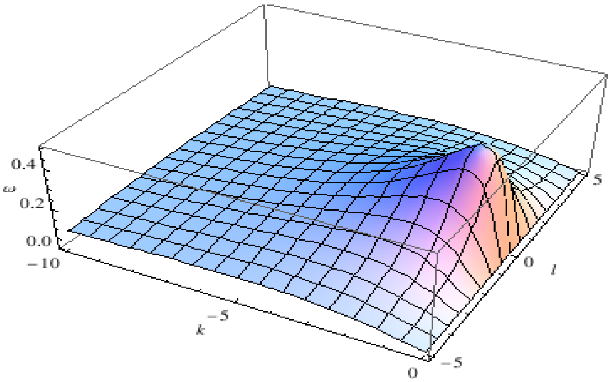
Linearisation :

$$\partial_t \eta - \partial_t \vec{\nabla}^2 \eta - \partial_x \eta = 0. \quad (56)$$

Waves $\eta \propto e^{i(kx+ly-\omega t)}$ with dispersion relation

$$\omega = -\frac{k}{k^2 + l^2 + 1}. \quad (57)$$

Dispersion relation of barotropic Rossby waves on the β- plane



Negative values $\omega < 0$ are not shown.

2-layer QG model with a rigid lid

Linearisation in the limit of weak stratification $r \to 1$

$$\begin{cases} \partial_t \left[\nabla^2 \pi_1 + F_1 (\pi_2 - \pi_1) \right] + \partial_x \pi_1 = 0, \\ \partial_t \left[\nabla^2 \pi_2 - F_2 (\pi_2 - \pi_1) \right] + \partial_x \pi_2 = 0. \end{cases} \tag{58}$$

Looking for wave solutions $\pi_i = A_i e^{i(\mathbf{k}\cdot\mathbf{x} - \omega t)} + c.c.$ we get the dispersion relation :

$$\omega = -\frac{k_x}{2\mathbf{k}^2(\mathbf{k}^2 + F_1 + F_2)} \left[(2\mathbf{k}^2 + F_1 + F_2) \pm (F_1 + F_2) \right]. \tag{59}$$

Two solutions correspond to :

- a faster barotropic mode : $\omega_{bt} = -\frac{k_x}{\mathbf{k}^2}$,
- a slower baroclinic mode : $\omega_{bc} = -\frac{k_x}{(\mathbf{k}^2 + F_1 + F_2)}$.

As in the one-layer case, these waves are Rossby waves arising due to the β - effect.

Baroclinic Rossby waves : continuous stratification

Formal linearisation

$$\partial_t \left[\nabla_h^2 \pi - \partial_z \left(\frac{1}{\rho_s'(z)} \partial_z \pi \right) \right] + \partial_x \pi = 0, \quad \partial_{tz}^2 \pi \Big|_{z=0,1} = 0. \tag{60}$$

Separation of variables

$$\pi(x, y, z; t) = p(x, y; t) S(z) \Rightarrow \tag{61}$$

$$\partial_t \nabla_h^2 p(x, y; t) S(z) - \partial_t p(x, y; t) \left[\frac{1}{\rho_s'(z)} S'(z) \right]' + \partial_x p(x, y; t) S(z) = 0 \Rightarrow$$

Equations in z and in x, y, t :

- $$\frac{1}{S(z)}\left[\frac{1}{\rho_s'(z)}S'(z)\right]' = \kappa^2 \qquad (62)$$

- $$\partial_t \nabla_h^2 p(x,y;t) - \kappa^2 \partial_t p(x,y;t) + \partial_x p(x,y;t) = 0, \qquad (63)$$

κ - separation constant

Vertical modes

Sturm - Liouville problem :

$$\left[\frac{1}{\rho_s'(z)} S'(z)\right]' - \kappa^2 S(z) = 0, \quad S'(z)\big|_{z=0,1} = 0 \qquad (64)$$

Eigenfunctions $S_n(z)$ and eigenvalues $\kappa_n, n = 0, 1, 2, ...$

Rossby waves : $p(x, y; t) \propto e^{i(\mathbf{k}\cdot\mathbf{x} - \omega t)}$:

$$\omega = -\frac{k_x}{\mathbf{k}^2 + \kappa_n^2}. \qquad (65)$$

The larger is the vertical wavenumber n (stronger vertical shear) $\rightarrow$ the slower is the propagation. f - plane : no waves.

What have we seen :

- Hierarchy of the GFD models : from PE to QG, passing through RSW
- Inertia-gravity wave-vortex dichotomy in the f- plane approximation, and their time-scale separation (spectral gap) : waves - fast, vortices - slow
- Rossby waves appearing in the vortex-motion sector in the β-plane

What we have not seen :

Waves in the presence of

- boundaries
- non-trivial topography
- mean flow
- at the equator, where there is no f_0

All this is coming up !