Waves and related processes in Geophysical Fluid Dynamics

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Plan

Non-linear Rossby waves in the QG RSW model on the β-plane

Non-dimensional equations of motion :

$$\nabla^2\psi_t - \psi_t + \epsilon\mathcal{J}(\psi, \nabla^2\psi) + \psi_x = 0,$$

ϵ - non-linearity parametre, $\epsilon \to 0$. Asymptotic expansion in non-linearity parametre. Solution - asymtotic series :

$$\psi = \psi^{(0)} + \epsilon\psi^{(1)} + \dots$$

Order zero : linear Rossby waves.

$$\nabla^2\psi_t^{(0)} - \psi_t^{(0)} + \psi_x^{(0)} = 0, \;\Rightarrow$$

$$\psi^{(0)} = \sum_i A_i e^{i(\mathbf{k}_i\cdot\mathbf{x} - \omega(\mathbf{k}_i)t)} + c.c., \quad \omega(\mathbf{k}) = -\frac{k}{k^2 + l^2 + 1}, \;\; \mathbf{k} = (k, l). \tag{1}$$

Order one : first non-linear correction :

$$\nabla^2 \psi_t^{(1)} - \psi_t^{(1)} + \psi_x^{(1)} = -\mathcal{J}(\psi^{(0)}, \nabla^2 \psi^{(0)}), \qquad (2)$$

Term in the r.h.s. :

$$\sum_{i,j} A_i A_j \left[\left(k_i l_j - k_j l_i \right) \mathbf{k}_j^2 \right] e^{i\left[(\mathbf{k}_i + \mathbf{k}_j)\cdot \mathbf{x} - (\omega(\mathbf{k}_i) + \omega(\mathbf{k}_j)) t \right]} \qquad (3)$$

$$- \sum_{i,j} A_i A_j^* \left[\left(k_i l_j - k_j l_i \right) \mathbf{k}_j^2 \right] e^{i\left[(\mathbf{k}_i - \mathbf{k}_j)\cdot \mathbf{x} - (\omega(\mathbf{k}_i) - \omega(\mathbf{k}_j)) t \right]} + c.c.$$

Integrability conditions : solution $\psi^{(1)}$ should be bounded

Integrability conditions :

$$
\begin{aligned}
\forall \hat{\psi} : \nabla^2 \hat{\psi}_t - \hat{\psi}_t + \hat{\psi}_x &= 0, \\
\int_{-\infty}^{\infty} dt \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy \hat{\psi}^* \left(\nabla^2 \psi_t^{(1)} - \psi_t^{(1)} + \psi_x^{(1)} \right) &= 0.
\end{aligned}
$$

Therefore :

$$
\int_{-\infty}^{\infty} dt \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy \hat{\psi}^* \left(\mathcal{J}(\psi, \nabla^2 \psi) \right) = 0. \qquad (4)
$$

- orthogonality of the r.h.s. to the eigen-vectors of the zero-order linear operator.

Resonances

Necessarily : $\hat{\psi} \propto e^{i(\hat{\mathbf{k}}\cdot\mathbf{x}-\omega(\hat{\mathbf{k}})t)}$, and (4) becomes :

$$\sum_{i,j} A_i A_j \left[\left(k_i l_j - k_j l_i\right) \mathbf{k}_j^2\right] \cdot \int_{-\infty}^{\infty} dt\, dx\, dy e^{i\left[(\mathbf{k}_i+\mathbf{k}_j-\hat{\mathbf{k}})\cdot\mathbf{x}-(\omega(\mathbf{k}_i)+\omega(\mathbf{k}_j)-\omega(\hat{\mathbf{k}}))t\right]} - \sum_{i,j} A_i A_j^* \left[\left(k_i l_j - k_j l_i\right) \mathbf{k}_j^2\right] \cdot \int_{-\infty}^{\infty} dt\, dx\, dy e^{i\left[(\mathbf{k}_i-\mathbf{k}_j-\hat{\mathbf{k}})\cdot\mathbf{x}-(\omega(\mathbf{k}_i)-\omega(\mathbf{k}_j)-\omega(\hat{\mathbf{k}}))t\right]} + c.c. = 0$$

Calculus of generalised functions :

$$\int_{-\infty}^{\infty} dx e^{ikx} = \delta(k) - \text{Dirac's delta-function.} \qquad (5)$$

Generalisation of

$$\int_{0}^{2\pi} dx e^{ikx} = \delta_{k0} - \text{tensor delta of Kronecker} \qquad (6)$$

for periodic boundary conditions.

Resonances :

Non-zero contributions :

$$\mathbf{k}_i \pm \mathbf{k}_j = \hat{\mathbf{k}}, \quad \omega(\mathbf{k}_i) \pm \omega(\mathbf{k}_j) = \omega(\hat{\mathbf{k}}). \qquad (7)$$

three-wave resonances, resonant triads.

Elimination of resonances

If $\exists\, \hat{\mathbf{k}}$ which verifies (7) the first non-linear correction is not bounded $\Rightarrow$ asymptotic procedure is not self-consistent : resonances should be "killed".

Introducing slow evolution of the amplitudes :

$$\begin{aligned} \partial_t \to \partial_t + \epsilon \partial_T \quad \Rightarrow \qquad (8) \\ \nabla^2 \psi_t^{(1)} - \psi_t^{(1)} + \psi_x^{(1)} = -\nabla^2 \psi_T^{(0)} - \psi_T^{(0)} - \mathcal{J}(\psi^{(0)}, \nabla^2 \psi^{(0)}) \end{aligned}$$

New contribution in the r.h.s. :

$$\sum_i A_{i_T} e^{i(\mathbf{k}_i \cdot \mathbf{x} - \omega(\mathbf{k}_i) t)} + c.c. \quad \Rightarrow \qquad (9)$$

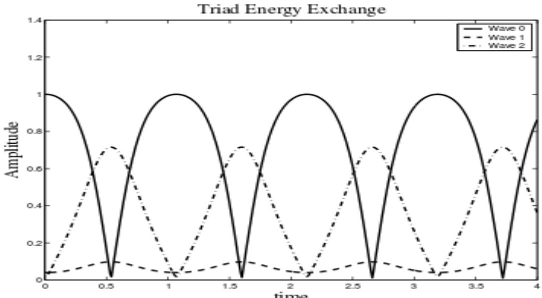
Possibility of compensation of resonant contributions by slow evolution of amplitudes.

A resonant triad :

$$\mathbf{k}_1 + \mathbf{k}_2 = \mathbf{k}_3, \quad \omega(\mathbf{k}_1) + \omega(\mathbf{k}_2) = \omega(\mathbf{k}_3), \tag{10}$$

$$\begin{aligned} \dot{A}_3 &= c(\mathbf{k}_1, \mathbf{k}_2) A_1 A_2, \\ \dot{A}_2 &= c(\mathbf{k}_3, -\mathbf{k}_1) A_3 A_1^*, \\ \dot{A}_1 &= c(\mathbf{k}_3, -\mathbf{k}_2) A_3 A_2^*, \end{aligned} \tag{11}$$

where $c(\mathbf{k}_1, \mathbf{k}_2) = \hat{\mathbf{z}} \cdot (\mathbf{k}_1 \wedge \mathbf{k}_2) \mathbf{k}_2^2$ - interaction coefficients . This is an integrable system (in elliptic fonctions). Energy is conserved and redistributed among three waves.



Waveguides in RSW

Equator

- Trapped waves - baroclinic : Kelvin (nondispersive), IG, Rossby, Yanai (dispersive),
- Free waves : barotropic Rossby (dispersive)

Coast/Escarpment/Ridge

- Trapped waves
  - Coast : shelf and edge waves (dispersive) and Kelvin (nondispersive for vertical edge),
  - Escarpment/Ridge : topographic Rossby(dispersive)
- Free waves (incident + reflected + transmitted) : IG (dispersive).

The ideology

- Interactions between free waves and trapped waves (waveguide modes) may be resonant.
- If so the incoming free waves may resonantly excite waveguide modes.
- The resonant growth, if confirmed by "naive" asymptotic expansions, should be nonlinearly or dissipatively saturated in one way or another in the "improved" expansions leading to coherent structures formation.

Remark :

One of the waveguide modes forming a resonant triad may have infinite wavelength, i.e. be a mean flow.

Workflow

- Take a waveguide in (multi-layer) RSW
- Analyse the dispersion relations for free and trapped waves and check if resonances between them are possible (algebra may be tough)
- Apply straightforward ("naive") asymptotic expansions in nonlinearity parameter to the flow consisting of resonating waves and "eliminate" resonances. Caveat : *Normalizations are different for trapped and free waves - should be careful*.
- Check a possibility of parametric resonance in slow-time evolution equations and growth of amplitude(s) of the trapped wave(s).

Workflow - continued

- If yes, rearrange the asymptotic expansions to account for possible nonlinear saturation and get nonlinear amplitude equations (Landau-type for dispersive waves). Check saturation.
- Introduce (weak) spatial modulation and obtain Ginzburg-Landau (or nonlinear Schrödinger) type (dispersive waves), or Burgers-type (non-dispersive waves) evolution equations for wave amplitudes.
- Look for pattern formation in modulated amplitudes.

Resonances leading to excitation of the trapped waves

- Free wave $\rightarrow$ Trapped wave + Trapped wave
"Naive" perturbation theory $\Rightarrow$

$$A_{TT}^{(t)} = C\left|A^{(f)}\right|^2 A^{(t)}, \quad C > 0, \quad A^{(t)} \propto e^{cT}$$

- Free wave + Free wave $\rightarrow$ Trapped wave
"Naive" perturbation theory $\Rightarrow$

$$A_T^{(t)} = CA_1^{(f)}A_2^{(f)}, \quad A^{(t)} \propto T$$

- Free wave + Mean flow $\rightarrow$ Trapped wave
"Naive" perturbation theory $\Rightarrow$

$$A_T^{(t)} = CA^{(mf)}A^{(f)}, \quad A^{(t)} \propto T$$

RSW in the presence of the coast

Non-dimensional equations of motion

$$\begin{aligned} u_t - v + h_x &= -\epsilon(uu_x + vu_y) \\ v_t + u + h_y &= -\epsilon(uv_x + vv_y) \\ h_t + u_x + v_y &= -\epsilon\left((hu)_x + (hv)_y\right). \end{aligned} \tag{12}$$

Boundary condition : $x \geq 0, \quad u|_{x=0} = 0$.

Linear waves

Kelvin wave (KW)

$$(u, v, h) = (0, K(y + t), -K(y + t))\, e^{-x}; \qquad (13)$$

Inertia-gravity waves (IGW) :

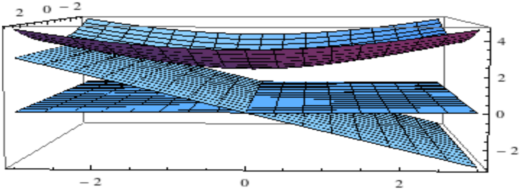
Incident wave, $\sigma = \sqrt{1 + k^2 + l^2}$:

$$(u_i, v_i, h_i) = A_i \left(\frac{-k\sigma + il}{\sigma^2 - 1}, \frac{l\sigma + ik}{\sigma^2 - 1}, 1 \right) e^{i(ly - kx - \sigma t)} + c.c. \qquad (14)$$

Reflected wave, $\sigma = \sqrt{1 + k^2 + l^2}$:

$$(u_r, v_r, h_r) = A_r \left(\frac{k\sigma + il}{\sigma^2 - 1}, \frac{l\sigma - ik}{\sigma^2 - 1}, 1 \right) e^{i(ly + kx - \sigma t)} + c.c., \qquad (15)$$

Dispersion diagram



IGW - KW resonance

A pair of IGW of the form is in resonance with a KW if

$$\sigma_1 - \sigma_2 = -l, \quad l_1 - l_2 = l, \ l \neq 0. \tag{16}$$

We choose $l < 0$:

$$|l| = \sqrt{1 + k_1^2 + l_1^2} - \sqrt{1 + k_2^2 + l_2^2}, \ \ l_2 = l_1 + |l|\,, \tag{17}$$

and

$$\sqrt{1 + k_1^2 + l_1^2} - |l| = \sqrt{1 + k_2^2 + (l_1 + |l|)^2}. \tag{18}$$

Algorithm for finding resonances :

1. Take any l and l_1, then $l_2 = l_1 + |l|$,
2. Take arbitrary k_1 satisfying $k_1^2 \geq 2|l| \left(\sqrt{1 + l_2^2} + l_2\right)$,
3. Define k_2 from

$$k_2^2 = k_1^2 - 2|l| \left(\sqrt{1 + k_1^2 + l_1^2} + l_1\right).$$

Therefore, a KW with wavenumber l may be resonantly excited by *a continuum* of incident IGW with wavenumbers l_1 and $|k_1| > \sqrt{2|l| \left(\sqrt{1 + (l_1 + |l|)^2} + l_1 + |l|\right)}$ interacting with another incident wave with k_2, l_2 :

$$k_2^2 = k_1^2 - 2|l| \left(\sqrt{1 + k_1^2 + l_1^2} + l_1\right), \quad l_2 = l_1 + |l|. \quad (19)$$

Removal of resonances condition in RSW equations with coast :

$$\int_0^\infty dx\, e^{-x}(R_h - R_v) = 0, \tag{20}$$

where $R_{h,v}$ - r.h.s. of h- and v - equations.

Evolution equation for the amplitude of the Kelvin wave

$$K_T + KK_\eta = Se^{il\eta} + S^* e^{-il\eta}, \tag{21}$$

where $\eta = y + t$,

$$\begin{aligned} S &= \int_0^\infty dx\, e^{-x} \left[(H_1 U_2^* + U_1 H_2^*)_x - U_1 V_{2_x}^* - V_{1_x} U_2 \right. \\ &+ \left. il(H_1 V_2^* + V_1 H_2^* - V_1 V_2^*) \right] \end{aligned} \tag{22}$$

and (U_i, V_i, H_i), $i = 1, 2$ are amplitudes of two IGW.

Final form of the evolution equation

From the polarisation relations get :

$$S = iA_1 A_2 s, \quad Ims = 0 \tag{23}$$

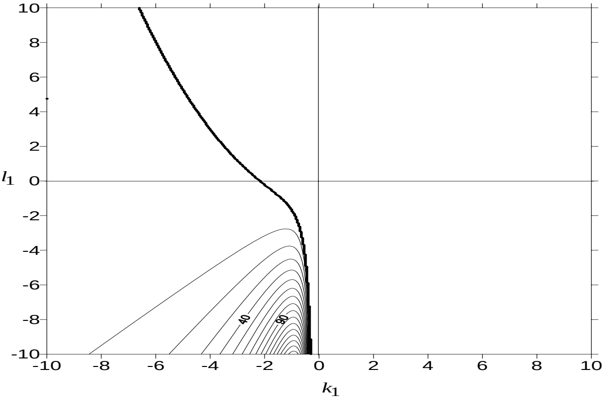
and hence

$$K_T + KK_\eta = -2sA_1 A_2 \sin l\eta, \tag{24}$$

where A_i - amplitudes of the two IGW,

$$\begin{aligned} s \;=\; & \frac{4l}{(k_1^2+1)(k_2^2+1)[1+(k_1+k_2)^2][1+(k_1-k_2)^2]} \cdot \\ & \Big[(\sigma_1 l_2 + \sigma_2 l_1 - l_1 l_2)(1+k_1^2+k_2^2) \\ & +\sigma_2 l_1 k_1(1+k_1^2-k_2^2) + \sigma_1 l_2 k_2(1+k_2^2-k_1^2) \\ & +2k_1 k_2(l_1 l_2 - (1+k_1^2)(1+k_2^2))\Big] \end{aligned} \tag{25}$$

Isopleths of the interaction coefficient $s(l, k_1, l_1)$ for $l = -1$ at the interval 10



Integrability of the KW evolution equation

Forced simple-wave equation (after renormalizations) :

$$K_\tau + KK_\chi = -\sin\chi. \tag{26}$$

Lagrangian (characteristics) approach :

$$K = \mathcal{U} = \dot{\mathcal{X}}; \quad (\dot{...}) = \partial_\tau + \mathcal{U}\partial_\chi(...) \Rightarrow \tag{27}$$

$$\ddot{\mathcal{X}} + \sin\mathcal{X} = 0 \tag{28}$$

Pendulum equation : integrable. Shock formation $\leftrightarrow$ Lagrangian clustering (known in statistical physics : mean-field limit of the kinetics of particles with repulsive long-range interaction on the circle) $\Rightarrow$ Implications for transport and mixing.

Ramifications :

- Including small dissipation $\Rightarrow$ harmonically forced Burgers equation. Cole-Hopf change of variables $\rightarrow$ Mathieu equation for Laplace transform in time $\rightarrow$ integrable.
- Including small dispersion (long waves near a steep, but not vertical border) $\Rightarrow$ harmonically forced KdV equation.

Summary

Resonant excitation of Kelvin waves by pairs of inertia - gravity waves near the coast is possible for a continuum of IGW - should be ubiquitous. The mechanism generates KW "from nothing". Subsequent slow evolution of KW leads to nontrivial transport and mixing properties.

Equations of motion

2-layer rotating shallow water on the equatorial tangent plane :

$$\partial_t \mathbf{u}_i + \mathbf{u}_i \cdot \nabla \mathbf{u}_i + \beta y \hat{\mathbf{z}} \times \mathbf{u}_i + \frac{1}{\rho_i} \nabla \pi_i = 0\,,\, i = 1, 2; \tag{29}$$

$$\partial_t h_i + \nabla \cdot (\mathbf{u}_i h_i) = 0\,,\, i = 1, 2, \tag{30}$$

$\mathbf{u}_i = (u_i(x, y, t),\ v_i(x, y, t))$ - velocity fields, h_i - depths of the layers, $h_1 + h_2 = H$ - rigid lid upper b.c..

Barotropic and baroclinic velocities :

$$\mathbf{u}_{bt} = \frac{h_1 \mathbf{u}_1 + h_2 \mathbf{u}_2}{H},\ \mathbf{u}_{bc} = \mathbf{u}_1 - \mathbf{u}_2. \tag{31}$$

Parameters and characteristic scales

$$q = \frac{H - 2H_1}{H}, \quad s = \frac{H_s}{H}, \quad \epsilon = \frac{\Delta H}{H_s}, \tag{32}$$

ΔH - typical variation of the interface, and $H_s = \frac{H_1(H-H_1)}{H}$.
Equatorial scaling :

$$L = \frac{(g' H_s)^{\frac{1}{4}}}{\sqrt{\beta}}; \; T = \frac{1}{\beta L}; \quad U = \frac{g' \Delta H}{\beta L^2}. \tag{33}$$

Reduced gravity : $g' = g(\rho_2 - \rho_1)/\rho_1$

Baroclinic/barotropic decomposition

$$\nabla^2\psi_t + \psi_x = \epsilon\left[-J(\psi,\nabla^2\psi) - s(\partial_{xx}-\partial_{yy})\left[(1+\epsilon qh)(uv)\right] + s\partial_{xy}\left[(1+\epsilon qh)(u^2-v^2)\right]\right)\right]$$

$$\mathbf{u}_t + \nabla h + y\hat{\mathbf{z}}\times\mathbf{u} = \epsilon\left[-J(\psi,\mathbf{u}) + \mathbf{u}\cdot\nabla(\hat{\mathbf{z}}\times\nabla\psi) - q\mathbf{u}\cdot\nabla\mathbf{u} + \epsilon s\left(2h\mathbf{u}\cdot\nabla\mathbf{u} + \mathbf{u}\,\mathbf{u}\cdot\nabla h\right)\right],$$

$$h_t + \nabla\cdot\mathbf{u} = \epsilon\left[-J(\psi,h) - q\nabla\cdot(\mathbf{u}h) + \epsilon s\nabla\cdot\left(h^2\mathbf{u}\right)\right].$$

where ψ - barotropic streamfunction, $\mathbf{u}=(u,v)$ - baroclinic velocity and $h_1=h$ - depth of the upper layer.

Linear wave spectrum

- Free barotropic Rossby waves propagating at any angle :

$$\psi_0 = A_\psi e^{i(\theta + ly)} + c.c.;\ \theta = kx - \sigma t; \qquad (34)$$

with the dispersion relation

$$\sigma = -k/(k^2 + l^2). \qquad (35)$$

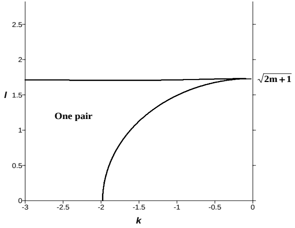
- Trapped baroclinic waves

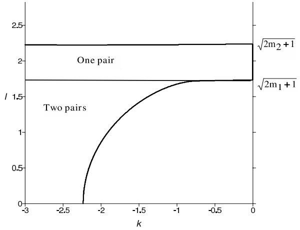
$$(u, v, h) = (iU_m, \phi_m, iH_m)\, Ae^{i\theta_m} + c.c.;\ \theta_m = \hat{k}x - \sigma_m t \qquad (36)$$

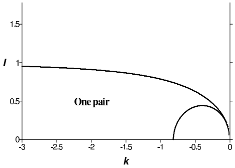
with strongly localized at $y = 0$ amplitudes and dispersion relation

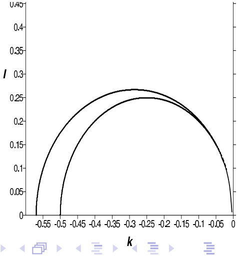
$$\hat{\sigma}_m^3 - (\hat{k}^2 + 2m + 1)\hat{\sigma}_m - \hat{k} = 0;\ m = 0, 1, 2, \dots . \qquad (37)$$

Solutions to the synchronism conditions









First nonlinear correction to the triad :

$$
\begin{aligned}
\nabla^2 \psi_t^{(1)} + \psi_x^{(1)} &= -\nabla^2 \psi_T^{(0)} + N_\psi, \\
u_t^{(1)} - yv^{(1)} + h_x^{(1)} &= -u_T^{(0)} + N_u, \\
v_t^{(1)} + yu^{(1)} + h_y^{(1)} &= -v_T^{(0)} + N_v, \\
h_t^{(1)} + u_x^{(1)} + v_y^{(1)} &= -h_T^{(0)} + N_h,
\end{aligned}
$$

Nonlinear source-terms

$$
\begin{aligned}
N_\psi &= -J(\psi^{(0)}, \nabla^2\psi^{(0)}) - s(\partial_{xx} - \partial_{yy})(u^{(0)}v^{(0)}) \\
&+ s\partial_{xy}(u^{(0)2} - v^{(0)2}), \\
N_u &= \underline{-J(\psi^{(0)}, u^{(0)}) + \mathbf{u}^{(0)} \cdot \nabla\psi_y^{(0)}} - q\mathbf{u}^{(0)} \cdot \nabla u^{(0)} \\
N_v &= \underline{-J(\psi^{(0)}, v^{(0)}) - \mathbf{u}^{(0)} \cdot \nabla\psi_x^{(0)}} - q\mathbf{u}^{(0)} \cdot \nabla v^{(0)} \\
N_h &= \underline{-J(\psi^{(0)}, h^{(0)})} - q\nabla \cdot (\mathbf{u}^{(0)}h^{(0)}).
\end{aligned}
$$

Only underlined terms may contain resonances.

Removal of resonances :

$$A_{1_T} = L_1 A_\psi \bar{A}_2, \;\; \bar{A}_{2_T} = \bar{L}_2 \bar{A}_\psi A_1,$$

These equations are reduced to a single one :

$$A_{\alpha_{TT}} = C \left| A_\psi \right|^2 A_\alpha, \;\; C = L_1 \bar{L}_2.$$

C is real. An important property is $C > 0 \Rightarrow$ exponential growth.

Energy balance

$$E = E_{bt} + E_{bc} = const,$$

where E_{bt}, E_{bc} are the barotropic and the baroclinic energies :

$$E_{bt} = \int_{-\infty}^{\infty} dy \, \langle (\nabla\psi)^2 \rangle_x, \;\; E_{bc} = \frac{s}{2} \int_{-\infty}^{\infty} dy \, \langle (1+\epsilon q h)(u^2+v^2+h^2) \rangle_x.$$

Quadratic energy form conserved in the lowest order :

$$E_0 = \frac{s}{2} \int_{-\infty}^{\infty} dy \, \langle (u^{(0)2}+v^{(0)2}+h^{(0)2}) \rangle_x + \int_{-\infty}^{\infty} dy \, \langle \nabla\psi^{(0)} . \nabla\psi^{(1)} \rangle_x,$$

First barotropic correction (the barotropic response of the equator) is crucial

"Pure" parametric resonance $\sigma = 2\hat{\sigma},\ k = 2\hat{k}$.

Amplitude (Landau) equation :

$$A_{T_2} + LA_{\psi}\bar{A} + (P + iQ)\left|A\right|^2 A = 0.$$

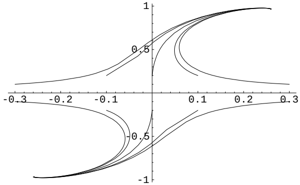
$Q = Q_0 + qQ_1 + sQ_2,\ L, P, Q$ are real, $P \geq 0,\ T_2 = \epsilon^2 t$.
Term $\propto L$: the interaction of the primary barotropic wave with the zero-order baroclinic one,
Term $\propto P + iQ_0$: interaction between the secondary barotropic mode and the zero-order baroclinic mode,
Term with Q_1 : interaction between zero- and first-order baroclinic fields,
Term with Q_2 : cubic interaction of the zero-order baroclinic mode.

Stationary solutions :

$$A_0 = 0,\ A_{\pm}^2 = -\frac{LA_{\psi}}{P + iQ}.$$

A_0 is unstable, $A_{\pm}$ are stable and attractive if $P > 0$ (neutrally stable if $P = 0$). $\Rightarrow$ nonlinear saturation always takes place.

Typical behaviour of the trajectories of solutions of the Landau equation in the phase-space $ReA - ImA$



Two different baroclinic waves.

$$
\begin{aligned}
A_{1_{T_2}} + \alpha_1 \bar{A}_2 + \beta_1 |A_2|^2 A_1 + \gamma_1 |A_1|^2 A_1 &= 0 \\
A_{2_{T_2}} + \alpha_2 \bar{A}_1 + \beta_2 |A_1|^2 A_2 + \gamma_2 |A_2|^2 A_2 &= 0.
\end{aligned}
$$

The following properties of the coefficients may be established :

$$\alpha_1 \bar{\alpha}_2 > 0,\ Re(\beta_1 + \beta_2) \geq 0,\ Re(\gamma_1) \geq 0,\ Re(\gamma_2) \geq 0.$$

$\alpha_{1,2}$ are either both real or both imaginary, and at least one of $Re(\gamma_1), Re(\gamma_2)$ is > 0. $\Rightarrow$ saturation.
Solution - attracting limit cycle :

$$A_{1,2} = B_{1,2} e^{\pm i\omega T_2}$$

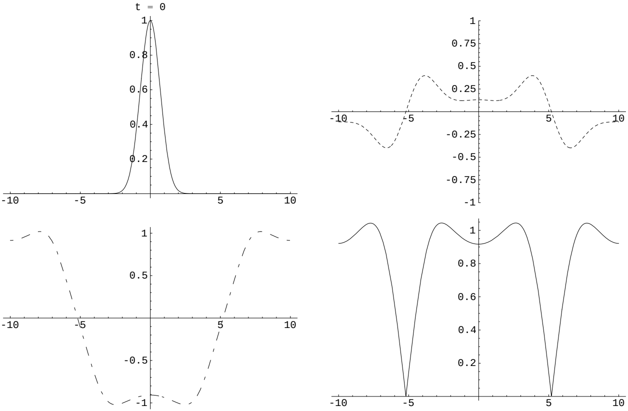
"Pure" parametric resonance $\sigma = 2\hat{\sigma},\ k = 2\hat{k}$ with spatial modulation

Amplitude equation in the reference frame moving with the trapped wave :

$$A_{T_2} - \frac{i}{2}\hat{\sigma}''(\hat{k})A_{X_1X_1} - LA_{\psi}\bar{A} + (P + iQ)\left|A\right|^2 A = 0.$$

$\hat{\sigma}''(\hat{k})$ is the derivative of the group velocity with respect to $\hat{k}$, $X_1 = \epsilon x$.
Equation of the Ginzburg - Landau (GL) type : resonantly forced GL equations known in optics and in the Faraday effect.
The stationary solutions are still solutions, stable for $P > 0$. Two different stationary states - characteristic Bloch-type domain wall structures appear, forming a bound state (a so-called bubble, or "dark soliton")

Domain walls and dark soliton



Initial (top left), and final profiles of real (dashed) and imaginary (dash-dotted) parts of A, and $|A|$ (solid)

A pair of baroclinic waves

The amplitude equations in the lowest order :

$$A_{i_{T_1}} + c_{g_i} A_{i_{X_1}} = 0, \; i = 1, 2,$$

c_{g_i} - group velocities of the respective waves. Next order :

$$\begin{aligned} A_{1_{T_2}} + c_{g_1} A_{1_{X_2}} - \frac{i}{2}\sigma_1''(k_1) A_{1_{X_1 X_1}} + \alpha_1 \bar{A}_2 &+ \beta_1 |A_2|^2 A_1 \\ &+ \gamma_1 |A_1|^2 A_1 = 0 \\ A_{2_{T_2}} + c_{g_2} A_{2_{X_2}} - \frac{i}{2}\sigma_2''(k_2) A_{2_{X_1 X_1}} + \alpha_2 \bar{A}_1 &+ \beta_2 |A_1|^2 A_2 \\ &+ \gamma_2 |A_2|^2 A_2 = 0, \end{aligned}$$

$X_2 = \epsilon^2 x$.

Comments

- Resonant excitation of trapped waves by non-trapped ones in *semitransparent* waveguide = parametric instability of the barotropic wave with respect to baroclinic perturbations.
- New type of GL equation : spatio-temporal organization ("dark solitons")
- Coupled GL-type equations : slow variability, fronts.

Summary of the results

What we have seen :

- Incident IGW can generate coastal Kelvin waves "from nothing"
- Barotropic planetary waves passing through equator excite equatorial Rossby and Yanai waves

What we have not seen, but can be shown :

- Shelf and edge waves can be generated by IGW similarly to Kelvin waves
- Generation of trapped waves both in coastal and equatorial wave-guides can be provided by interaction of a free wave with a mean current, although resonance conditions are more restrictive.

Discussion

Importance

- Teleconnections midlatitudes-tropics
- Slow variability in the tropical atmosphere
- Transport of energy in the ocean from the large to the coasts $\Rightarrow$ a route for dissipation
- Transport and mixing properties in the coastal zones

Realisability

Weakly nonlinear phenomen : efficiency conditioned by persistent of free-wave flux towards the wave-guide.