On the relation between distributional and Leray-Hopf solutions to the Navier-Stokes equations

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Abstract

Consider the three-dimensional Cauchy problem for the Navier-Stokes equations

$$\left.\begin{array}{l}\partial_t v + v\cdot\nabla v = \Delta v - \nabla p\\ \operatorname{div} v = 0\end{array}\right\}\ \text{in }\ \mathbb{R}^3\times(0,\infty) \qquad (1)$$
$$v(x,0) = v_0(x)\ \ x\in\mathbb{R}^3\,.$$

It is well known that for any $v_0 \in L^2_\sigma(\mathbb{R}^3)$ one can construct a global Leray-Hopf solution to (1), namely, a function v in the $\mathscr{L}\mathscr{H}$-class defined by

$$v \in L^\infty(0,T;L^2_\sigma(\mathbb{R}^3)) \cap L^2(0,T;H^1(\mathbb{R}^3)) \quad \text{all } T>0\,,$$

solving (1) in a distributional sense. It is likewise well known that, to date, in order to ensure further relevant properties for such a solution, one has to assume *in addition* that v satisfies some *extra requirements*. In particular, for a given Leray-Hopf solution v, the validity of the energy equality is ensured if also $v \in L^{4,4}$ (J.-P.Lions), while uniqueness and regularity are secured if either $v \in L^{\frac{2s}{s-3},s}$, $s>3$, or $v \in C([0,T];L^3)$ (Prodi-Serrin-Ladyzhenskaya).

Objective of this talk is to show that a generic distributional solution corresponding to the data v_0 and satisfying any of these conditions must, in fact, be in the $\mathscr{L}\mathscr{H}$-class, so that the *assumption* in the above results that v is in such a class becomes *redundant*.

Keywords: Navier–Stokes equations, Leray-Hopf solution, distributional solution.